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by

Qikeng Lu, Shikun Wang,
Wu Ke

Qikeng Lu
Institute of Mathematics, AMSS, CAS
Shikun Wang
Institute of Applied Mathematics, AMSS, CAS
Wu Ke
Dept. Math., Capital Normal University

Department of Mathematics
Faculty of Science and Technology
Keio University

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Global Solutions of Einstein-Dirac Equation on the Conformal Space [1]

Qikeng Lu
Institute of Mathematics, AMSS, CAS
Beijing 100080, China
e-mail: luqik@public.bta.net.cn

Shikun Wang
Institute of Applied Mathematics, AMSS, CAS
Beijing 100080, China
e-mail: xyswsk@pku.edu.cn

Wu Ke
Dept. Math., Capital Normal University
Beijing 100080, China
e-mail: wuke@itp.ac.cn

ABSTRACT

The conformal space $\mathfrak{M}$ was introduced by Dirac in 1936. It is an algebraic manifold with a spin structure and possesses naturally an invariant Lorentz metric. By carefully studying the birational transformations of $\mathfrak{M}$, we obtain explicitly the transition functions of the spin bundle over $\mathfrak{M}$. Since the transition functions are closely related to the propagator in physics, we get a kind of solutions of the Dirac equation by integrals constructed from the propagator. Moreover, we prove that the invariant Lorentz metric together with one of such solutions satisfies the Einstein-Dirac combine equation.

§1. The main results

In general relativity the 4-dimensional Lorentz manifold is used. It is Penrose [1] who began to apply 2-component spinor analysis for studying Einstein equation. It implied that the spin group $Spin(1,3)$ of a Lorentz spin manifold $\mathfrak{M}$ is locally isomorphic to the group $SL(2,\mathbb{C})$ such that there is a Lie group homeomorphism

$$\iota: \ SL(2,\mathbb{C}) \longrightarrow SO(1,3)$$

which is a two to one covering map. Then a two component Dirac operator $\mathfrak{D}: V_2(x) \to V_2^*(x)$ and $\mathfrak{D}: V_2^*(x) \to V_2(x)$ can be defined, where $V_2(x)$ is the vector space of spinors at $x \in \mathfrak{M}$ and $V_2^*(x)$ is the conjugate vector space of $V_2(x)$.

We will use the following lemma for studying the Dirac equation.

Lemma 1 If ψ is a two component spinor field on $\mathfrak{M}$ and satisfies

$$\mathfrak{D}^2\psi = \mathfrak{D}\mathfrak{D}\psi = -m^2\psi \tag{1.1}$$

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then

$$\Psi = \begin{pmatrix} \psi \\ \varphi^* \end{pmatrix}, \quad \varphi^* = \frac{i}{m}\mathfrak{D}\psi \tag{1.2}$$

is a 4-component spinor on $\mathfrak{M}$ and satisfies the Dirac equation

$$\not{\mathcal{D}}\Psi = \begin{pmatrix} 0 & \mathfrak{D} \\ \mathfrak{D} & 0 \end{pmatrix}\Psi = -im\Psi. \tag{1.3}$$

The first purpose of this paper is to solve the equation (1.1) in the case that $\mathfrak{M}$ is the conformal space.

The conformal space $\mathfrak{M}$ was introduced by Dirac [2]. It is a quadratic algebraic 4-dimensional manifold defined by

$$\mathfrak{x}_1^2 + \mathfrak{x}_2^2 - \mathfrak{x}_3^2 - \mathfrak{x}_4^2 - \mathfrak{x}_5^2 - \mathfrak{x}_6^2 = 0,$$

where $\mathfrak{x} = (\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_6)$ is the homogeneous coordinate of the real project space $\mathbb{RP}^5$, and it is the boundary of the 5-dimensional anti-de-Sitter space AdS_5:

$$\mathfrak{x}_1^2 + \mathfrak{x}_2^2 - \mathfrak{x}_3^2 - \mathfrak{x}_4^2 - \mathfrak{x}_5^2 - \mathfrak{x}_6^2 > 0.$$

So to study the field theory of the conformal space would be useful to study the problem of AdS/CFT corresponding, a research hot point in recent years (see the references in [3]). It should be noted that AdS is also introduced by Dirac [4] and is one kind of space-time studied in [5].

We use heavily the birational transformations of algebraic geometry to study in detail the transition functions of the Lorentz spin manifold $\mathfrak{M}$ so that the solutions Ψ of the Dirac equation can be expressed explicitly by integrals.

Let

$$ds^2 = g_{jk}ds^j ds^k = \sum_{j,k=0}^{3} g_{jk}dx^j dx^k = \eta_{ab}\omega^a\omega^b \tag{1.4}$$

be a Lorentz metric on $\mathfrak{M}$, where $(\eta_{ab}) = \{1, -1, -1, -1\}$ is a diagonal matrix and

$$\omega^a = e_j^{(a)}dx^j, \ (a = 0, 1, 2, 3); \quad \text{and} \quad X_a = e_{(a)}^j \frac{\partial}{\partial x^j} \quad (a = 0, 1, 2, 3) \tag{1.5}$$

are the Lorentz coframe and the dual frame respectively.

The second purpose of this paper is to find solutions of g_{jk} and Ψ which satisfy the Einstein-Dirac equation

$$R_{jk} - \frac{1}{2}Rg_{jk} - \Lambda g_{jk} = \mathcal{X}T_{jk}, \quad \not{\mathcal{D}}\Psi = -im\Psi \tag{1.6}$$

where Λ, $\mathcal{X}$ and $m(> 0)$ are constants and T_{jk} is the energy-momentum tensor of Ψ such that

$$T_{jk} = \frac{1}{2}[\eta_{ab}\overline{\Psi}^{*\prime}\gamma^b(e_j^{(a)} \nabla_k \Psi + e_k^{(a)} \nabla_j \Psi) - \eta_{ab}(e_j^{(a)}\overline{\nabla_k\Psi^*}' + e_k^{(a)}\overline{\nabla_j\Psi^*}')\gamma^b\Psi]. \tag{1.7}$$

Here we denote $\overline{A}$ the complex conjugate of a matrix A and A' the transpose of A and

$$\Psi^* = \begin{pmatrix} \varphi^* \\ \psi \end{pmatrix}. \tag{1.8}$$

Besides, $\gamma^a(a = 0, 1, 2, 3,)$ are Dirac matrices and ∇_j is the covariant differentiation of 4-component spinor such that

$$\not{\mathcal{D}} = \gamma^a e_{(a)} \nabla_j \,. \tag{1.9}$$

We at first map the conformal space $\mathfrak{M}$ by birational transformation into the compactized Minkowski space $\overline{M}$, which can be mapped by birational transformation [6] to the group manifold $U(2) \cong U(1) \times SU(2)$, and we will prove that $SU(2) \cong \overline{M} \cap \mathbf{P}_0$, where $\mathbf{P}_0$ is a hyperplane. It known that $U(1) \cong S^1$ and $SU(2) \cong S^3$. So we can introduce a Lorentz metric ds^2 on $\mathfrak{M}$ such that

$$ds^2 = ds_1^2 - ds_3^2 \tag{1.10}$$

where

$$ds_1^2 = (dx^0)^2 \quad \text{and} \quad ds_3^2 = \frac{\delta_{\alpha\beta}}{(1+xx')^2} dx^\alpha dx^\beta \tag{1.11}$$

are the Riemann metrics of $U(1)$ and $SU(2) \cong S^3$ respectively.

Since $SU(2) \cong \overline{M} \cap \mathbf{P}_0$ is a Riemann spin manifold, there is a principal bundle

$$Spin\{\overline{M} \cap \mathbf{P}_0, SU(2)\}$$

with base manifold $\overline{M} \cap \mathbf{P}_0$ and structure group $SU(2)$. The transition functions of this principal bundle can be written out explicitly.

Lemma 2 The isometric automorphism $T_u : \overline{M} \cap \mathbf{P}_0 \to \overline{M} \cap \mathbf{P}_0$ can be expressed by admissible local coordinates such that

$$y^\alpha \sigma_\alpha = U_0^{-1} \Phi(x,u) U_0, \quad \Phi(x,u) = (\sigma_0 + x^\mu u^\nu \sigma_\mu \sigma_\nu)^{-1} (x^\alpha - u^\alpha) \sigma_\alpha$$

where $U_0 \in SU(2)$, σ_0 is the 2×2 identity matrix and $\sigma_\alpha (\alpha = 1,2,3)$ are Pauli matrices. The transition function associated to T_u is

$$\mathfrak{A}_{T_u}(x) = U_0^{-1} U(x,u)^{-1},$$

where

$$U(x,u) = [(1+xu')^2 + xx'uu' - (xu')^2]^{-\frac{1}{2}} \left[(1+xu')\sigma_0 + i x^\mu u^\nu \delta^{123}_{\mu\nu\alpha} \sigma_\alpha\right],$$

which belongs to $SU(2)$ and $xu' = \delta_{\alpha\beta} x^\alpha u^\beta$.

U(x,u) is called the propagator.

With the metric (1.10) the 2-component Dirac operator of $S^1 \times S^3$ is

$$\mathfrak{D} = \frac{\partial}{\partial x^0} - \not{D}_{s^3} \tag{1.12}$$

where $\not{D}_{s^3}$ is the Dirac operator of the Riemann spin manifold of S^3 and x^0 the local coordinate of S^1 and $x = (x^1, x^2, x^3)$ the admissible local coordinate of S^3. Hence, if the spinor $\widehat{\psi}(x)$ satisfies the equation

$$\not{D}_{s^3}^2 \widehat{\psi} = -(n^2 - m^2)\widehat{\psi} \tag{1.13}$$

then $e^{inx^0}\widehat{\psi}(x)$ is a solution of the equation

$$\mathfrak{D}^2[e^{inx^0}\widehat{\psi}(x)] = -m^2 e^{inx^0}\widehat{\psi}(x). \tag{1.14}$$

By Weitzenböck formula of S^3,

$$\not{D}_{s^3}^2 = \Delta - \frac{1}{4} R_{S^3} \sigma_0 \tag{1.15}$$

where R_{S^3} is the scalar curvature of ds_3^2 and Δ is an elliptic differential operator. Hence to solve the equation (1.1) on $S^1 \times S^3$ is reduced to solve the equation on S^3,

$$\not{D}_{S^3}^2 \widehat{\psi}(x) = -\lambda \widehat{\psi}(x) \tag{1.16}$$

where $\lambda = n^2 - m^2$ should be an eigen-value of $\not{\mathcal{D}}^2_{S^3}$. The λ-eigen kernel is defined by

$$\mathcal{K}_\lambda(x,u) = \sum_{\xi=0}^{N_\lambda} \widehat{\psi}_\xi(x)\overline{\widehat{\psi}_\xi(u)'} \tag{1.17}$$

where $\{\widehat{\psi}_\xi(x)\}_{\xi=1,2,\cdots,N_\lambda}$ is an orthonormal basis of the vector space of λ-eigen functions of $\not{\mathcal{D}}^2_{S^3}$. The eigen values of (1.16) and the corresponding dimensions N_λ are known(c. f. [7]) Then for any spinor $\widehat{\psi}_0$ on S^3,

$$\widehat{\psi}(x) = \int_{S^3} \mathcal{K}_\lambda(x,u)\widehat{\psi}_0(u)\dot{u}, \tag{1.18}$$

where $\dot{u}$ is the volume element associated to ds_3^2, is a solution of the equation (1.16). The problem to solve the Dirac equation on the conformal space $\mathfrak{M} \cong S^1 \times S^3$ is reduced to construct the λ-eigen kernel $\mathcal{K}_\lambda$ of $\not{\mathcal{D}}^2_{S^3}$ on S^3 explicitly.

THEOREM 1 If we choose on $S^3 \cong SU(2)$ the metric

$$ds_3^2 = \frac{\delta_{\alpha\beta}}{(1+xx')^2} dx^\alpha dx^\beta, \tag{1.19}$$

then the λ-eigen kernel of $\not{\mathcal{D}}^2_{S^3}$ is

$$\mathcal{K}_\lambda(x,u) = U(x,u)\big[f(\rho^2(x,u))\sigma_0 + h(\rho^2(x,u))\Phi(x,u)\big],$$

where $U(x,u)$ and $\Phi(x,u)$ are defined by Lemma 2,

$$\rho^2(x,u) = \frac{(x-u)(x-u)'}{1+2xu'+xx'uu'},$$

and $f(t) = \overline{f(t)}$ and $h(t) = -\overline{h(t)}$ are functions which satisfy respectively the following differential equations

$$4t(1+t)^2\frac{d^2f}{dt^2} + (1+t)[6(1+t)-4t)]\frac{df}{dt} - (2t+6)f = -\lambda f$$

and

$$4t(1+t)^2\frac{d^2h}{dt^2} + (1+t)[10(1+t)-4t)]\frac{dh}{dt} - 4h = -\lambda h.$$

In fact, the solutions of the equations are respectively

$$f(t) = c_0F_0(t) + c_1F_1(t) \quad \text{and} \quad h(t) = c_2F_2(t) + ic_3F_3(t) \tag{1.20}$$

where $c_j (j=0,1,2,3)$ are real constants,

$$F_0(t) = (1+t)^{1+\sqrt{\lambda}/2}F(\frac{\sqrt{\lambda}}{2},\frac{3}{2}+\frac{\sqrt{\lambda}}{2},1+\frac{\sqrt{\lambda}}{2},1+t), \quad F_1(t) = (1+t)^{1-\sqrt{\lambda}/2}F(-\frac{\sqrt{\lambda}}{2},\frac{3}{2}-\frac{\sqrt{\lambda}}{2},1-\frac{\sqrt{\lambda}}{2},1+t) \tag{1.21}$$

and

$$F_2(t) = (1+t)^{1+\sqrt{\lambda}/2}F(\frac{3}{2}+\frac{\sqrt{\lambda}}{2},1+\frac{\sqrt{\lambda}}{2},1+\frac{\sqrt{\lambda}}{2},1+t), \quad F_3(t) = (1+t)^{1-\sqrt{\lambda}/2}F(\frac{3}{2}-\frac{\sqrt{\lambda}}{2},1-\frac{\sqrt{\lambda}}{2},1-\frac{\sqrt{\lambda}}{2},1+t). \tag{1.22}$$

Here $F(\alpha,\beta,\gamma,x)$ is the hypergeometric function. The constants $c_j (j=0,1,2,)$ are determined from the equality

$$\int_{S^3} \mathcal{K}_\lambda(a,x)\mathcal{K}_\lambda(x,b)\dot{x} = \mathcal{K}_\lambda(a,b). \tag{1.23}$$

Since

$$\overline{\mathcal{K}_\lambda(a,b)}' = \mathcal{K}_\lambda(b,a), \tag{1.24}$$

there are four independent equations in (1.23) for determining the four constants $c_j (j = 0, 1, 2, 3)$.

A spinor $\widehat{\psi}_0(x)$ on S^3 is said to be orthogonal invariant if $\widehat{\psi}_0(x\Gamma) = U\widehat{\psi}_0(x)$, where $\Gamma \in SO(3)$ and $U \in SU(2)$ such that Γ is the image of U by group homeomorphism ι restricted to the group $SU(2)$. The two component spinor

$$\psi(x_1) = e^{inx^0}\widehat{\psi}(x), \quad x_1 = (x^0, x), \tag{1.25}$$

where $\widehat{\psi}(x)$ defined by (1.19), is orthogonal invariant, provided that $\widehat{\psi}_0(x)$ is orthogonal invariant. By Lemma 1, the 4-component spinor on $S^1 \times S^3$

$$\Psi = \begin{pmatrix} \psi \\ \varphi^* \end{pmatrix}, \qquad \varphi^* = \frac{i}{m}\mathfrak{D}\psi \tag{1.26}$$

satisfies the Dirac equation and it is orthogonal invariant in the sense that $\varphi^*(x^0, x\Gamma) = U\varphi^*(x_1)$ whenever ψ is orthogonal. So

$$\Psi(x^0, x\Gamma) = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \Psi(x^0, x).$$

THEOREM 2. If g_{ij} are defined by

$$g_{00} = 1, g_{0\alpha} = g_{\alpha 0} = 0, g_{\alpha\beta} = -\frac{\delta_{\alpha\beta}}{(1 + xx')^2}, \quad \alpha, \beta = 1, 2, 3,$$

and Ψ is defined by the integral (1.20) and is orthogonal invariant and the energy-momentum tensor T_{jk} of Ψ is not identically zero, then the pair $\{g_{jk}, \Psi\}$ satisfy the Einstein-Dirac equation with the constants

$$\mathcal{X} = \frac{R_{11}(0)}{T_{00}(0) + T_{11}(0)}, \quad \Lambda = \frac{-T_{00}}{T_{00}(0) + T_{11}(0)} - \frac{1}{2}R(0)$$

and m is non-negative and satisfies

$$m^2 = n^2 - \lambda$$

where n is a positive integer and λ is an eigen value of the operator $\mathcal{D}^2_{s^3}$.

§2. The relation between the Dirac operators of 2-component spinor and 4-component spinor

Let $\mathfrak{M}$ be a four-dimensional Lorentz spin manifold with the Lorentz metric

$$ds^2 = g_{ij}dx^j dx^j = \eta_{ab}\omega^a\omega^b \tag{2.1}$$

where $x = (x^0, x^1, x^2, x^3)$ is an admissible local coordinate of $\mathfrak{M}$, η_{ab} is a diagonal matrix with diagonal elements $\{1, -1, -1, -1\}$ and

$$\omega^a = e^{(a)}_j dx^j, \qquad a = 0, 1, 2, 3 \tag{2.2}$$

is a Lorentz co-frame. Let the dual frame of $\{\omega^a\}$ be

$$\mathbb{X}_a = e^j_{(a)}\frac{\partial}{\partial x^j}. \tag{2.3}$$

¿From the Christoffel symbol associated to ds^2

$$\left\{ \begin{array}{c} l \\ j \quad k \end{array} \right\} = \frac{1}{2} g^{li} \left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right), \tag{2.4}$$

which is an $\mathfrak{gl}(4, \mathbb{R})$-connection, there is a Lorentz connection

$$\Gamma^a_{bj} = e^{(a)}_k \frac{\partial e^k_{(b)}}{\partial x^j} + e^{(a)}_l \left\{ \begin{array}{c} l \\ k \quad j \end{array} \right\} e^k_{(b)}. \tag{2.5}$$

We denote the matrix

$$\Gamma_j = \left(\Gamma^a_{bj}\right)_{0 \le a,b \le 3}. \tag{2.6}$$

If we change the local coordinate $\widetilde{x}^\alpha = \widetilde{x}^\alpha(x)$ and the corresponding Lorentz co-frame as follows

$$\widetilde{\omega}^a(\widetilde{x}) = \ell^a_b(x)\omega^b(x), \qquad L(x) = (\ell^a_b(x))_{0 \le a,b \le 3} \in O(1,3) \tag{2.7}$$

then the Lorentz connection $\widetilde{\Gamma}_j$ satisfies the relation

$$\widetilde{\Gamma}_j = \left(L\Gamma_k L^{-1} - \frac{\partial L}{\partial x^k} L^{-1} \right) \frac{\partial x^k}{\partial \widetilde{x}^j}. \tag{2.8}$$

Since Γ_j for each j belongs to the of Lie algebra of $O(1,3)$ and this algebra is $\mathfrak{so}(1,3)$, we have

$$Tr(\Gamma_j) = 0. \tag{2.9}$$

There is a Lie group homeomorphism

$$\iota : SL(2, \mathbb{C}) \to SO(1,3) \tag{2.10}$$

which is defined by the following manner. Let

$$\sigma_0 = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \quad \sigma_1 = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma_2 = \left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \sigma_3 = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \tag{2.11}$$

which forms a base of the vector space of all 2×2 Hermitian matrices. For any $\mathfrak{A} \in SL(2, \mathbb{C})$ we denote the transpose matrix and the complex conjugate matrix of $\mathfrak{A}$ by $\mathfrak{A}'$ and $\overline{\mathfrak{A}}$ respectively. Each matrix $\mathfrak{A}\sigma_j\overline{\mathfrak{A}}'$ is a Hermitian matrix, so it can be expressed as a linear combination of σ_k. That is

$$\mathfrak{A}\sigma_j\overline{\mathfrak{A}}' = \ell^k_j \sigma_k. \tag{2.12}$$

It is proved (see [8] Th. 2.4.1) that the corresponding matrix

$$L = \left(\ell^j_k\right)_{0 \le j,k \le 3} \in SO(1,3) \tag{2.13}$$

and the homeomorphism ι is a two to one covering map and hence a local isomorphism. Especially, when $\mathfrak{A} \in SU(2)$, the corresponding L is of the form

$$L = \left(\begin{array}{cc} 1 & 0 \\ 0 & K \end{array} \right), \qquad K \text{ is a } 3 \times 3 \text{ orthogonal matrix.} \tag{2.14}$$

Moreover, according to Th. 2.4.2 in [8], associated to the $\mathfrak{so}(1,3)$-connection Γ_j, there is locally a $\mathfrak{sl}(2, \mathbb{C})$-connection

$$\mathfrak{B}_j = \frac{1}{4} \eta^{cb} \Gamma^a_{cj} . \sigma_a \sigma^*_b, \quad \sigma^*_b = \epsilon \overline{\sigma}_b \epsilon', \quad \epsilon = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right). \tag{2.15}$$

This means that, when Γ_j suffers the transformation relation (2.8), the corresponding relation of $\mathfrak{B}$ is

$$\widetilde{\mathfrak{B}}_j = (\mathfrak{A}\mathfrak{B}_k\mathfrak{A}^{-1} - \frac{\partial \mathfrak{A}}{\partial x^k}\mathfrak{A}^{-1})\frac{\partial x^k}{\partial \widetilde{x}^j} \tag{2.16}$$

where $\mathfrak{A}$ corresponds to the matrix L defined by (2.12). When $\mathfrak{M}$ is a Lorentz spin manifold $\mathfrak{B}_j$ is globally defined on $\mathfrak{M}$. We call $\mathfrak{B}_j$ the 2-component spinor connection derived from the Lorentz connection of the spin manifold $\mathfrak{M}$.

A two component spinor ψ on a Lorentz spin manifold $\mathfrak{M}$ is a vector

$$\psi(x) = \begin{pmatrix} \psi^1(x) \\ \psi^2(x) \end{pmatrix}$$

on each admissible local coordinate neighborhood $\mathfrak{V}$ and x is the local coordinate of this neighborhood. Let $\widetilde{\psi}(\widetilde{x})$ is the vector defined on another admissible local coordinate neighborhood $\widetilde{\mathfrak{V}}$ and $\widetilde{x}$ is the corresponding local coordinate of $\widetilde{\mathfrak{V}}$. When $\mathfrak{V} \cap \widetilde{\mathfrak{V}} \neq \emptyset$, there a matrix $\mathfrak{A} \in SL(2, \mathbb{C})$ such that

$$\widetilde{\psi}(\widetilde{x}) = \mathfrak{A}(x)\psi(x). \tag{2.17}$$

The matrix $\mathfrak{A}(x)$ is the transition function of the spin manifold $\mathfrak{M}$.

A spinor ψ corresponds to a conjugate spinor

$$\psi^* = \epsilon\overline{\psi}. \tag{2.18}$$

Then under the coordinate transformation between two admissible local coordinates,

$$\widetilde{\psi}^*(\widetilde{x}) = \overline{\mathfrak{A}}'^{-1}\psi^*(x) \tag{2.19}$$

because for any 2×2 matrix A

$$A\epsilon A' = (det A)\epsilon. \tag{2.20}$$

Now we can define the covariant differential $\mathfrak{D}_j$ of a spinor ψ by the connection $\mathfrak{B}_j$ such that

$$\mathfrak{D}_j\psi = \frac{\partial \psi}{\partial x^j} + \mathfrak{B}_j\psi. \tag{2.21}$$

which satisfies

$$\widetilde{\mathfrak{D}}_j\widetilde{\psi} = \frac{\partial x^k}{\partial \widetilde{x}^j}\mathfrak{A}\mathfrak{D}_k\psi. \tag{2.22}$$

under admissible coordinate transformation. This means that $\mathfrak{D}_j\psi$ is still a spinor, but a covariant vector with respect to the index j. If we operate again to $\mathfrak{D}_j\psi$ by $\mathfrak{D}_k$ and wish $\mathfrak{D}_k\mathfrak{D}_j\psi$ still be covariant, then it needs in addition a $\mathfrak{gl}(4, \mathbb{R})$ connection to define the covariant differentiation of $\mathfrak{D}_j\psi$. In usual tensor calculus, a covariant differentiation ∇_j of a contravariant vector can be extended to operate on any mixed tensors. We can do the same to define $\mathfrak{D}_j$ such that it can operate on mixed tensors.

Since

$$\mathfrak{B}_j = (\mathfrak{B}^A_{Bj})_{1 \leq A, B \leq 2} \tag{2.23}$$

is derived from the $\mathfrak{so}(1,3)$-connection Γ^a_{bj} by (2.15) and Γ^a_{bj} is derived from the $\mathfrak{gl}(4, \mathbb{R})$-connection $\left\{ \begin{array}{c} l \\ j\,k \end{array} \right\}$ by (2.5) and (2.4). $\mathfrak{D}_j$ can be extended to operate on mixed tensor of $SL(2, \mathbb{C})$-,$SO(1,3)$- and $GL(4, \mathbb{R})$-type. For example, the components of the spinor ψ are ψ^A $(A = 1, 2)$. (2.21) can be rewritten into

$$\mathfrak{D}_j\psi^A = \frac{\partial \psi^A}{\partial x^j} + \mathfrak{B}^A_{Bj}\psi^B \tag{2.24}$$

which is contravariant with respect to the spinor index A and covariant with respect to the index j. Then $\mathfrak{D}_k\mathfrak{D}_j\psi^A$ is defined as

$$\mathfrak{D}_k\mathfrak{D}_j\psi^A = \frac{\partial}{\partial x^k}\mathfrak{D}_j\psi^A + \mathfrak{B}^A_{Bk}\mathfrak{D}_j\psi^B - \left\{ \begin{array}{c} l \\ kj \end{array} \right\} \mathfrak{D}_l\psi^A, \tag{2.25}$$

which is still a mixed tensor, contravariant with respect to spin index A and $GL(2,\mathbb{R})$ covariant with respect to the indices j and k. Moreover, if

$$T^{jA\overline{C}}_{aB\overline{D}}$$

is a tensor $GL(4,\mathbb{R})$-contravariant w.r.t. j, $SO(1,3)$-covariant w.r.t. a, spin tensor w.r.t. A, B, C, D, then its covariant differentiation is defined as follows

$$\begin{aligned} \mathfrak{D}_k T^{jA\overline{C}}_{aB\overline{D}} &= \frac{\partial}{\partial x^k} T^{jA\overline{C}}_{aB\overline{D}} + \mathfrak{B}^A_{Ek} T^{jE\overline{C}}_{aB\overline{D}} - \mathfrak{B}^E_{Bk} T^{jA\overline{C}}_{aE\overline{D}} \\ &+ \overline{\mathfrak{B}}^C_{Ek} T^{jA\overline{E}}_{aB\overline{D}} - \overline{\mathfrak{B}}^E_{Dk} T^{jA\overline{C}}_{aB\overline{E}} - \Gamma^b_{ak} T^{jA\overline{C}}_{bB\overline{D}} + \left\{ \begin{array}{c} j \\ lk \end{array} \right\} T^{lA\overline{C}}_{aB\overline{D}} \end{aligned} \tag{2.26}$$

which is a mixed tensor of the same type plus $GL(4,\mathbb{R})$-covariant w.r.s. to the index k.

If ψ is a spinor,

$$\psi^* = \epsilon\overline{\psi} \tag{2.27}$$

is called the conjugate spinor of ψ. The covariant differentiation can be also extended to the conjugate spinor ψ^* such that

$$\mathfrak{D}_j\psi^* = \frac{\partial\psi^*}{\partial x^j} + \mathfrak{B}^*_j\psi^* \qquad \mathfrak{B}^*_j = \epsilon\overline{\mathfrak{B}}_j\epsilon'. \tag{2.28}$$

After this extension of the definition of covariant differentiation we can find its application. Since the following formula

$$\eta_{ab} = \eta_{cd}\ell^c_a\ell^d_b, \quad \text{for any} \quad L = (\ell^a_b)_{0\le a,b\le 3} \in SO(1,3)$$

means that η_{ab} is an $SO(1,3)$-covariant with respect to indices a and b, we have

$$\mathfrak{D}_j\eta_{ab} = \frac{\partial}{\partial x^j}\eta^{ab} - \Gamma^c_{aj}\eta_{cb} - \Gamma^c_{bj}\eta_{ac} = 0.$$

Similarly, let

$$\sigma_a = \left(\sigma^{A\overline{B}}_a\right)_{1\le A,B\le 2}, \quad a = 0,1,2,3, \quad \mathfrak{A} = \left(\mathfrak{A}^A_B\right)_{1\le A,B\le 2}.$$

(2.12) can be written as

$$\sigma^{A\overline{B}}_a = \sigma^{C\overline{D}}_b (L^{-1})^b_a \mathfrak{A}^A_C \overline{\mathfrak{A}}^B_D$$

which is $SO(1,3)$-covariant w.r.t. a, spin contravariant w.r.t. to A and complex conjugate spin contravariant w.r.t. $\overline{B}$. Then

$$\mathfrak{D}_j\sigma^{A\overline{B}}_a = \frac{\partial}{\partial x^j}\sigma^{A\overline{B}}_a - \Gamma^b_{aj}\sigma^{A\overline{B}}_b + \mathfrak{B}^A_{Cj}\sigma^{C\overline{B}}_a + \overline{\mathfrak{B}}^B_{Cj}\sigma^{A\overline{C}}_a = 0.$$

The 2-component Dirac operator is defined by

$$\mathfrak{D} = \eta^{ab}e^j_{(a)}\sigma^*_b\mathfrak{D}_j \tag{2.29}$$

If ψ is a spinor on $\mathfrak{M}$, then according to the definition of σ_b^* and the formula (2.22), we have

$$\widetilde{\mathfrak{D}}\widetilde{\psi} = \overline{\mathfrak{A}}'^{-1}\mathfrak{D}\psi, \quad \widetilde{\mathfrak{D}}\widetilde{\psi}^* = \mathfrak{A}\mathfrak{D}\psi^*. \tag{2.30}$$

This means that $\mathfrak{D}$ is a map

$$\mathfrak{D} : V_2(x) \to V_2^*(x) \quad \text{and} \quad \mathfrak{D} : V_2^*(x) \to V_2(x)$$

where $V_2(x)$ is the vector space of 2-component spinors of $\mathfrak{M}$ at x and $V_2^*(x)$ the conjugate vector space. Obviously,

$$\mathfrak{D}^2 = \mathfrak{D}\mathfrak{D} : \quad V_2(x) \to V_2(x) \quad \text{and} \quad \mathfrak{D}^2 : V_2^*(x) \to V_2^*(x). \tag{2.31}$$

The equation

$$\mathfrak{D}^2\psi = -m^2\psi \tag{2.32}$$

is called the wave equation of spinor on $\mathfrak{M}$.

A solution ψ of the wave equation will give a solution of the 4-component Dirac equation. Before proving this assertion, we at first make clear the relation between the 2-component spinor and 4-component spinor.

Let

$$\gamma^a = \eta^{ab}\begin{pmatrix} 0 & \sigma_b \\ \sigma_b^* & 0 \end{pmatrix}, \qquad a, b = 0, 1, 2, 3. \tag{2.33}$$

According to the relation

$$\sigma_a\sigma_b^* + \sigma_b\sigma_a^* = 2\eta_{ab}\sigma_0. \tag{2.34}$$

we have the relation

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}I \tag{2.35}$$

where I is the 4×4 identity matrix and according to (2.12)

$$\gamma^a\mathcal{R}(\mathfrak{A}) = \ell_b^a(\mathfrak{A})\mathcal{R}(\mathfrak{A})\gamma^b \tag{2.36}$$

where $\ell_b^a(\mathfrak{A})$ is the element corresponding to $\mathfrak{A}$ by (2.12) and

$$\mathcal{R}(\mathfrak{A}) = \begin{pmatrix} \mathfrak{A} & 0 \\ 0 & \overline{\mathfrak{A}}'^{-1} \end{pmatrix} \tag{2.37}$$

is a representation of the group $SL(2, \mathbb{C})$. The relation (2.35) shows that $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ is a set of Dirac matrices and the relation (2.36) means that the group

$$Spin(1, 3) = \{\mathcal{R}(\mathfrak{A})\}_{\mathfrak{A} \in SL(2,\mathbb{C})} \tag{2.38}$$

is an 2 to 1 homeomorphism to the group $SO(1, 3)$. The 4-component vector

$$\Psi = \begin{pmatrix} \psi \\ \varphi^* \end{pmatrix}, \tag{2.39}$$

where ψ is a 2-component spinor and φ^* a conjugate spinor, obviously satisfies the relation

$$\widetilde{\Psi} = \mathcal{R}(\mathfrak{A})\Psi \tag{2.40}$$

and conversely any $Spin(1, 3)$ 4-component spinor must be of the form (2.39).

The Dirac operator $\mathcal{D}\!\!\!/$ is defined by

$$\mathcal{D}\!\!\!/ = \begin{pmatrix} 0 & \mathfrak{D} \\ \mathfrak{D} & 0 \end{pmatrix} \quad \text{and} \quad \nabla_j = \begin{pmatrix} \mathfrak{D}_j & 0 \\ 0 & \mathfrak{D}_j \end{pmatrix} \tag{2.41}$$

and the Dirac equation is

$$\mathcal{D}\!\!\!/\Psi = -im\Psi. \tag{2.42}$$

If the 2-component spinor ψ is a solution of the wave equation (2.32), then we set

$$\varphi^* = \frac{i}{m}\mathfrak{D}\psi \tag{2.43}$$

and obtain

$$\mathfrak{D}\varphi^* = \frac{i}{m}\mathfrak{D}^2\psi = -im\psi \tag{2.44}$$

or

$$\psi = \frac{i}{m}\mathfrak{D}\varphi^* \tag{2.45}$$

and

$$\mathfrak{D}\psi = \frac{i}{m}\mathfrak{D}^2\varphi^* = \frac{-1}{m^2}\mathfrak{D}^3\psi = \mathfrak{D}\psi = -im\varphi^*. \tag{2.46}$$

Hence Ψ defined by (2.39) satisfies the Dirac equation

$$\mathcal{D}\!\!\!/\Psi = -im\Psi. \tag{2.47}$$

This proves **Lemma 1** in §1.

It should be noted that

$$\begin{aligned} \mathcal{D}\!\!\!/\Psi &= \begin{pmatrix} \mathfrak{D}\varphi^* \\ \mathfrak{D}\psi \end{pmatrix} = \begin{pmatrix} (\mathfrak{D}\varphi)^* \\ \mathfrak{D}\psi \end{pmatrix} = \begin{pmatrix} \eta^{ab}e^j_{(a)}\sigma_b\mathfrak{D}_j\varphi^* \\ \eta^{ab}e^j_{(a)}\sigma^*_b\mathfrak{D}_j\varphi \end{pmatrix} \\ &= \eta^{ab}e^j_{(a)}\begin{pmatrix} 0 & \sigma_b \\ \sigma^*_b & 0 \end{pmatrix}\begin{pmatrix} \mathfrak{D}_j\psi \\ \mathfrak{D}_j\varphi^* \end{pmatrix}. \end{aligned}$$

That is

$$\mathcal{D}\!\!\!/\Psi = \gamma^a e^j_{(a)} \nabla_j \Psi \tag{2.48}$$

when we define the covariant differentiation of the 4-component spinor $\Psi = \begin{pmatrix} \psi \\ \varphi^* \end{pmatrix}$ by

$$\nabla_j\Psi = \begin{pmatrix} \mathfrak{D}_j\psi \\ \mathfrak{D}_j\varphi^* \end{pmatrix}. \tag{2.49}$$

§3. The spin structure of S^3

It is well-known that S^3 is a Riemann spin manifold. For solving the Dirac equation on S^3 we need to describe the transition functions of the principal bundle $Spin\{S^3, SU(2)\}$ explicitly.

$$S^3 = \{(a,b) \in \mathbf{C}^2 | \quad |a|^2 + |b|^2 = 1\}$$

is equivalent to $SU(2)$ by the map

$$(a,b) \rightarrow \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}.$$

The unitary group $U(2)$ is the characteristic manifold of the classical domain

$$\mathfrak{R}_I(2.2) = \{W \in \mathbf{C}^{2\times 2} | \quad I - WW^\dagger > 0\}$$

where $W^\dagger = \overline{W}'$. Since $\mathfrak{R}_I(2,2)$ is a domain in the complex Grassmann manifold $\mathfrak{F}(2,2)$, $U(2)$ is a submanifold of $\mathfrak{F}(2,2)$. Since $SU(2)$ is a subgroup of $U(2)$, $SU(2)$ is also a submanifold of $\mathfrak{F}(2,2)$. The complex Grassmann manifold can be described by complex matrix homogeneous coordinate $\mathfrak{Z}$, which is a 2×4 complex matrix satisfying

$$\mathfrak{Z}\mathfrak{Z}^\dagger = I,$$

and two matrix homogeneous coordinates $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ represent a same point of $\mathfrak{F}(2,2)$ iff there is a 2×2 unitary matrix U such that $\mathfrak{Z}_1 = U\mathfrak{Z}_2$.

$\mathfrak{F}(2,2)$ is a complex spin manifold because for any $T \in SU(4)$ there is a holomorphic automorphism defined by

$$\mathfrak{W} = U_T \mathfrak{Z} T, \quad U_T \in U(2) \qquad (3.1),$$

where U_T is the transition function of the principal bundle $E\{\mathfrak{F}(2,2), U(2)\}$ (c.f.[9]), and the transition function of the reduced bundle $Spin\{\mathfrak{F}(2,2), SU(2)\}$ is

$$\mathfrak{A}_T = (detU_T)^{-\frac{1}{2}} U_T. \qquad (3.2)$$

Without lose of generality we assume that in $\mathfrak{Z} = (Z_1, Z_2)$ and $\mathfrak{W} = (W_1, W_2)$ the submatrices Z_1 and W_1 are non-singular. We write

$$T = \begin{pmatrix} A & C \\ B & D \end{pmatrix}, \qquad (3.3)$$

where A, B, C, D are 2×2 matrices satisfying

$$AA^\dagger + CC^\dagger = I, \quad AB^\dagger + CD^\dagger = 0, \quad BB^\dagger + DD^\dagger = I. \qquad (3.4)$$

Comparing the submatrices of (3.1) we obtain

$$U_T = W_1(Z_1 A + Z_2 B)^{-1} = W_1(A + ZB)^{-1} Z_1^{-1}, \qquad (3.5)$$

where

$$Z = Z_1^{-1} Z_2 \quad \text{and} \quad W = W_1^{-1} W_2 \qquad (3.6)$$

are the local coordinates. From

$$\mathfrak{Z}\mathfrak{Z}^\dagger = Z_1 Z_1^\dagger + Z_2 Z_2^\dagger = Z_1(I + ZZ^\dagger) Z_1^\dagger = I$$

we have a unique positively definite Hermitian matrix $Z_1 = (I + ZZ^\dagger)^{-\frac{1}{2}}$ satisfies the above equation, so that the transition function

$$U_T = (I + WW^\dagger)^{-\frac{1}{2}} (A + ZB)^{-1} (I + ZZ^\dagger)^{\frac{1}{2}}. \qquad (3.7)$$

When the transformation (3.1) is expressed in local coordinates

$$W = (A + ZB)^{-1}(C + ZD), \qquad (3.8)$$

we have

$$I + WW^\dagger = (A + ZB)^{-1}(I + ZZ^\dagger)(A + ZB)^{\dagger -1}. \tag{3.9}$$

The classical domain $\mathfrak{R}_I(2,2)$ can be transformed to the Siegel domain

$$\mathfrak{H}_I(2,2) = \{Z \in \mathbf{C}^{2\times 2}| \quad \frac{1}{2i}(Z - Z^\dagger) > 0\} \tag{3.10}$$

by the transformation

$$W = (I + iZ)^{-1}(I - iZ) \tag{3.10}$$

such that the characteristic manifold $U(2)$ is transformed to $\overline{M}$ by

$$U = (I + iH)^{-1}(I - iH), \quad H^\dagger = H. \tag{3.11}$$

Let $\mathcal{G}$ be the subgroup of $SU(4)$ such that the submatrices in (3.3) satisfy

$$C = -B, \quad D = A, \quad A^\dagger A + B^\dagger B = I, \quad B^\dagger A = A^\dagger B. \tag{3.12}$$

The transformation for $T \in \mathcal{G}$

$$K = (A + HB)^{-1}(-B + HA) \tag{3.13}$$

is an automorphism of $\overline{M}$ i.e., $K^\dagger = K$. This transformation must map a certain point, say $H = H_0$, to the point $K = 0$. Then the condition (3.12) becomes

$$B = H_0 A, \quad A = (I + H_0^2)^{-\frac{1}{2}} U_0, \quad U_0 \in SU(2) \tag{3.14}$$

and (3.13) can be written into

$$K = U_0^{-1}(I + H_0^2)^{\frac{1}{2}}(I + HH_0)^{-1}(H - H_0)(I + H_0^2)^{-\frac{1}{2}} U_0. \tag{3.15}$$

$SU(2)$ is a subgroup of $U(2)$. The transformation (3.11) must map $SU(2)$ into a submanifold of $\overline{M}$.

Lemma 3 The necessary and sufficient that $U \in SU(2)$ in transformation (3.11) is $Tr(H) = 0$.

Proof. Since the Hermitian matrix H can be written into $H = x^j \sigma_j$, the condition

$$Tr(H) = 0 \quad \text{equivalent} \quad x^0 = 0. \tag{3.16}$$

When the above condition is satisfied we write

$$H = H_x = x^\alpha \sigma_\alpha$$

which satisfies the relations

$$\det H_x = -xx' \quad \text{and} \quad H_x^2 = xx'\sigma_0, \quad x = (x^1, x^2, x^3). \tag{3.17}$$

The above relation implies that the characteristic roots of H_x are $\sqrt{xx'}$ and $-\sqrt{xx'}$ so that there is a $V \in SU(2)$ such that

$$H_x = \sqrt{xx'} V \sigma_3 V^\dagger. \tag{3.18}$$

According to (3.11)

$$\det U = \det[V(I + i\sqrt{xx'}\sigma_3)^{-1}(I - i\sqrt{xx'}\sigma_3)V^\dagger] = 1.$$

This means that $U \in SU(2)$. Conversely, if $U \in SU(2)$, then the inverse of (3.11) is

$$H = -i(I+U)^{-1}(I-U) = \frac{-i}{|1+a|^2+|b|^2}\begin{pmatrix} 1+\bar{a} & b \\ -\bar{b} & 1+a \end{pmatrix}\begin{pmatrix} 1-a & b \\ -\bar{b} & 1-\bar{a} \end{pmatrix} \qquad (3.19)$$

so that $Tr(H) = 0$ because $|a|^2 + |b|^2 = 1$. The lemma is proved.

Since $x^0 = 0$ is a hyperplane $\mathbf{P}_0$ in $\overline{M}$, Lemma 3 implied that $SU(2) \cong \overline{M} \cap \mathbf{P}_0$ and we can use the admissible local coordinate of $\overline{M} \cap \mathbf{P}_0$ as the local coordinate of $SU(2) \cong S^3$. Consequently,

$$\mathfrak{M} \cong \overline{M} \cong U(2) \cong U(1) \times SU(2) \cong S^1 \times S^3 \cong U(1) \times \mathbf{M}_1$$

where we set

$$\mathbf{M}_1 = \overline{M} \cap \mathbf{P}_0. \qquad (3.20)$$

Now we take in the transformation (3.15)

$$H_0 = H_a = a^\alpha \sigma_\alpha. \quad a = (a^1, a^2, a^3), \qquad (3.21)$$

Since $H_0^2 = aa'\sigma_0$, the transformation becomes

$$K = U_0^{-1}(I + HH_a)^{-1}(H - H_a)U_0. \qquad (3.22)$$

Lemma 4 The transformation (3.22) is an automorphism of $\overline{M}_1$, in other words, it transforms $Tr(H) = 0$ to $Tr(K) = 0$.

Proof. Since $Tr(H) = 0$, it can be written into $H_x = x^\alpha \sigma_\alpha$ and

$$H_x H_a = x^\mu a^\nu \sigma_\mu \sigma_\nu = \frac{1}{2} x\mu a\nu [(\sigma_\mu\sigma_\nu + \sigma_\nu\sigma_\mu) + (\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu)]$$

$$= x^\mu a^\nu [\delta_{\mu\nu}\sigma_0 + i x^\mu a^\nu \delta^{123}_{\mu\nu\alpha}\sigma_\alpha] = xa'\sigma_0 + if^\alpha(x,a)\sigma_\alpha, \qquad (3.23)$$

where

$$f^\alpha(x,a) = x^\mu a^\nu \delta^{123}_{\mu\nu\alpha}. \qquad (3.24)$$

Since

$$(I + H_x H_a)((I + H_x H_a)^\dagger = [(1 + xa')I + iH_f)][(1 + xa')I + iH_f]^\dagger$$
$$= (1 + xa')^2 I + H_f^2 = [(I + xa')^2 + ff']I = \chi^2 I$$

where

$$\chi = \chi(x,a) = [(1 + xa')^2 + xx'aa' - xa'xa']^{\frac{1}{2}}, \qquad (3.25)$$

the matrix

$$U(x,a) = \chi^{-1}(I + H_x H_a) \qquad (3.26)$$

is a unitary matrix with $detU(x,a) = 1$ and

$$(I + H_x H_a)^{-1}(H_x - H_a) = \chi^{-2}((1 + xa')I - iH_f)H_{(x-a)}$$

$$= \chi^{-2}[(1 + xa')H_{(x-a)} - if(x,a)(x-a)'\sigma_0 + f^\alpha(f(x,a), x-a)\sigma_\alpha]. \qquad (3.27)$$

Hence

$$Tr(K) = 0$$

because

$$f(x,a)(x-a)' = x^\mu a^\nu \delta^{123}_{\mu\nu\alpha}(x^\alpha - a^\alpha) = 0.$$

The lemma is proved.

By Lemma 4, we can write

$$K = H_y = y^\alpha \sigma_\alpha$$

and according to (3.27) the transformation (3.22) can be written into usual manner

$$y^\nu = \chi^{-2}\{x^\mu - a^\mu + xa'(x^\mu - a^\mu) + [x(x-a)'a^\mu - a(x-a)'x^\mu]\}\gamma^\nu_\mu, \tag{3.28}$$

where $(\gamma^\alpha_\beta) \in SO(3)$. Moreover, all such transformations form a group ,which is a group of automorphism of $\mathbf{M}_1$, or all the matrices of the form

$$T_a = (1 + aa')^{-\frac{1}{2}} \begin{pmatrix} I & -H_a \\ H_a & I \end{pmatrix} \begin{pmatrix} U_0 & 0 \\ 0 & U_0 \end{pmatrix} \tag{3.29}$$

form a group $\mathcal{G}_1$ which is a subgroup of $\mathcal{G}$. So when $T_a \in \mathcal{G}_1$ the transition function (3.7) becomes, according to (3.9) and (3.26),

$$U_{T_a} = [(A+H_xB)^{-1}(I+H_x^2)(A+H_xB)^{\dagger-1}]^{-\frac{1}{2}}(A+H_xB)^{-1}(I+H_x^2)^{\frac{1}{2}} = U_0^\dagger U(x,a)^{-1}, \tag{3.30}$$

and $detU_{T_a} = 1$.. Hence

$$\mathfrak{A}_{T_a} = U_{T_a} = U_0^\dagger U(x,a)^{-1}. \tag{3.31}$$

This proves Lemma 2 in §1.

In S^3 there is a natural Riemann metric

$$ds_3^2 = \frac{1}{4}(|da|^2 + |db|^2) = \frac{1}{8}Tr(dUdU^\dagger), \tag{3.32}$$

where

$$U = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \quad |a|^2 + |b|^2 = 1.$$

Differentiating (3.11) and substituting dU into (3.32) we have

$$ds_3^2 = \frac{1}{2}Tr[(I+H_x^2)^{-1}dH_x(I+H_x^2)^{-1}dH_x] = \frac{\delta_{\mu\nu}}{(1+xx')^2}dx^\mu dx^\nu. \tag{3.33}$$

Differentiating (3.13) we have

$$dH_y = (A + H_xB)^{-1}dH_x(A + H_xB)^{-1}. \tag{3.34}$$

Applying (3.9) and (3.34) we obtain

$$\begin{aligned} ds_3^2 &= \frac{\delta_{\mu\nu}}{(1+yy')^2}dy^\mu dy^\nu \frac{1}{2}Tr[(I+H_y^2)^{-1}dH_y(I+H_y^2)^{-1}dH_y] \\ &= \frac{1}{2}Tr[(I+H_x^2)^{-1}dH_x(I+H_x^2)^{-1}dH_x]\frac{\delta_{\mu\nu}}{(1+xx')^2}dx^\mu dx^\nu. \qquad (3.35). \end{aligned}$$

This means that the ds_3^2 is invariant under the group $\mathcal{G}_1$. When we set

$$a = \xi^0 - i\xi^3, \quad b = \xi^1 - i\xi^2 \tag{3.36}$$

and use (3.11)

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} = (I+iH_x)^{-1}(I-iH_x) = (1+xx')^{-1}\begin{pmatrix} 1-xx'-2ix^3 & -2x^2-2ix^1 \\ 2x^2-2ix^1 & 1-xx'+2ix^3 \end{pmatrix},$$

we obtain the coordinate transformation

$$\xi^0 = \frac{1 - xx'}{1 + xx'}, \quad \xi^\alpha = \frac{2x^\alpha}{1 + xx'}, \quad \alpha = 1, 2, 3 \tag{3.37}$$

such that

$$ds_3^2 = \frac{1}{4}\delta_{jk} d\xi^j d\xi^k = \frac{\delta_{\mu\nu}}{(1 + xx')^2} dx^\mu dx^\nu. \tag{3.38}$$

§4. The harmonic analysis of Dirac spinors on $S^1 \times S^3$

Now we discuss the case that $\mathfrak{M} \cong S^1 \times S^3$ with the metric (1.4) as its Lorentz metric. It is obvious that $S^1 \times S^3$ is a Lorentz spin manifold and S^3 a Riemann spin manifold with the metric

$$ds_3^2 = \frac{\delta_{\mu\nu}}{(1 + xx')^2} dx^\mu dx^\nu. \tag{4.1}$$

Since in S^1

$$ds_1^2 = (dx^0)^2 \tag{4.2}$$

the tensor g_{jk} in (1.4) is of the form

$$\begin{cases} g_{00} = 1, \quad g_{0\mu} = g_{\mu 0} = 0, \quad \mu = 1, 2, 3, \\ g_{\mu\nu} = \dfrac{-1}{[1 + r^2(x_1)]^2}\delta_{\mu\nu}, \quad \mu, \nu = 1, 2, 3 \end{cases} \tag{4.3}$$

and the Christoffel symbol is

$$\left\{ \begin{matrix} l \\ j\,k \end{matrix} \right\} = 0, \quad \text{when one of the indices } l, j, k \text{ equals to } 0 \tag{4.4}$$

and

$$\left\{ \begin{matrix} \lambda \\ \mu\,\nu \end{matrix} \right\}, \qquad \lambda, \mu, \nu = 1, 2, 3$$

is the Christoffel symbol of ds_3^2. The coefficients of the Lorentz coframe of ds^2 are

$$e_0^{(0)} = 1, \quad e_\mu^{(0)} = 0, \quad \mu = 1, 2, 3$$

and

$$e_\nu^{(\alpha)} = (1 + xx')^{-1}\delta_\nu^\alpha, \quad (\alpha, \nu = 1, 2, 3). \tag{4.5}$$

The later ones are the coefficients of the Riemann co-frame of ds_3^2. Since $g_{\mu\nu}$ do not depend on the coordinate x_0 , so the Lorentz connection

$$\Gamma_{bj}^a = 0 \quad \text{when one of the indices } a, b, j \text{ equal to } 0$$

and is a $\mathfrak{so}(1, 3)$-connection. So the connection defined by (2.15) is

$$\mathfrak{B}_j = \frac{1}{4}\sigma_\alpha\sigma_\beta\Gamma_{\beta j}^\alpha \quad \text{because} \quad \sigma_\alpha^* = -\sigma_\alpha. \tag{4.7}$$

So

$$\mathfrak{B}_0 = 0, \qquad \mathfrak{B}_\mu = \frac{1}{4}\sigma_\alpha\sigma_\beta\Gamma_{\beta\mu}^\alpha. \tag{4.8}$$

Then the covariant differentiation defined by (2.21) is

$$\mathfrak{D}_0\psi = \frac{\partial\psi}{\partial x^0}, \qquad \mathfrak{D}_\mu\psi = \frac{\partial\psi}{\partial x^\mu} + \mathfrak{B}_\mu\psi \tag{4.9}$$

where $\mathfrak{B}_\mu$ is an $su(2)$-connection on S^3, so from (1.12)

$$\mathfrak{D} = \frac{\partial}{\partial x^0} - \not{D}_{S^3}, \qquad \not{D}_{S^3} = e^\mu_{(\alpha)}\sigma_\alpha\mathfrak{D}_\mu \tag{4.10}$$

where $\not{D}_{S^3}$ is the Dirac operator of the Riemann spin manifold of S^3. Hence

$$\mathfrak{D}^2\psi = \frac{\partial^2\psi}{(\partial x^0)^2} - \not{D}^2_{S^3}\psi. \tag{4.11}$$

where $\not{D}^2_{S^3}$ does not depend on the coordinate x^0. So we use the method of separating variables to solve (1.1). Let

$$\psi^{(n)}(x_1) = e^{inx^0}\widehat{\psi}(x) \tag{4.12}$$

where $\widehat{\psi}$ is a spinor on S^3 and e^{inx^0} is defined on S^1, then e^{inx^0} should be a periodic function with n being an integer and $\widehat{\psi}$ should satisfies

$$\not{D}^2_{S^3}\widehat{\psi} = -(n^2 - m^2)\widehat{\psi} \tag{4.13}$$

if ψ satisfies (1.1). Since the eigen value of $\not{D}^2_{S^3}$ is known[7] to be of the form

$$n^2 - m^2 = (l + \frac{1}{2})^2 \tag{4.14}$$

where l is a positive integer. So the integer n must be sufficiently large so that

$$n^2 - m^2 > 0. \tag{4.15}$$

Using Weitzenböck formulae for Riemann spin manifold S^3, we have

$$\not{D}^2_{S^3}\widehat{\psi} = \Delta\widehat{\psi} - \frac{1}{4}R_{S^3}\widehat{\psi} \tag{4.16}$$

where

$$\begin{aligned}\Delta\widehat{\psi} = g^{\mu\nu}\left(\frac{\partial^2\widehat{\psi}}{\partial x^\mu\partial x^\nu} - \left\{ \begin{array}{c}\lambda\\ \mu\,\nu\end{array} \right\}\frac{\partial\widehat{\psi}}{\partial x^\lambda}\right) + g^{\mu\nu}\left(\frac{\partial\mathfrak{B}_\mu}{\partial x^\nu} - \left\{ \begin{array}{c}\lambda\\ \mu\,\nu\end{array} \right\}\mathfrak{B}_\lambda\right)\widehat{\psi} \\ + g^{\mu\nu}\left(\mathfrak{B}_\mu\frac{\partial\widehat{\psi}}{\partial x^\nu} + \mathfrak{B}_\nu\frac{\partial\widehat{\psi}}{\partial x^\mu}\right) + g^{\mu\nu}\mathfrak{B}_\mu\mathfrak{B}_\nu\widehat{\psi}\end{aligned} \tag{4.17}$$

and R_{S^3} is the scalar curvature of S^3. It is known $R_{S^3} = 24$. Hence, to solve the equation (1.1) is reduced to solve the following equation

$$\not{D}^2_{S^3}\widehat{\psi} = -(n^2 - m^2)\widehat{\psi}. \tag{4.18}$$

Since $\not{D}^2_{S^3}$ is an elliptic differential operator and S^3 is compact, there is ,in general, no solution of (4.18) for arbitrary $m > 0$ unless $\lambda = n^2 - m^2$ is an eigenvalue of the operator $\not{D}^2_{S^3}$. In this case the linear independent solutions of (4.18) is finite. Let

$$\widehat{\psi}_\xi(\lambda, x^1, x^2, x^3), \quad \xi = 1, 2, \cdots N_\lambda \tag{4.19}$$

be an orthonormal base of the λ-eigen function space such that

$$\int_{S^3} \overline{\widehat{\psi}'}_\xi \widehat{\psi}_\eta \sqrt{-g} dx^1 dx^2 dx^3 = \delta_{\xi\eta}, \tag{4.20}$$

where $g = det(g_{ij})_{0 \le i,j \le 3} = -det(g_{\alpha\beta})_{1 \le \alpha,\beta \le 3}$.

Now we let

$$x_1 = (x^0, x) \quad \text{and} \quad H_{x_1} = x^j \sigma_j$$

and construct the kernel of λ-eigen space

$$\mathcal{H}_\lambda(H_{x_1}, H_{y_1}) = \sum_{\xi=1}^{N_\lambda} \psi_\xi^{(n)}(\lambda, x_1) \overline{\psi_\xi^{(n)}(\lambda, x_1)}' \tag{4.21}$$

which is an 2×2 matrix of matrix variables H_{x_1} and H_{y_1}. We set

$$\psi_\xi^{(n)}(\lambda, x_1) == \frac{1}{\sqrt{2\pi}} e^{inx^0} \widehat{\psi}_\xi(\lambda, x). \tag{4.22}$$

It should be noted that

$$\lambda = n^2 - m^2 \tag{4.23}$$

is positive.

According to the (3.22) given in §3, the transformation T_{a_1}

$$y^0 = x^0 - a^0, \quad H_y = (A + H_x B)^{-1}(-B + H_x A), \quad B = H_a A, \tag{4.24}$$

is an automorphism of $S^1 \times S^3$ and it transforms the point $x_1 = a_1$ to $y_1 = 0$. Since ds^2 is invariant under the transformation, the co-frame is changed as follows:

$$\omega^0 = 1, \quad \omega^\alpha(y) = \omega^\alpha(x) \ell_\alpha^\beta(x), \quad \left(\ell_\beta^\alpha(x)\right)_{0 \le \alpha,\beta \le 3} \in SO(3)$$

and the spinor

$$\psi_{T_{a_1}}(y_1) = \mathfrak{A}_{T_{a_1}}(x_1) \psi(x_1) \tag{4.25}$$

where $\mathfrak{A}_{T_{a_1}}(x_1) = \mathfrak{A}_{T_a}(x)$ is defined by (3.31) and belongs to $SU(2)$. Let

$$\psi_{T_{a_1},\xi}(y_1) = \mathfrak{A}_{T_{a_1}}(x_1) \psi_\xi(x_1).$$

Since

$$\overline{\psi_{T_{a_1},\xi}^{(n)}(\lambda, y_1)}' \psi_{T_{a_1},\eta}^{(n)}(\lambda, y_1) = \overline{\psi_\xi^{(n)}(\lambda, x_1)}' \psi_\eta^{(n)}(\lambda, x_1), \tag{4.26}$$

the

$$\left\{ \psi_{T_{a_1},\eta}^{(n)}(\lambda, y_1) \right\} \tag{4.27}$$

is a base of spinors of λ-eigenvalue in $S^1 \times S^3$. If $u_1 \in S^1 \times S^3$ is another point which is mapped to the point v_1 under the same transformation T_{a_1}, we have

$$\mathcal{H}_\lambda(H_{y_1}, H_{v_1}) = \mathfrak{A}_{T_{a_1}}(x_1) \mathcal{H}_\lambda(H_{x_1}, H_{u_1}) \mathfrak{A}_{T_{a_1}}(u_1)^{-1}. \tag{4.28}$$

According to the definition (4.21), we have

$$\mathcal{H}_\lambda(H_{x_1}, H_{u_1}) = e^{in(x^0 - u^0)} \mathcal{K}_\lambda(H_x, H_u) \tag{4.29}$$

where

$$\mathcal{K}_\lambda(H_x, H_u) = \sum_{\xi=1}^{N_\lambda} \widehat{\psi}_\xi(\lambda, x) \overline{\widehat{\psi}_\xi(\lambda, u)}' \tag{4.30}$$

is the kernel of λ-eigen functions of the operator $\mathcal{D}^2_{S^3}$ of the Riemann manifold S^3 with the metric ds_3^2 . Under the transformation (4.24),

$$\mathcal{K}_\lambda(H_y, H_v) = \mathfrak{A}_{T_a}(x)\mathcal{K}_\lambda(H_x, H_u)\mathfrak{A}_{T_a}(u)^{-1}. \tag{4.31}$$

Since $\mathcal{D}^2_{S^3}$ is a covariant differentiation, we have

$$\mathcal{D}^2_{S^3}(y)\mathcal{K}_\lambda(H_y, H_v) = \mathfrak{A}_{T_a}(x)\mathcal{D}^2_{S^3}(x)\mathcal{K}_\lambda(H_x, H_u)\mathfrak{A}_{T_a}(u)^{-1}, \tag{4.32}$$

where $\mathcal{D}_{S^3}(x)$ means that $\mathcal{D}_{S^3}$ operates with respect to the variable x.

Since

$$T_a : \quad H_y = U_0^\dagger(I + H_x H_a)^{-1}(H_x - H_a)U_0 \tag{4.33}$$

we have

$$\begin{aligned}\left[\mathcal{D}^2_{S^3}(y)\mathcal{K}_\lambda(H_y, H_v)\right]_{v=0} &= \mathfrak{A}_{T_a}(x)\left[\mathcal{D}^2_{S^3}(x)\mathcal{K}_\lambda(x, a)\right]\mathfrak{A}_{T_a}(a)^{-1} \\ &= -\lambda\mathfrak{A}_{T_a}(x)\mathcal{K}_\lambda(x, a)\mathfrak{A}_{T_a}(a)^{-1}.\end{aligned} \tag{4.34}$$

Since $\mathfrak{A}_{T_a}(x)$ is known explicitly by (3.31) and (3.26), it remains to calculate $\mathcal{D}^2_{S^3}(x)\mathcal{K}_\lambda(H_x, 0)$ in (4.34).

We expand $\mathcal{K}_\lambda(H_x, 0)$ into power series of the matrix variable H_x such that

$$\begin{aligned}\mathcal{K}_\lambda(H_x, 0) &= \sum_{n=0}^\infty C_n H_x^n = \sum_{n=0}^\infty C_{2n} H_x^{2n} + \sum_{n=0}^\infty C_{2n+1} H_x^{2n+1} \\ &= \sum_{n=0}^\infty C_{2n} r^{2n}(x) I + \sum_{n=0}^\infty C_{2n+1} r^{2n}(x) H_x = f(r^2(x))I + h(r^2(x))H_x,\end{aligned} \tag{4.35}$$

where C_n are complex constants $r^2(x) = xx'$ and f and h are functions of $r^2(x)$ but not real values in general.

We set $u = a$ in (4.31) and have by Lemma 2

$$\mathcal{K}_\lambda(H_x, H_a) = \mathfrak{A}_{T_a}(x)^{-1}\mathcal{K}_\lambda(H_y, 0)U_0^{-1} = U(x, a)[fI + h\Phi(x, a,)], \tag{4.36}$$

where we have written in (3.22) that

$$H = H_x \quad \text{and} \quad K = H_y$$

so that (3.22) becomes

$$H_y = U_0^{-1}\Phi(x, a)U_0, \quad \Phi(x, a) = (I + H_x H_a)^{-1} H_{x-a}. \tag{4.37}$$

By the definition of $\mathcal{K}_\lambda$,

$$\mathcal{K}_\lambda(H_x, H_a)^\dagger = \mathcal{K}_\lambda(H_a, H_x) \tag{4.38}$$

and, by (3.26) and $H_x^\dagger = H_x$,

$$U(x, a)^\dagger = U(a, x). \tag{4.39}$$

So from (4.36) we have the equality

$$\overline{f}U(a, x) + \overline{h}\Phi(x, a)^\dagger U(a, x) = fU(a, x) + hU(a, x)\Phi(a, x)$$

or

$$\overline{f}I + \overline{h}\Phi(x, a)^\dagger = fI + hU(a, x)\Phi(a, x)U(a, x)^{-1}. \tag{4.40}$$

According to Lemma 4 $\Phi(x, a)$ is Hermitian and $Tr[\Phi(x, a)] = 0$. So the trace of (4.40) implies

$$\overline{f} = f \tag{4.41}$$

and then

$$\overline{h}\Phi(x,a)^{\dagger} = hU(a,x)\Phi(a,x)U(a,x)^{-1}. \tag{4.42}$$

We let $x = 0$ in (4.42) and have

$$\overline{h}(aa')H_{-a} = h(aa')H_a$$

or

$$\overline{h} = -h. \tag{4.43}$$

Moreover, we have the following formulas

$$\frac{\partial \mathcal{K}_\lambda(H_x,0)}{\partial x^\mu} = 2f'x^\mu I + 2h'x^\mu H_x + h\sigma_\mu \tag{4.44}$$

and

$$\frac{\partial^2 \mathcal{K}_\lambda(H_x,0)}{\partial x^\mu x^\nu} = (4f''x^\mu x^\nu + 2f'\delta_{\mu\nu})I + (4h''x^\mu x^\nu + 2h'\delta_{\mu\nu})H_x + 2(h'x^\mu\sigma_\nu + h'x^\nu\sigma_\mu). \tag{4.45}$$

The Christoffel symbol associated to ds_3^2 is

$$\left\{ \begin{matrix} \alpha \\ \beta\mu \end{matrix} \right\} = -\frac{2}{1+xx'}(x^\mu\delta^\alpha_\beta + x^\beta\delta^\alpha_\mu - x^\alpha\delta_{\beta\mu}), \tag{4.46}$$

and

$$g^{\beta\mu}\left\{ \begin{matrix} \alpha \\ \beta\mu \end{matrix} \right\} = 2(1+xx')x^\alpha. \tag{4.47}$$

The Riemann connection is

$$\Gamma^\alpha_{\beta\mu} = \frac{2}{1+xx'}(x^\alpha\delta^\beta_\mu - x^\beta\delta^\alpha_\mu) \tag{4.48}$$

And the spin connection is

$$\mathfrak{V}_\mu = \frac{1}{2(1+xx')}[H_x,\sigma_\mu] = \frac{1}{2(1+xx')}(H_x\sigma_\mu - \sigma_\mu H_x). \tag{4.49}$$

We have the following formulae:

(i)

$$g^{\mu\nu}\frac{\partial^2 \mathcal{K}_\lambda(H_x,0)}{\partial x^\mu x^\nu} = (1+r^2)^2[(4f''r^2 + 6f')I + (4h''r^2 + 10h')H_x];$$

(ii)

$$g^{\mu\nu}\left\{ \begin{matrix} \alpha \\ \mu\,\nu \end{matrix} \right\}\frac{\partial \mathcal{K}_\lambda(H_x,0)}{\partial x^\alpha} = 2(1+r^2)[2f'r^2 I + (2h'r^2 + h)H_x];$$

(iii)

$$g^{\mu\nu}\frac{\partial \mathfrak{V}\mu}{\partial x^\nu} = 0;$$

(iv)

$$-g^{\mu\nu}\left\{ \begin{matrix} \alpha \\ \mu\,\nu \end{matrix} \right\}\mathfrak{V}_\alpha\mathcal{K}_\lambda(H_x,0) = 0;$$

(v)

$$g^{\mu\nu}\left[\mathfrak{B}_\mu \frac{\partial}{\partial x^\nu}\mathcal{K}_\lambda(H_x,0) + \mathfrak{B}_\nu \frac{\partial}{\partial x^\mu}\mathcal{K}_\lambda(H_x,0)\right] = 4(1+r^2)hH_x;$$

(vi)

$$g^{\mu\nu}\mathfrak{B}_\mu\mathfrak{B}_\nu\mathcal{K}_\lambda(H_x,0) = -2r^2 fI - 2hr^2 H_x.$$

¿From (i) to (vi) and the Weizenböck formula we have

$$\begin{aligned}
&\not{D}^2_{S^3}\mathcal{K}_\lambda(H_x,0) = \Delta\mathcal{K}_\lambda(H_x,0) - 6(fI + hH_x)\\
&= \left\{4r^2(1+r^2)^2 f'' + (1+r^2)[6(1+r^2) - 4r^2]f' - (2r^2+6)f\right\} I\\
&+ \left\{4r^2(1+r^2)^2 h'' + (1+r^2)[10(1+r^2) - 4r^2]h' + [2(1+r^2) - 2r^2 - 6]h\right\} H_x\\
&= -\lambda(fI + hH_x).
\end{aligned}$$

This means that $f(t)$ and $h(t)$ $(t = r^2)$should satisfy the following differential equations respectively

$$4t(1+t)^2 f'' + (1+t)[6(1+t) - 4t]f' - (2t+6)f = -\lambda f, \tag{4.50}$$

and

$$4t(1+t)^2 h'' + (1+t)[10(1+t) - 4t]h' - 4h = -\lambda h. \tag{4.51}$$

For simplicity we write $\mathcal{K}_\lambda(x,a) = \mathcal{K}_\lambda(H_x, H_a)$.
Theorem 1 in §1 is proved.

§5. The solution of the Einstein-Dirac equation

Let $\widehat{\psi}_0(x)$ be a spinor in S^3 which is orthogonal invariant. Obviously, the spinor

$$\widehat{\psi}(x) = \int_{S^3}\mathcal{K}_\lambda(x,u)\widehat{\psi}_0(u)\dot{u}, \quad \dot{u} = \sqrt{-g}du^1du^2du^3 \tag{5.1}$$

is orthogonal invariant. This spinor satisfies

$$\not{D}^2_{S^3}\widehat{\psi}(x) = -\lambda\widehat{\psi}(x) \tag{5.2}$$

where λ is a eigenvalue of $\not{D}^2_{S^3}$, and the spinor

$$\psi(x) = e^{inx^0}\widehat{\psi}(x_1) \tag{5.4}$$

satisfies

$$\mathfrak{D}^2\psi(x) = -m^2\psi(x) \tag{5.4}$$

when m is taken as $\lambda = n^2 - m^2$. Moreover, according to **Theorem 1**, the 4-component spinor obtained by the following formula

$$\Psi = \begin{pmatrix}\psi\\ \varphi^*\end{pmatrix}, \qquad \varphi^* = \frac{i}{m}\mathfrak{D}\psi \tag{5.5}$$

satisfies the Dirac equation

$$\not{D}\Psi = -im\Psi \tag{5.6}$$

If the energy-momentum tensor T_{jk} of Ψ is not identically zero, then the tensor at $x=0$ must be of the form

$$(T_{jk}(0)) = \begin{pmatrix} c_0 & 0\\ 0 & c_1 I\end{pmatrix}. \tag{5.7}$$

In fact, since the metric ds^2 is invariant under $\mathcal{G}_1$, the tensor T_{jk} must be invariant under $\mathcal{G}_1$. That is

$$T_{jk}(y_1) = T_{pq}(x_1)\frac{\partial x^p}{\partial y^j}\frac{\partial x^q}{\partial y^k} \tag{5.8}$$

where $y^0 = x^0 - a^0$ and y^μ is defined by (3.28). Especially, if we choose $a_1 = (a^0, a) = 0$, we have

$$T_{jk}(0) = T_{pq}(0)\ell^p_j\ell^q_k \tag{5.9}$$

where

$$L = \left(\ell^j_k\right) = \begin{pmatrix} 1 & 0 \\ 0 & \Gamma \end{pmatrix}, \quad \Gamma \in SO(3).$$

Therefore, (5.9) can be written into matrix form

$$(T_{jk}(0)) = \begin{pmatrix} 1 & 0 \\ 0 & \Gamma \end{pmatrix} (T_{jk}(0)) \begin{pmatrix} 1 & 0 \\ 0 & \Gamma' \end{pmatrix}$$

for arbitrary Γ. Hence $T_{jk}(0)$ must be the form (5.7)

We assert $c_0 + c_1 \neq 0$. In fact, c_0 and c_1 can be not zero simultaneously, otherwise $T_{jk}(x_1) \equiv 0$ according to (5.8), because $\mathcal{G}_1$ acts transitively on $S^1 \times S^3$. Moreover, according to the definition of T_{jk}, we have

$$\begin{aligned} & g^{jk}T_{jk} \\ &= \frac{1}{2}\left[\overline{\Psi}^{*\prime}\eta_{ab}\gamma^b\left(\eta^{ac}e^k_{(c)}\nabla_k\Psi + \eta^{ac}e^j_{(c)}\nabla_j\Psi\right) - \eta_{ab}\left(\eta^{ac}e^k_{(c)}\overline{\nabla_k\Psi^{*\prime}} + \eta^{ac}e^j_{(c)}\overline{\nabla_j\Psi^{*\prime}}\right)\gamma^b\Psi\right] \\ &= \left[\overline{\Psi}^{*\prime}\not{\nabla}\Psi - \overline{(\not{\nabla}\Psi^*)}'\Psi\right] = -im\left[\overline{\Psi}^{*\prime}\Psi - \overline{\Psi}^{*\prime}\Psi\right] = 0. \end{aligned} \tag{5.10}$$

Especially,

$$\left(g^{jk}T_{jk}\right)_{x=0} = c_0 - 3c_1 = 0, \quad \text{or} \quad c_0 = 3c_1. \tag{5.11}$$

So $c_0 + c_1 = 4c_1 \neq 0$.

Hence the Einstein equation at $x = 0$ is

$$R_{jk}(0) - \frac{1}{2}g_{jk}(0)R(0) - \Lambda g_{jk}(0) = \chi T_{jk}(0) \tag{5.12}$$

According to the orthogonal invariant of $R_{jk}(0)$ and $R_{0j} = R_{j0} = 0$, we have (5.12) in form of matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & R_{11}(0)I \end{pmatrix} - \frac{1}{2}R(0)\begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix} - \Lambda\begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix} = \chi\begin{pmatrix} c_0 & 0 \\ 0 & c_1 I \end{pmatrix}$$

or

$$\begin{cases} -\frac{1}{2}R(0) - \Lambda = \chi c_0 \\ R_{11}(0) + \frac{1}{2}R(0) + \Lambda = \chi c_1 \end{cases}$$

If we choose

$$\chi = \frac{1}{c_0 + c_1}R_{11}(0), \qquad \Lambda = \frac{-c_0}{c_0 + c_1}R_{11}(0) - \frac{1}{2}R(0)$$

then (5.12) is satisfied and the Einstein equation is also satisfied at any point of $S^1 \times S^3$ because it is invariant under $\mathcal{G}_1$.

Theorem 2 given in §1 is proved.

References

[1] R. Penrose, A spinor approach to general relativity, Ann. Phys., (N.Y.), **10**(1960), 171-201.

[2] P.A.M. Dirac, Wave equation in the conformal space, Ann. Math. **37**(1936),429-442.

[3] E. Witten, Anti-de-Sitter space and holograph, Adv. Theor. Math. Phys., **2**(1998),253-291.

[4] P.A.M. Dirac,The electron wave equation in de-Sitter space, Ann Math., **36**(1935),657-669.

[5] K. H. Look(=Qikeng Lu),G. L. Tsou(=Zhenlong Zhuo), H.Y. Kou(=Hanying Guo), The kinematic effect in the classical domains and the red-shift phenomena of ectra-galactic objects(in Chinese), Acta Physica Sinica, **23**(1974), 225-238.

[6] Qikeng Lu, Classical Manifolds and Classical Domains(in Chinese), Science Press, Beijing, 1963.

[7] R. Canporesi and A.Higuchi, On the eigen functions of the Dirac operator on sphere and real hyperbolic space, J. Geom. Phys., **20**(1996),1-43.

[8] Qi-keng Lu, Differential Geometry and its application to physics (in Chinese), Science Press, Beijing, 1982.

[9] Qikeng Lu, A global solution of Einstein-Yang-Mills equation on the conformal space, Science in China, Series A, **5**(2002),343-355.