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A LIMIT THEOREM FOR WEIGHTED SUMS OF INFINITE VARIANCE RANDOM VARIABLES WITH LONG-RANGE DEPENDENCE

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Abstract
Let $\{\xi_j\}_{j\in\mathbb{Z}}$ be a sequence of random variables with long-range dependence belonging to the domain of attraction of a linear fractional stable motion $\{\Delta_{H,\alpha}(t)\}$ with infinite variance. We propose a class of deterministic functions f for which stochastic integrals with respect to linear fractional stable motion are well-defined and provide sufficient conditions for the convergence of $n^{-H}\sum_{j\in\mathbb{Z}} f\left(j/n\right)\xi_j$ to $\int_{-\infty}^{\infty} f(u)d\Delta_{H,\alpha}(u)$ in distribution as $n\to\infty$.

Keywords: Linear fractional stable motion. Long-range dependence. Integral with respect to linear fractional stable motion.

1. Introduction

Let $\{Z_\alpha(t), t\in\mathbb{R}\}$ be a symmetric α-stable Lévy process $(0<\alpha\le 2)$ and $\{\xi_j\}_{j\in\mathbb{Z}}$ i.i.d. random variables, and suppose that $n^{-1/\alpha}\sum_{j=1}^{n}\xi_j \xrightarrow{d} Z_\alpha(1)$ as $n\to\infty$, where $\xrightarrow{d}$ means convergence in law. Then, it is proved in Kasahara and Maejima (1986) that for any deterministic function f bounded on any bounded intervals,

$$\frac{1}{n^{1/\alpha}}\sum_{j\in\mathbb{Z}} f\left(\frac{j}{n}\right)\xi_j \xrightarrow{d} \int_{-\infty}^{\infty} f(u)dZ_\alpha(u) \qquad \text{as } n\to\infty. \tag{1.1}$$

Let $1/2<H<1$ and let $\{B_H(t), t\in\mathbb{R}\}$ be the fractional Brownian motion with exponent H in the sense that it is a centered Gaussian H-selfsimilar process with stationary increments. Let $\{\xi_j, j\in\mathbb{Z}\}$ be a stationary sequence of square integrable random variables such that $E[\xi_j]=0$ and $|E[\xi_0\xi_k]|\le C|k|^{2H-2}, k\in\mathbb{N}$, for some $C>0$, and such that all finite dimensional distributions of $n^{-H}\sum_{j=1}^{[nt]}\xi_j$ converge to

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those of $B_H(t)$ as $n \to \infty$. Let

$$F = \left\{ f \ \Big| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(u)||f(v)||u-v|^{2H-2} du dv < \infty \right\}.$$

Then, it is shown in Pipiras and Taqqu (2000) that for any $f \in F$,

$$\frac{1}{n^H} \sum_{j \in \mathbb{Z}} f\left(\frac{j}{n}\right) \xi_j \xrightarrow{d} \int_{-\infty}^{\infty} f(u) dB_H(u) \qquad \text{as } n \to \infty. \tag{1.2}$$

In (1.1), if $\alpha < 2$, the variances of $\{\xi_j\}$ and $\{Z_\alpha(t)\}$ are infinite, but the increments of $\{Z_\alpha(t)\}$ are independent. On the other hand, in (1.2), $\{\xi_j\}$ and $\{B_H(t)\}$ have long-range dependence, but their variances are finite. The purpose of this paper is to establish a limit theorem of type (1.1) or (1.2) in the case when $\{\xi_j\}$ have infinite variances and long-range dependence. For that purpose, we take linear fractional stable motions $\{\Delta_{H,\alpha}(t)\}$ (defined below) instead of $\{Z_\alpha(t)\}$ or $\{B_H(t)\}$.

The rest of the paper is organized as follows. In section 2, we propose a class of deterministic functions f for which stochastic integrals with respect to linear fractional stable motion are well-defined. In section 3, we prove an auxiliary result on approximation of some Riemann integral which is needed for the proof of our main theorem, and in section 4, we present and prove our main theorem of type (1.1) or (1.2), where $\{Z_\alpha(t)\}$ or $\{B_H(t)\}$ is replace by $\{\Delta_{H,\alpha}(t)\}$.

2. Stochastic integrals with respect to linear fractional stable motions

We call $\{X(t), t \in \mathbb{R}\}$ a Lévy process if it has independent and stationary increments, $X(0) = 0$ a.s., and it is stochastically continuous. If, furthermore, the distribution of $X(1)$ is symmetric α-stable, $0 < \alpha \le 2$, in the sense that $E[e^{i\theta X(1)}] = e^{-c|\theta|^\alpha}, \theta \in \mathbb{R}$, for some $c > 0$, we call $\{X(t)\}$ a symmetric α-stable Lévy process, and denote it by $\{Z_\alpha(t), t \in \mathbb{R}\}$ throughout this paper. Without loss of generality, we also assume $c = 1$. When $\alpha = 2$, it is nothing but the standard Brownian motion.

Let $0 < H < 1, 0 < \alpha \le 2$ and $H \neq 1/\alpha$. We define linear fractional stable motion $\{\Delta_{H,\alpha}(t), t \in \mathbb{R}\}$ by

$$\Delta_{H,\alpha}(t) = \int_{-\infty}^{\infty} ((t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha}) dZ_\alpha(s), \tag{2.1}$$

where $x_+ = \max(x, 0)$, $x_- = \max(-x, 0)$ and $0^s = 0$ even for $s \le 0$. When $\alpha = 2$, it is fractional Brownian motion.

In the following, we always assume $\alpha < 2$, because we want to extend a known story for fractional Brownian motion to the case of infinite variances. In this section, we introduce stochastic integrals with linear fractional stable motions with $\alpha < 2$. It is known that if $\alpha < 2$ and $H < 1/\alpha$, sample paths of $\Delta_{H,\alpha}(t)$ are nowhere bounded (Maejima, 1983). On the other hand, we cannot expect integrals with respect to such processes with irregular paths. Therefore, we throughout assume $H > 1/\alpha$. Since

$H < 1$, necessarily $\alpha > 1$. In the following, we put $\beta = H - 1/\alpha$, for notational convenience. β satisfies $0 < \beta < 1 - 1/\alpha$.

When $f(u)$ is a simple function with the form

$$f(u) = \sum_{k=1}^{n} f_k 1_{[u_k, u_{k+1})}(u), \qquad f_k \in \mathbb{R}\,, \quad u_k < u_{k+1}, \quad k = 1, \dots, n,$$

we define

$$I(f) := \int_{-\infty}^{\infty} f(u) d\Delta_{H,\alpha}(u) = \sum_{k=1}^{n} f_k(\Delta_{H,\alpha}(u_{k+1}) - \Delta_{H,\alpha}(u_k)). \tag{2.2}$$

Denote by $\mathcal{E}$ the set of all simple functions.

Same as (2.5) in Pipiras and Taqqu (2000), we note that for $0 < \beta < 1 - 1/\alpha$

$$(t-s)_+^{\beta} - (-s)_+^{\beta} = \beta \int_s^{\infty} 1_{[0,t)}(u)(u-s)^{\beta-1} du = (I_\beta 1_{[0,t)})(s), \tag{2.3}$$

where

$$(I_\beta f)(s) = \beta \int_s^{\infty} f(u)(u-s)^{\beta-1} du, \quad s \in \mathbb{R}\,.$$

Then, if follows from (2.1) and (2.3) that

$$\Delta_{H,\alpha}(t) = \int_{-\infty}^{\infty} (I_\beta 1_{[0,t)})(s) dZ_\alpha(s). \tag{2.4}$$

We also have from (2.2) and (2.4) that for $f \in \mathcal{E}$,

$$\int_{-\infty}^{\infty} f(u) d\Delta_{H,\alpha}(u) = \int_{-\infty}^{\infty} (I_\beta f)(s) dZ_\alpha(s).$$

Let

$$\Lambda = \left\{ f \,\middle|\, \int_{-\infty}^{\infty} |(I_\beta |f|)(s)|^\alpha ds < \infty \right\}.$$

It is easily seen by an ordinary argument that Λ is a normed linear space with the norm

$$\|f\| = \left(\int_{-\infty}^{\infty} |(I_\beta |f|)(s)|^\alpha ds \right)^{1/\alpha}$$

and $\mathcal{E}$ is a dense subset of Λ.

Theorem 2.1. *For every $f \in \Lambda$, the stochastic integral*

$$\int_{-\infty}^{\infty} f(u) d\Delta_{H,\alpha}(u) = \int_{-\infty}^{\infty} (I_\beta f)(s) dZ_\alpha(s)$$

can be well-defined in the sense of convergence in probability.

Proof. For any $f \in \Lambda$, choose $\{f_n\} \subset \mathcal{E}$ such as $\|f_n - f\| \to 0$. We have

$$I(f_n) - I(f_m) = \int_{-\infty}^{\infty} (I_\beta(f_n - f_m))(s) dZ_\alpha(s),$$

and it follows from a known fact on stable integrals that

$$E\left[e^{i\theta(I(f_n)-I(f_m))}\right] = \exp\left\{|\theta|^\alpha \int_{-\infty}^{\infty} |(I_\beta(f_n - f_m))(s)|^\alpha ds\right\} = \exp\left\{|\theta|^\alpha \|f_n - f_m\|^\alpha\right\}.$$

(see, e.g, Samorodnitsky and Taqqu, 1994, Proposition 3.4.1). Since $\|f_n - f_m\| \to 0$ as $n, m \to \infty$, we have $I(f_n) - I(f_m) \to 0$ in distribution and thus in probability. Therefore $I(f_n)$ converges in probability to a limit $I(f)$, say, which we write as $\int_{-\infty}^{\infty} f(u) d\Delta_{H,\alpha}(u)$. Actually, the value of the limit of $I(f_n)$ does not depend on the choice of $\{f_n\}$, which can easily be shown by a standard argument. This completes the proof of the theorem. □

3. An auxiliary statement

In this section, we prepare a statement on Riemann integral, which will be needed for the proof of our main theorem in the next section.

Theorem 3.1. *Let f be a real valued differentiable function on $\mathbb{R}$. Let $\alpha > 1$, $1/\alpha < \gamma < 1$ and suppose that*

$$|f(u)| \ll (1 + |u|)^{-1}, \qquad |f'(u)| \ll (1 + |u|)^{-1}, \qquad \text{for any } u \in \mathbb{R},$$

where $|f(u)| \ll |h(u)|$ means that there exists $K > 0$ satisfying $|f(u)| \le K|h(u)|$ for any $u \in \mathbb{R}$. Then we have

$$\lim_{n\to\infty} \sum_{i\in\mathbb{Z}} \left| \sum_{j>i} f\left(\frac{j}{n}\right) \left(\frac{j}{n} - \frac{i}{n}\right)^{-\gamma} \frac{1}{n} \right|^\alpha \frac{1}{n} = \int_{-\infty}^{\infty} \left| \int_s^\infty f(u)(u-s)^{-\gamma} du \right|^\alpha ds.$$

For the proof, we need several lemmas.

Lemma 3.2. *Let $g(s,u) = f(u)(u-s)^{-\gamma}$. For a sufficiently small $\varepsilon > 0$, choose s, s', u, u' such as $|s - s'| < \varepsilon$, $|u - u'| < \varepsilon$.*
(i) *If $u - 1 \le s < s' < u \le s + 1$, then*

$$|g(s,u) - g(s',u)| \ll (1 + |s|)^{-1}(u - s')^{-\gamma-1}|s - s'|.$$

(ii) *If $s < u \le u' \le s + 1$, then*

$$|g(s,u) - g(s,u')| \ll (1 + |s|)^{-1}(u - s)^{-\gamma-1}|u - u'|.$$

(iii) *If $u \ge s + 1$, then*
(iii-1) $|g(s,u) - g(s',u)| \ll (1 + |u|)^{-1}(u - s)^{-\gamma}|s - s'|$
and
(iii-2) $|g(s,u) - g(s,u')| \ll (1 + |u|)^{-1}(u - s)^{-\gamma}|u - u'|$.

Proof. (i) We have

$$|g(s,u)-g(s',u)| = \left|\frac{\partial g}{\partial s}(s'',u)\right| |s-s'|$$

for some s'' such that $|s-s''|<|s-s'|<\varepsilon$, where

$$\left|\frac{\partial g}{\partial s}(s'',u)\right| \le |f(u)|(u-s'')^{-\gamma-1} \ll (1+|u|)^{-1}(u-s')^{-\gamma-1} \ll (1+|s|)^{-1}(u-s')^{-\gamma-1}.$$

(ii) We have

$$|g(s,u)-g(s,u')| = \left|\frac{\partial g}{\partial u}(s,u'')\right| |u-u'|$$

for some u'' such that $|u-u''|<|u-u'|<\varepsilon$, where

$$\begin{aligned}\left|\frac{\partial g}{\partial u}(s,u'')\right| &\le |f'(u'')|(u''-s)^{-\gamma}+|f(u'')|(u''-s)^{-\gamma-1}\\ &\ll (1+|u|)^{-1}\left((u-s)^{-\gamma}+(u-s)^{-\gamma-1}\right) \ll (1+|s|)^{-1}(u-s)^{-\gamma-1}.\end{aligned}$$

(iii) We have

$$|g(s,u)-g(s',u)| = \left|\frac{\partial g}{\partial s}(s'',u)\right| |s-s'|$$

for some s'' such that $|s-s''|<|s-s'|<\varepsilon$, where

$$\left|\frac{\partial g}{\partial s}(s'',u)\right| \le |f(u)|(u-s'')^{-\gamma-1} \ll (1+|u|)^{-1}(u-s)^{-\gamma-1} \le (1+|u|)^{-1}(u-s)^{-\gamma},$$

since $(u-s)^{-1}\le 1$. Also, we have

$$|g(s,u)-g(s,u')| = \left|\frac{\partial g}{\partial u}(s,u'')\right| |u-u'|$$

for some u'' such that $|u-u''|<|u-u'|<\varepsilon$, where

$$\begin{aligned}\left|\frac{\partial g}{\partial u}(s,u'')\right| &\le |f'(u'')|(u''-s)^{-\gamma}+|f(u'')|(u''-s)^{-\gamma-1}\\ &\ll (1+|u|)^{-1}\left((u-s)^{-\gamma}+(u-s)^{-\gamma-1}\right) \ll (1+|u|)^{-1}(u-s)^{-\gamma},\end{aligned}$$

since $(u-s)^{-1}\le 1$. The proof is thus completed. □

In the following, for any $\varepsilon>0$, choose n such that $\varepsilon\in[1/n,2/n]$. For $a\in\mathbb{R}$, $[a]$ denotes the largest integer less than or equal to a.

Lemma 3.3. *For any $\delta>0$,*

$$\left|\int_s^\infty g(s,u)du - \sum_{j>[sn]+1} g\left(s,\frac{j}{n}\right)\frac{1}{n}\right| \ll (1+|s|)^{-\gamma+\delta}\varepsilon^{1-\gamma},$$

Proof.

$$
\begin{aligned}
&\left|\int_s^\infty g(s,u)du - \sum_{j>[sn]+1} g\left(s,\frac{j}{n}\right)\frac{1}{n}\right| \\
&= \left|\int_s^{([sn]+2)/n} g(s,u)du + \int_{([sn]+2)/n}^\infty g(s,u)du - \sum_{j>[sn]+1} g\left(s,\frac{j}{n}\right)\frac{1}{n}\right| \\
&\le \left|\int_s^{([sn]+2)/n} g(s,u)du\right| + \left|\sum_{j>[sn]+1}\int_{j/n}^{(j+1)/n}\left(g(s,u)-g\left(s,\frac{j}{n}\right)\right)du\right| \\
&\le \int_s^{([sn]+2)/n} |g(s,u)|du \\
&\qquad + \sum_{[sn]+1<j\le[(s+1)n]}\int_{j/n}^{(j+1)/n}\left|g(s,u)-g\left(s,\frac{j}{n}\right)\right|du \\
&\qquad + \sum_{j>[(s+1)n]}\int_{j/n}^{(j+1)/n}\left|g(s,u)-g\left(s,\frac{j}{n}\right)\right|du, \\
&=: I_1 + I_2 + I_3,
\end{aligned}
$$

say. We first estimate I_1. We have

$$
\begin{aligned}
I_1 &= \int_s^{([sn]+2)/n} |f(u)|(u-s)^{-\gamma}du \ll \int_s^{s+2/n}(1+|u|)^{-1}(u-s)^{-\gamma}du \\
&\ll (1+|s|)^{-1}\int_s^{s+2/n}(u-s)^{-\gamma}du \ll (1+|s|)^{-1}\left(\frac{2}{n}\right)^{-\gamma+1}
\end{aligned}
$$

Since $1/n \le \varepsilon \le 2/n$, we have

$$
I \ll (1+|s|)^{-1}\varepsilon^{1-\gamma} \ll (1+|s|)^{-\gamma+\delta}\varepsilon^{1-\gamma}.
$$

We next estimate I_2. In I_2, since $sn+1 < [sn]+2 \le j \le [(s+1)n] \le (s+1)n$, we have $s+1/n < j/n \le (s+1)$ and $1/n < j/n - s \le 1$. By Lemma 3.2 (ii) with $u = j/n, u' = u$

$$
\left|g(s,u)-g\left(s,\frac{j}{n}\right)\right| \ll (1+|s|)^{-1}\left(\frac{j}{n}-s\right)^{-\gamma-1}\left(u-\frac{j}{n}\right) \ll (1+|s|)^{-1}\left(\frac{j}{n}-s\right)^{-\gamma-1}\varepsilon,
$$

since $0 \le u - j/n \le 1/n < \varepsilon$. Hence

$$
\begin{aligned}
I_2 &\ll (1+|s|)^{-1}\sum_{[sn]+1<j\le[(s+1)n]}\left(\frac{j}{n}-s\right)^{-\gamma-1}\frac{\varepsilon}{n} \\
&\ll (1+|s|)^{-1}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{-\gamma-1}\frac{\varepsilon}{n} \ll (1+|s|)^{-1}n^\gamma\varepsilon \ll (1+|s|)^{-\gamma+\delta}\varepsilon^{1-\gamma},
\end{aligned}
$$

since $\varepsilon \le 2/n$.

In I_3, note that $j/n \ge s+1$. Also, since $u \in [j/n, (j+1)/n]$, it follows that $s+1 \le j/n \le u$. Thus by Lemma 3.2 (iii-2) with $u = u, u' = j/n$, noting that $|u - j/n| < 1/n$, we have

$$I_3 \ll \frac{1}{n} \sum_{j > [(s+1)n]} \int_{j/n}^{(j+1)/n} (1+|u|)^{-1}(u-s)^{-\gamma} du \ll \frac{1}{n} \int_{s+1}^{\infty} (1+|u|)^{-1}(u-s)^{-\gamma} du.$$

We now need to estimate depending on the range of s.

When $s < -2$, we have

$$\int_{s+1}^{\infty} (1+|u|)^{-1}(u-s)^{-\gamma} du = \left(\int_{s+1}^{s/2} + \int_{s/2}^{|s|/2} + \int_{|s|/2}^{\infty} \right) (1+|u|)^{-1}(u-s)^{-\gamma} du$$
$$=: \text{(i)} + \text{(ii)} + \text{(iii)},$$

say. Here by the change of variables $u = st$ $(t > 0)$,

$$\text{(i)} \le \int_{1/2}^{1+1/s} (1+|st|)^{-1}|s|^{-\gamma}|t-1|^{-\gamma}|s| dt \ll |s|^{-\gamma} \int_{1/2}^{1} t^{-1}(1-t)^{-\gamma} dt \ll |s|^{-\gamma}.$$

As to (ii), by noticing $(u-s)^{-\gamma} \ll |s|^{-\gamma}$ for $|u| < |s|/2$,

$$\text{(ii)} \ll |s|^{-\gamma} \int_{s/2}^{|s|/2} (1+|u|)^{-1} du \ll |s|^{-\gamma} \log |s|.$$

Finally, as to (iii), by the change of variables $u = |s|t$ $(t > 0)$,

$$\text{(iii)} = \int_{1/2}^{\infty} (1+|st|)^{-1}|s|^{-\gamma}(t+1)^{-\gamma}|s| dt \ll |s|^{-\gamma} \int_{1/2}^{\infty} t^{-1-\gamma} dt \ll |s|^{-\gamma}.$$

When $|s| \le 2$, we have

$$\int_{s+1}^{\infty} (1+|u|)^{-1}(u-s)^{-\gamma} du = \left(\int_{s+1}^{3} + \int_{3}^{\infty} \right) (1+|u|)^{-1}(u-s)^{-\gamma} du =: \text{(iv)} + \text{(v)},$$

say. Here, since $u - s \ge 1$, (iv) $\le \int_{s+1}^{3} du \ll 1$, and (v) $\ll \int_{3}^{\infty} u^{-1-\gamma} du \ll 1$.

When $s > 2$, we have, by $u = st$,

$$\int_{s+1}^{\infty} (1+|u|)^{-1}(u-s)^{-\gamma} du = \int_{1+1/s}^{\infty} (1+st)^{-1}s^{-\gamma}(t-1)^{-\gamma} s dt$$
$$\ll s^{-\gamma} \int_{1+1/s}^{\infty} t^{-1}(t-1)^{-\gamma} dt \ll s^{-\gamma}.$$

Altogether, we have, for any $\delta > 0$,

$$I_3 \ll \frac{1}{n}(1+|s|)^{-\gamma} \log(e+|s|) < \varepsilon(1+|s|)^{-\gamma+\delta},$$

since $1/n \le \varepsilon$, and thus $I_3 \ll (1+|s|)^{-\gamma+\delta}\varepsilon^{1-\gamma}$. This completes the proof of the lemma. □

Lemma 3.4. *For any* $\delta > 0$,

$$\left|\int_s^\infty g(s,u)du\right| \ll (1+|s|)^{-\gamma+\delta} \tag{3.1}$$

and

$$\left|\sum_{j>[sn]+1} g\left(s, \frac{j}{n}\right)\frac{1}{n}\right| \ll (1+|s|)^{-\gamma+\delta}. \tag{3.2}$$

Due to Lemma 3.3, we only need to show (3.1). However, the proof can essentially be found in the proof of Lemma 3.3, and so we omit the proof.

Lemma 3.5. *For any* $\delta > 0$, *we have*

$$\left|\sum_{j>[sn]+1} g\left(s, \frac{j}{n}\right)\frac{1}{n} - \sum_{j>[sn]+1} g\left(\frac{[sn]+1}{n}, \frac{j}{n}\right)\frac{1}{n}\right| \ll \left(1+\left|\frac{[sn]+1}{n}\right|\right)^{-\gamma+\delta}\varepsilon^{1-\gamma}.$$

Proof. We have

$$\begin{aligned}
&\left|\sum_{j>[sn]+1} g\left(s, \frac{j}{n}\right)\frac{1}{n} - \sum_{j>[sn]+1} g\left(\frac{[sn]+1}{n}, \frac{j}{n}\right)\frac{1}{n}\right| \\
&\quad \le \sum_{j>[sn]+1}\left|g\left(s, \frac{j}{n}\right) - g\left(\frac{[sn]+1}{n}, \frac{j}{n}\right)\right|\frac{1}{n} \\
&\quad = \left(\sum_{[sn]+1<j\le[(s+1)n]} + \sum_{j>[(s+1)n]}\right)\left|g\left(s, \frac{j}{n}\right) - g\left(\frac{[sn]+1}{n}, \frac{j}{n}\right)\right|\frac{1}{n} \\
&\quad =: I_1 + I_2,
\end{aligned}$$

say. In I_1, since $s < ([sn]+1)/n < j/n \le s+1$, by Lemma 3.2 (i) with $s' = ([sn]+1)/n$ and $u = j/n$, we have

$$\left|g\left(s, \frac{j}{n}\right) - g\left(\frac{[sn]+1}{n}, \frac{j}{n}\right)\right| \ll (1+|s|)^{-1}\left(\frac{j}{n} - \frac{[sn]+1}{n}\right)^{-\gamma-1}\left|s - \frac{[sn]+1}{n}\right|.$$

Thus, since $|s - ([sn]+1)/n| \ll \varepsilon$,

$$\begin{aligned}
I_1 &\ll \sum_{[sn]+1<j\le[(s+1)n]} (1+|s|)^{-1}\left(\frac{j}{n} - \frac{[sn]+1}{n}\right)^{-\gamma-1}\frac{\varepsilon}{n} \\
&\ll (1+|s|)^{-1}\frac{\varepsilon}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{-\gamma-1} \ll (1+|s|)^{-1}n^\gamma\varepsilon \ll (1+|s|)^{-1}\varepsilon^{1-\gamma}
\end{aligned}$$

since $n^\gamma \ll \varepsilon^{-\gamma}$. In I_2, note again that $j/n \ge s+1$. Thus we have by Lemma 3.2 (iii-1) with $u = j/n, s = s, s' = ([sn]+1)/n$,

$$\left|g\left(s, \frac{j}{n}\right) - g\left(\frac{[sn]+1}{n}, \frac{j}{n}\right)\right| \ll \left(1+\left|\frac{j}{n}\right|\right)^{-1}\left(\frac{j}{n} - s\right)^{-\gamma}\left|s - \frac{[sn]+1}{n}\right|.$$

Thus, since $|s-([sn]+1)/n| \ll \varepsilon$,

$$I_2 \ll \sum_{j>[(s+1)n]} \left(1+\left|\frac{j}{n}\right|\right)^{-1} \left(\frac{j}{n}-s\right)^{-\gamma} \frac{\varepsilon}{n} \ll \varepsilon \int_{s+1}^{\infty} (1+|u|)^{-1}(u-s)^{-\gamma} du$$
$$\ll \varepsilon(1+|s|)^{-\gamma} \log(e+|s|) \ll \varepsilon(1+|s|)^{-\gamma+\delta}.$$

Thus

$$I_1+I_2 \ll (1+|s|)^{-\gamma+\delta}\varepsilon^{1-\gamma} \ll \left(1+\left|\frac{[sn]+1}{n}\right|\right)^{-\gamma+\delta} \varepsilon^{1-\gamma},$$

which completes the proof. □

Lemma 3.6. *For any $\delta > 0$,*

$$\sum_{j>[sn]+1} g\left(s,\frac{j}{n}\right)\frac{1}{n} \ll \left(1+\left|\frac{[sn]+1}{n}\right|\right)^{-\gamma+\delta},$$

$$\sum_{j>[sn]+1} g\left(\frac{[sn]+1}{n},\frac{j}{n}\right)\frac{1}{n} \ll \left(1+\left|\frac{[sn]+1}{n}\right|\right)^{-\gamma+\delta}.$$

Proof. By Lemmas 3.3 and 3.4,

$$\left|\sum_{j>[sn]+1} g\left(\frac{[sn]+1}{n},\frac{j}{n}\right)\frac{1}{n}\right|$$
$$\leq \left|\sum_{j>[sn]+1} g\left(\frac{[sn]+1}{n},\frac{j}{n}\right)\frac{1}{n} - \sum_{j>[sn]+1} g\left(s,\frac{j}{n}\right)\frac{1}{n}\right| + \left|\sum_{j>[sn]+1} g\left(s,\frac{j}{n}\right)\frac{1}{n}\right|$$
$$\ll \left(1+\left|\frac{[sn]+1}{n}\right|\right)^{-\gamma+\delta}\varepsilon^{1-\gamma} + (1+|s|)^{-\gamma+\delta} \ll \left(1+\left|\frac{[sn]+1}{n}\right|\right)^{-\gamma+\delta},$$

completing the proof of the lemma. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. We have

$$\begin{aligned}
I := & \left| \int_{-\infty}^{\infty} \left| \int_s^{\infty} f(u)(u-s)^{-\gamma} du \right|^{\alpha} ds - \sum_{i\in\mathbb{Z}} \left| \sum_{j>i} f\left(\frac{j}{n}\right) \left| \frac{j}{n} - \frac{i}{n} \right|^{-\gamma} \frac{1}{n} \right|^{\alpha} \frac{1}{n} \right| \\
\le & \left| \int_{-\infty}^{\infty} \left| \int_s^{\infty} g(s,u) du \right|^{\alpha} ds - \int_{-\infty}^{\infty} \left| \sum_{j>[sn]+1} g\left(s, \frac{j}{n}\right) \frac{1}{n} \right|^{\alpha} ds \right| \\
& + \left| \int_{-\infty}^{\infty} \left| \sum_{j>[sn]+1} g\left(s, \frac{j}{n}\right) \frac{1}{n} \right|^{\alpha} ds - \sum_{i\in\mathbb{Z}} \left| \sum_{j>i} g\left(\frac{i}{n}, \frac{j}{n}\right) \frac{1}{n} \right|^{\alpha} \frac{1}{n} \right| \\
=: & I_1 + I_2,
\end{aligned}$$

say. Since $||x|^{\alpha} - |y|^{\alpha}| \ll |x-y| \max\{|x|^{\alpha-1}, |y|^{\alpha-1}\}$ for $\alpha > 1$, we have, by Lemmas 3.3 and 3.4, and by choosing a sufficiently small $\delta > 0$,

$$I_1 \ll \int_{-\infty}^{\infty} (1+|s|)^{-\gamma+\delta} \varepsilon^{1-\gamma} (1+|s|)^{(-\gamma+\delta)(\alpha-1)} ds \ll \varepsilon^{1-\gamma}.$$

As to I_2,

$$I_2 = \left| \sum_{i\in\mathbb{Z}} \int_{(i-1)/n}^{i/n} \left(\left| \sum_{j>[sn]+1} g\left(s, \frac{j}{n}\right) \frac{1}{n} \right|^{\alpha} - \left| \sum_{j>i} g\left(\frac{i}{n}, \frac{j}{n}\right) \frac{1}{n} \right|^{\alpha} \right) ds \right|.$$

Note here that for $s \in [(i-1)/n, i/n), i = [sn]+1$. Thus

$$\begin{aligned}
I_2 \le & \sum_{i\in\mathbb{Z}} \int_{(i-1)/n}^{i/n} \left| \left| \sum_{j>i} g\left(s, \frac{j}{n}\right) \frac{1}{n} \right|^{\alpha} - \left| \sum_{j>i} g\left(\frac{i}{n}, \frac{j}{n}\right) \frac{1}{n} \right|^{\alpha} \right| ds \\
\ll & \sum_{i\in\mathbb{Z}} \int_{(i-1)/n}^{i/n} \varepsilon^{1-\gamma} \left(1 + \left|\frac{i}{n}\right|\right)^{-\gamma+\delta} \left(1 + \left|\frac{i}{n}\right|\right)^{(-\gamma+\delta)(\alpha-1)} ds \\
= & \varepsilon^{1-\gamma} \sum_{i\in\mathbb{Z}} \left(1 + \left|\frac{i}{n}\right|\right)^{-(\gamma-\delta)\alpha} \frac{1}{n} \\
\ll & \varepsilon^{1-\gamma} \int_{-\infty}^{\infty} (1+|t|)^{-(\gamma-\delta)\alpha} dt \ll \varepsilon^{1-\gamma},
\end{aligned}$$

provided that we choose a sufficiently small $\delta > 0$. We thus conclude that $I \ll \varepsilon^{1-\gamma}$, completing the proof. □

4. The main result

Our main theorem in the present paper is the following.

Theorem 4.1. *Let $\alpha > 1, 1/\alpha < H < 1, \beta = H - 1/\alpha \in (0, 1-1/\alpha)$, and define*

$$\Lambda_{\varepsilon} = \left\{ f \;\middle|\; \int_{-\infty}^{\infty} |(I_{\beta}|f|)(s)|^{\alpha-\varepsilon} ds < \infty \right\}$$

for some $0 < \varepsilon < \alpha - 1$*. For* $f \in \Lambda \cap \Lambda_\varepsilon$*, we assume that* f *is differentiable,* $|f(u)| \ll (1+|u|)^{-1}$ *and* $|f'(u)| \ll (1+|u|)^{-1}$*. Let* $\{X_j,\, j \in \mathbb{Z}\}$ *be i.i.d. random variables with* $E[X_j] = 0$ *such that*

$$\frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} X_j \xrightarrow{d} Z_\alpha(1), \tag{4.1}$$

and let

$$c_j = \begin{cases} 0, & j \le 0, \\ \beta j^{\beta-1}, & j > 0. \end{cases}$$

Define

$$\xi_j = \sum_{k \in \mathbb{Z}} c_k X_{j-k}, \quad j \in \mathbb{Z}\,.$$

Then,
(i) *for each* $n \in \mathbb{N}$*, the infinite series* $\sum_{j\in\mathbb{Z}} f(j/n)\xi_j$ *converges in the sense of the* $(\alpha-\varepsilon)$*-th mean, and*
(ii)

$$\frac{1}{n^H} \sum_{j\in\mathbb{Z}} f\left(\frac{j}{n}\right) \xi_j \xrightarrow{d} \int_{-\infty}^{\infty} f(u)\, d\Delta_{H,\alpha}(u), \qquad \text{as } n \to \infty. \tag{4.2}$$

Proof. For $m_1 < m_2$,

$$E\left[\left|\sum_{m_1<j\le m_2} f\left(\frac{j}{n}\right)\xi_j\right|^{\alpha-\varepsilon}\right] = E\left[\left|\sum_{m_1<j\le m_2} f\left(\frac{j}{n}\right)\sum_{k\in\mathbb{Z}} c_k X_{j-k}\right|^{\alpha-\varepsilon}\right] = E\left[\left|\sum_{l\in\mathbb{Z}} b_{n,l} X_l\right|^{\alpha-\varepsilon}\right],$$

where $b_{n,l} = \sum_{m_1<j\le m_2} f(j/n)\, c_{j-l}$. Note that (4.1) implies $E[|X_j|^{\alpha-\varepsilon}] < \infty$. Since $\{X_j, j \in \mathbb{Z}\}$ are i.i.d., $\alpha - \varepsilon > 1, E[|b_{n,l}X_l|^{\alpha-\varepsilon}] < \infty$ and $E[b_{n,l}X_l] = 0$, we have by Marcinkiewitz-Zygmund inequality,

$$I(m_1, m_2) := E\left[\left|\sum_{l\in\mathbb{Z}} b_{n,l}X_l\right|^{\alpha-\varepsilon}\right] \le CE\left[\left(\sum_{l\in\mathbb{Z}} b_{n,l}^2 X_l^2\right)^{\alpha-\varepsilon/2}\right], \tag{4.3}$$

where C denotes a positive constant which may differ from one to another. If we continue to estimate $I(m_1, m_2)$,

$$\begin{aligned} I(m_1,m_2) \le & CE\left[\sum_{l\in\mathbb{Z}} |b_{n,l}|^{\alpha-\varepsilon}|X_l|^{\alpha-\varepsilon}\right] = CE\left[|X_1|^{\alpha-\varepsilon}\right]\sum_{l\in\mathbb{Z}}|b_{n,l}|^{\alpha-\varepsilon} \\ = & C\sum_{l\in\mathbb{Z}}\left|\sum_{m_1<j\le m_2, j>l} f\left(\frac{j}{n}\right)\beta\,(j-l)^{\beta-1}\right|^{\alpha-\varepsilon} \\ \le & Cn^{\beta(\alpha-\varepsilon)+1}\sum_{l\in\mathbb{Z}}\left(\sum_{j>l}\left|f\left(\frac{j}{n}\right)\right|\left(\frac{j}{n}-\frac{l}{n}\right)^{\beta-1}\frac{1}{n}\right)^{\alpha-\varepsilon}\frac{1}{n} \end{aligned}$$

Here by Theorem 3.1 with $f = |f|, \gamma = 1-\beta, \alpha = \alpha - \varepsilon$, we have as $n \to \infty$,

$$\sum_{l\in\mathbb{Z}} \left| \sum_{j>l} \left| f\left(\frac{j}{n}\right)\right| \left(\frac{j}{n} - \frac{l}{n}\right)^{\beta-1} \frac{1}{n} \right|^{\alpha-\varepsilon} \frac{1}{n}$$
$$\longrightarrow \int_{-\infty}^{\infty} \left| \int_s^\infty |f(u)|(u-s)^{\beta-1} du \right|^{\alpha-\varepsilon} ds =: J,$$

say, which is finite by the assumption that $f \in \Lambda_\varepsilon$. Thus for any $\eta > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \sum_{l\in\mathbb{Z}} \left(\sum_{j>l} \left| f\left(\frac{j}{n_0}\right)\right| \left(\frac{j}{n_0} - \frac{l}{n_0}\right)^{\beta-1} \frac{1}{n_0} \right)^{\alpha-\varepsilon} \frac{1}{n_0} - J \right| < \eta.$$

Thus for any $m_1 < m_2$,

$$I(m_1, m_2) \le C(J+\eta) n_0^{\beta(\alpha-\varepsilon)+1} < \infty.$$

By letting $m_1 \to -\infty, m_2 \to \infty$, we get the statement (i).

We next prove the statement (ii). To this end, we show the convergence of the corresponding characteristic functions, namely,

$$E\left[\exp\left\{ i\theta \frac{1}{n^H} \sum_{j\in\mathbb{Z}} f\left(\frac{j}{n}\right) \xi_j \right\}\right] \longrightarrow E\left[\exp\left\{ i\theta \int_{-\infty}^{\infty} f(u) d\Delta_{H,\alpha}(u) \right\}\right]$$

Here, as we have almost seen above,

$$\frac{1}{n^H} \sum_{j\in\mathbb{Z}} f\left(\frac{j}{n}\right) \xi_j = \frac{\beta}{n^H} \sum_{l\in\mathbb{Z}} \left(\sum_{j>l} f\left(\frac{j}{n}\right) (j-l)^{\beta-1} \right) X_l,$$

and by Theorem 2.1 and (2.3) recall

$$\int_{-\infty}^{\infty} f(u) d\Delta_{H,\alpha}(u) = \beta \int_{-\infty}^{\infty} \left[\int_s^\infty f(u)(u-s)^{\beta-1} du \right] dZ_\alpha(s).$$

In general, we know that

$$E\left[\exp\left\{ i\theta \int_{-\infty}^{\infty} g(s) dZ_\alpha(s) \right\}\right] = \exp\left\{ -|\theta|^\alpha \int_{-\infty}^{\infty} |g(s)|^\alpha ds \right\}.$$

Hence

$$E\left[\exp\left\{ i\theta \int_{-\infty}^{\infty} f(u) d\Delta_{H,\alpha}(u) \right\}\right] = \exp\left\{ -|\theta|^\alpha \beta^\alpha \int_{-\infty}^{\infty} \left| \int_s^\infty f(u)(u-s)^{\beta-1} du \right|^\alpha ds \right\}.$$

On the other hand, if we put $\varphi(\theta) = E[\exp\{i\theta X_1\}]$, then

$$
\begin{aligned}
I := \lim_{n\to\infty} E\left[\exp\left\{i\theta\frac{\beta}{n^H}\sum_{l\in\mathbb{Z}}\left(\sum_{j>l} f\left(\frac{j}{n}\right)(j-l)^{\beta-1}\right)X_l\right\}\right] \\
= \lim_{n\to\infty}\prod_{l\in\mathbb{Z}} E\left[\exp\left\{i\theta\frac{\beta}{n^H}\sum_{j>l} f\left(\frac{j}{n}\right)(j-l)^{\beta-1}X_l\right\}\right] \\
= \lim_{n\to\infty}\prod_{l\in\mathbb{Z}}\varphi\left(\theta\frac{\beta}{n^H}\sum_{j>l} f\left(\frac{j}{n}\right)(j-l)^{\beta-1}\right).
\end{aligned}
$$

However, it is known that under (4.1), $1-\varphi(\theta) \sim |\theta|^\alpha$as $|\theta| \to 0$. Thus in general, if $\lim_{n\to\infty}\sup_l |a_l(n)| = 0$, then

$$
\lim_{n\to\infty}\prod_{l\in\mathbb{Z}}\varphi(a_l(n)) = \exp\left\{-\lim_{n\to\infty}\sum_{l\in\mathbb{Z}}|a_l(n)|^\alpha\right\}.
$$

Therefore, we need to show

$$
\lim_{n\to\infty}\sup_l\left|\frac{1}{n^H}\sum_{j>l} f\left(\frac{j}{n}\right)(j-l)^{\beta-1}\right| = 0
$$

in our case. However, this is true because (note $\beta = H - 1/\alpha$)

$$
\begin{aligned}
\sup_l\left|\frac{1}{n^H}\sum_{j>l} f\left(\frac{j}{n}\right)(j-l)^{\beta-1}\right| = \frac{1}{n^{1/\alpha}}\sup_l\frac{1}{n}\sum_{j>l}\left|f\left(\frac{j}{n}\right)\right|\left(\frac{j}{n}-\frac{l}{n}\right)^{\beta-1} \\
=\frac{1}{n^{1/\alpha}}\sup_s\frac{1}{n}\sum_{j>sn}\left|f\left(\frac{j}{n}\right)\right|\left(\frac{j}{n}-s\right)^{\beta-1} \ll \frac{1}{n^{1/\alpha}}
\end{aligned}
$$

by (3.2) in Lemma 3.4. Hence by Theorem 3.1,

$$
\begin{aligned}
I =\exp\left\{-|\theta|^\alpha\beta^\alpha\lim_{n\to\infty}\sum_{l\in\mathbb{Z}}\left|\frac{1}{n^H}\sum_{j>l} f\left(\frac{j}{n}\right)(j-l)^{\beta-1}\right|^\alpha\right\} \\
=\exp\left\{-|\theta|^\alpha\beta^\alpha\lim_{n\to\infty}\sum_{l\in\mathbb{Z}}\left|\sum_{j>l} f\left(\frac{j}{n}\right)(j-l)^{\beta-1}\frac{1}{n}\right|^\alpha\frac{1}{n}\right\} \\
=\exp\left\{-|\theta|^\alpha\beta^\alpha\int_{-\infty}^{\infty}\left|\int_s^\infty f(u)(u-s)^{\beta-1}du\right|^\alpha ds\right\},
\end{aligned}
$$

which completes the proof of Theorem 4.1. □

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