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Universal deformation formulae, symplectic Lie groups and symmetric spaces

by

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Universal Deformation Formulae, Symplectic Lie groups and Symmetric Spaces

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Abstract

We apply methods from strict quantization of solvable symmetric spaces to obtain universal deformation formulae for actions of a class of solvable Lie groups. We also study compatible co-products by generalizing the notion of smash product in the context of Hopf algebras.

1 Introduction

Let G be a group acting on a set M. Denote by $\tau : G \times M \to M : (g,x) \mapsto \tau_g(x)$ the (left) action and by $\alpha : G \times \mathrm{Fun}(M) \to \mathrm{Fun}(M)$ the corresponding action on the space of (complex valued) functions (or formal series) on M ($\alpha_g := \tau^\star_{g^{-1}}$). Assume that on a subspace $\mathbb{A} \subset \mathrm{Fun}(G)$, one has an associative $\mathbb{C}$-algebra product $\star^G_{\mathbb{A}} : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ such that

(i) $\mathbb{A}$ is invariant under the (left) regular action of G on $\mathrm{Fun}(G)$,

(ii) the product $\star^G_{\mathbb{A}}$ is left-invariant as well i.e. for all $g \in G; a, b \in \mathbb{A}$, one has

$$(L^\star_g a) \star^G_{\mathbb{A}} (L^\star_g b) = L^\star_g (a \star^G_{\mathbb{A}} b). \tag{1}$$

Given a function on M, $u \in \mathrm{Fun}(M)$, and a point $x \in M$, one denotes by $\alpha^x(u) \in \mathrm{Fun}(G)$ the function on G defined as

$$\alpha^x(u)(g) := \alpha_g(u)(x). \tag{2}$$

Then one readily observes that the subspace $\mathbb{B} \subset \mathrm{Fun}(M)$ defined as

$$\mathbb{B} := \{u \in \mathrm{Fun}(M) \,|\, \forall x \in M : \alpha^x(u) \in \mathbb{A}\} \tag{3}$$

becomes an associative $\mathbb{C}$-algebra when endowed with the product $\star^M_{\mathbb{B}}$ given by

$$u \star^M_{\mathbb{B}} v(x) := (\alpha^x(u) \star^G_{\mathbb{A}} \alpha^x(v))(e) \tag{4}$$

(e denotes the neutral element of G). Of course, all this can be defined for right actions as well.

Definition 1.1 *Such a pair* $(\mathbb{A}, \star^G_{\mathbb{A}})$ *is called a (left)* **universal deformation** *of* G, *while Formula (4) is called the associated* **universal deformation formula** *(briefly* **UDF***).*

In the present article, we will be concerned with the case where G is a Lie group. The function space $\mathbb{A}$ will be either
- a functional subspace (or a topological completion) of $C^\infty(G,\mathbb{C})$ containing the smooth compactly supported functions in which case we will talk about **strict deformation** (following Rieffel [Rie89]),
or,
- the space $\mathbb{A} = C^\infty(G)[[\hbar]]$ of formal power series with coefficients in the smooth functions on G in which case, we'll speak about **formal deformation**. In any case, we'll assume the product $\star^G_{\mathbb{A}}$ admits an asymptotic expansion of star-product type:

$$a \star^G_{\mathbb{A}} b \sim ab + \frac{\hbar}{2i}\mathbf{w}(\mathrm{d}u, \mathrm{d}v) + o(\hbar^2) \qquad (a, b \in C^\infty_c(G)),$$

where $\mathbf{w}$ denotes some (left-invariant) Poisson bivector on G[BFFLS]. In the strict cases considered here, the product will be defined by an integral three-point kernel $K \in C^\infty(G\times G\times G)$:

$$a \star^G_{\mathbb{A}} b(g) := \int_{G\times G} a(g_1)\, b(g_2) K(g, g_1, g_2) \mathrm{d}g_1\, \mathrm{d}g_2 \qquad (a, b \in \mathbb{A})$$

where $\mathrm{d}g$ denotes a normalized left-invariant Haar measure on G. Moreover, our kernels will be of **WKB type** [Wei94, Kar94] i.e.:

$$K = A\, e^{\frac{i}{\hbar}\Phi},$$

with A (the **amplitude**) and Φ (the **phase**) in $C^\infty(G\times G\times G,\mathbb{R})$ being invariant under the (diagonal) action by left-translations.
Note that in the case where the group G acts smoothly on a smooth manifold M by diffeomorphisms: $\tau : G\times M \to M : (g, x) \mapsto \tau_g(x)$, the first-order expansion term of $u \star^M_{\mathbb{B}} v, \quad u, v \in C^\infty(M)$ defines a Poisson structure $\mathbf{w}^M$ on M which can be expressed in terms of a basis $\{X_i\}$ of the Lie algebra $\mathfrak{g}$ of G as:

$$\mathbf{w}^M = [\mathbf{w}_e]^{ij}\, X_i^\star \wedge X_j^\star, \tag{5}$$

where $X^\star$ denotes the fundamental vector field on M associated to $X \in \mathfrak{g}$.

Strict deformation theory in the WKB context was initiated by Rieffel in [Rie93] in the cases where G is either Abelian or 1-step nilpotent. Rieffel's work has led to what is now called 'Rieffel's machinery'; producing a whole class of exciting non-commutative manifolds (in Connes sense) from the data of Abelian group actions on $C^\star$-algebras [CoLa01].
The study of formal UDF's for non-Abelian group actions in our context was initiated in [GiZh98] where the case of the group of affine transformations of the real line ('$ax + b$') was explicitly described.
In the strict (non-formal) setting, UDF's for Iwasawa subgroups of $SU(1, n)$ have been explicitly given in [BiMas01]. These were obtained by adapting a method developed by one of us in the symmetric space framework [Bie00].
This work has three parts. We first give some results about the structure of symplectic symmetric spaces for which the action of the linear holonomy algebra at a point o has isotropic range in the symplectic vector space tangent at o. In particular, we show that, when split (cf. Definition 4.3), such a space can be identified with the underlying manifold of a Lie subgroup S of its automorphism group. In particular, this yields a distinguished class of symplectic Lie groups in Lichnérowicz sense which we call *elementary*.
Secondly, we use the strict quantization method of [Bie00] to deduce universal deformation formulae for such symplectic Lie groups. We extend these results to a class of Abelian extensions of elementary symplectic Lie groups. We therefore obtain UDF's for a class of groups that can be thought of generalized $ax + b$ groups.
Finally, we generalize the classical definition of smash products in the context of (formal) bialgebras (or Hopf algebras) by defining a class of products constructed from the data of *bi*-module algebras; we call these products *L-R smash* products. We formally realize each of our UDF's as a L-R-smash product. As a corollary, we obtain compatible co-products.

2 General facts about symplectic symmetric spaces

Definition 2.1 *[Bie95] A* **symplectic symmetric space** *is a triple (M, ω, s), where (M, ω) is a smooth connected symplectic manifold, and where $s : M \times M \to M$ is a smooth map such that*

(i) for all x in M, the partial map $s_x : M \to M : y \mapsto s_x(y) := s(x, y)$ is an involutive symplectic diffeomorphism of (M, ω) called the **symmetry** *at x.*

(ii) For all x in M, x is an isolated fixed point of s_x.

(iii) For all x and y in M, one has $s_x s_y s_x = s_{s_x(y)}$.

Definition 2.2 *Two symplectic symmetric spaces (M, ω, s) and (M', ω', s') are* **isomorphic** *if there exists a symplectic diffeomorphism $\varphi : (M, \omega) \to (M', \omega')$ such that $\varphi s_x = s'_{\varphi(x)} \varphi$. Such a φ is called an* **isomorphism** *of (M, ω, s) onto (M', ω', s'). When $(M, \omega, s) = (M', \omega', s')$, one talks about* **automorphisms**. *The group of all automorphisms of the symplectic symmetric space (M, ω, s) is denoted by $\mathtt{Aut}(M, \omega, s)$.*

Proposition 2.1 *On a symplectic symmetric space (M, ω, s), there exists a unique affine connection ∇ which is invariant under the symmetries. Moreover, this connection satisfies the following properties.*

(i) For all smooth tangent vector fields X, Y, Z on M and all points x in M, one has

$$\omega_x(\nabla_X Y, Z) = \frac{1}{2} X_x.\omega(Y + s_{x_*} Y, Z).$$

(ii) (M, ∇) is an affine symmetric space. In particular ∇ is torsion free and its curvature tensor is parallel.

(iii) The symplectic form ω is parallel; ∇ is therefore a symplectic connection.

(iv) One has

$$\mathtt{Aut}(M, \omega, s) = \mathtt{Aut}(M, \omega, \nabla) = \mathtt{Aff}(\nabla) \cap \mathtt{Symp}(\omega).$$

The connection ∇ on the symmetric space (M, s) is called the **Loos connection**. The following facts are classical (see [Loo69]).

Theorem 2.1 *Let (M, ω, s) be a symplectic symmetric space and ∇ its Loos connection. Fix o in M and denote by H the stabilizer of o in $\mathtt{Aut}(M, \omega, s)$. Denote by G the* **transvection group** *of (M, s) (i.e. the subgroup of $\mathtt{Aut}(M, \omega, s)$ generated by $\{s_x \circ s_y\,;\, x, y \in M\}$) and set $K = G \cap H$. Then,*

(i) the transvection group G is a connected Lie transformation group of M. It is the smallest subgroup of $\mathtt{Aut}(M, \omega, s)$ which is transitive on M and stabilized by the conjugation $\tilde{\sigma} : \mathtt{Aut}(M, \omega, s) \to \mathtt{Aut}(M, \omega, s)$ defined by $\tilde{\sigma}(g) = s_o g s_o$.

(ii) The homogeneous space $M = G/_K$ is reductive. The Loos connection ∇ coincides with the canonical connection induced by the structure of reductive homogeneous space.

(iii) Denoting by $G^{\tilde{\sigma}}$ the set of $\tilde{\sigma}$-fixed points in G and by $G_0^{\tilde{\sigma}}$ the connected component of the identity, one has

$$G_0^{\tilde{\sigma}} \subset K \subset G^{\tilde{\sigma}}.$$

The Lie algebra $\mathfrak{k}$ of K is isomorphic to the holonomy algebra with respect to the canonical connection ∇.

(iv) Denote by σ the involutive automorphism of the Lie algebra $\mathfrak{g}$ of G induced by the automorphism $\tilde{\sigma}$. Denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the decomposition in ± 1-eigenspaces for σ. Then, identifying $\mathfrak{p}$ with $T_o(M)$, one has

$$exp(X) = s_{Exp_o(\frac{1}{2}X)} \circ s_o$$

for all X in a neighborhood of 0 in $\mathfrak{p}$. Here exp is the exponential map $exp : \mathfrak{g} \to G$ and Exp_o is the exponential map at point o with respect to the connection ∇.

Definition 2.3 *Let $(\mathfrak{g}, \sigma)$ be an* **involutive algebra**, *that is, $\mathfrak{g}$ is a finite dimensional real Lie algebra and σ is an involutive automorphism of $\mathfrak{g}$. Let Ω be a skewsymmetric bilinear form on $\mathfrak{g}$. Then the triple $(\mathfrak{g}, \sigma, \Omega)$ is called a* **symplectic triple** *if the following properties are satisfied.*

(i) Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ (resp. $\mathfrak{p}$) is the $+1$ (resp. -1) eigenspace of σ. Then $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$ and the representation of $\mathfrak{k}$ on $\mathfrak{p}$, given by the adjoint action, is faithful.

(ii) Ω is a Chevalley 2-cocycle for the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$ such that for any X in $\mathfrak{k}$, $i(X)\Omega = 0$. Moreover, the restriction of Ω to $\mathfrak{p} \times \mathfrak{p}$ is nondegenerate.

The dimension of $\mathfrak{p}$ defines the **dimension** *of the triple. Two such triples $(\mathfrak{g}_i, \sigma_i, \Omega_i)$ $(i = 1, 2)$ are* **isomorphic** *if there exists a Lie algebra isomorphism $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\psi \circ \sigma_1 = \sigma_2 \circ \psi$ and $\psi^*\Omega_2 = \Omega_1$.*

Theorem 2.1 associates to a symplectic symmetric space (M, ω, s) an involutive Lie algebra $(\mathfrak{g}, \sigma)$. Denoting by $\pi : G \to M$ the natural projection, one checks that the triple $(\mathfrak{g}, \sigma, \Omega = \pi^*(\omega_o))$ is a symplectic triple. This implies the next proposition.

Proposition 2.2 *There is a bijective correspondence between the isomorphism classes of simply connected symplectic symmetric spaces (M, ω, s) and the isomorphism classes of symmetric triples $(\mathfrak{g}, \sigma, \Omega)$.*

Since a symmetric symplectic manifold (M, ω, s) is a symplectic homogeneous space of its transvection group G, it seems natural, when possible, to relate (M, ω, s) to a coadjoint orbit of G in $\mathcal{G}^*$. Recall first the two following definitions.

Definition 2.4 *Let G be a Lie group of symplectomorphisms acting on a symplectic manifold (M, ω). For every element X in the Lie algebra $\mathfrak{g}$ of G, one denotes by X^* the* **fundamental vector field** *associated to X, i.e. for x in M,*

$$X^*_x = \frac{d}{dt} \exp(-tX)x|_{t=0}.$$

The action is called **weakly Hamiltonian** *if for all X in $\mathfrak{g}$ there exists a smooth function $\lambda_X \in C^\infty(M)$ such that*

$$i(X^*)\omega = d\lambda_X.$$

In this case, if the correspondence $\mathfrak{g} \to C^\infty(M) : X \mapsto \lambda_X$ is also a homomorphism of Lie algebras, one says that the action of G on (M, ω) is **Hamiltonian**. *(The Lie algebra structure on $C^\infty(M)$ is defined by the Poisson bracket.)*

Proposition 2.3 *Let $t = (\mathfrak{g}, \sigma, \Omega)$ be a symplectic triple and let (M, ω, s) be the associated simply connected symplectic symmetric space. The action of the transvection group G on M is Hamiltonian if and only if Ω is a Chevalley coboundary, that is, there exists an element ξ in $\mathfrak{g}^*$ such that $\Omega = \delta\xi$. In this case, (M, ω, s) is a G-equivariant symplectic covering of $\mathcal{O}$, the coadjoint orbit of ξ in $\mathfrak{g}^*$.*

The action of the transvection group G is in general not Hamiltonian. We therefore need to consider a one-dimensional central extension of G rather than G itself. At the infinitesimal level, this corresponds to extending the algebra $\mathfrak{g}$ by the 2-cocycle Ω. This way, one associates to any symplectic symmetric space an exact triple in the following sense (see [Bie98] for details).

Definition 2.5 *An* **exact triple** *is a triple $\tau = (\mathfrak{h}, \sigma, \mathbf{\Omega})$ where*

- *$(\mathfrak{h}, \sigma)$ is an involutive Lie algebra such that if $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{p}$ is the decomposition w.r.t. σ one has $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{l}$,*
- *$\mathbf{\Omega}$ is a Chevalley 2-coboundary (i.e. $\mathbf{\Omega} = \delta\xi$, $\xi \in \mathfrak{h}^*$) such that $i(\mathfrak{l})\mathbf{\Omega} = 0$ and $\mathbf{\Omega}|_{\mathfrak{p}\times\mathfrak{p}}$ is symplectic.*

Remark 2.1 *One can choose $\xi \in \mathfrak{h}^*$ such that $\xi(\mathfrak{p}) = 0$.*

Lemma 2.1 *Let $t = (\mathfrak{g}, \sigma, \mathbf{\Omega})$ be a symplectic triple. Assume that the triple t is non-exact. Consider the triple $\tau(t) = (\mathfrak{h}(\mathfrak{g}), \sigma_{\mathfrak{h}(\mathfrak{g})}, \mathbf{\Omega}_{\mathfrak{h}(\mathfrak{g})})$ constructed as follows:*

- *$0 \to \mathbb{R}E \to \mathfrak{h}(\mathfrak{g}) \xrightarrow{\pi} \mathfrak{g} \to 0$ is the central extension defined by $[X,Y]_{\mathfrak{h}(\mathfrak{g})} = \mathbf{\Omega}(X,Y)E \oplus [X,Y]_{\mathfrak{g}}$*
- $\sigma_{\mathfrak{h}(\mathfrak{g})} = id_{\mathbb{R}E} \oplus \sigma$
- *$\mathbf{\Omega}_{\mathfrak{h}(\mathfrak{g})}$ is the trivial extension of $\mathbf{\Omega}$ to $\mathfrak{h}(\mathfrak{g})$.*

Then, $\tau(t)$ is an exact triple.

Remark 2.2 Observe that, when associated to a (transvection) symplectic triple, the center $\mathfrak{z}(\mathfrak{h}(\mathfrak{g}))$ of the Lie algebra $\mathfrak{h}(\mathfrak{g})$ occurring in an exact triple is at most one dimensional. Indeed, on the one hand, exactness implies $\mathfrak{z}(\mathfrak{h}(\mathfrak{g})) \subset \mathfrak{k}$. One the other hand, faithfulness of the holonomy representation forces $\dim(\mathfrak{z}(\mathfrak{h}(\mathfrak{g})) \cap \mathfrak{k}) \leq 1$ since $\mathfrak{k}$ is either the holonomy algebra itself or a one dimensional central extension.

3 Elementary solvable symplectic symmetric spaces and their strict quantization

In Definition 3.1 below, we define a particular type of solvable symmetric spaces which we call elementary. It has been proven ([Bie98], Proposition 3.2) that every solvable symmetric space is realized through a sequence of split extensions by Abelian (flat) factors successively taken over an elementary solvable symmetric space. We therefore consider elementary solvable symmetric spaces as the "first induction step" when studying solvable symmetric spaces.

Definition 3.1 *A symplectic symmetric space (M, ω, s) is called an* **elementary solvable** *symplectic symmetric space if its associated exact triple $(\mathfrak{h}(\mathfrak{g}), \sigma, \Omega = \delta\xi)$ (see Lemma 2.1) is of the following type.*

(i) The Lie algebra $\mathfrak{h}(\mathfrak{g})$ is a split extension of Abelian Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$:

$$0 \to \mathfrak{b} \longrightarrow \mathfrak{h}(\mathfrak{g}) \longrightarrow \mathfrak{a} \to 0.$$

(ii) The automorphism σ preserves the splitting $\mathfrak{h}(\mathfrak{g}) = \mathfrak{b} \oplus \mathfrak{a}$.

Such an exact triple (associated to an elementary solvable symplectic symmetric space) is called an **elementary solvable exact triple**.

Observe that, since $\mathfrak{a} \cap \mathfrak{k} \subset \mathfrak{a} \cap [\mathfrak{h}(\mathfrak{g}), \mathfrak{h}(\mathfrak{g})] = 0$, one has $\mathfrak{a} \subset \mathfrak{p}$. Therefore $\mathfrak{b} = \mathfrak{k} \oplus \mathfrak{l}$, with $\mathfrak{l} \subset \mathfrak{p}$. Moreover, since $\mathfrak{l}$ and $\mathfrak{a}$ are Abelian and Ω is nondegenerate, the subspaces $\mathfrak{a}$ and $\mathfrak{l}$ of $\mathfrak{p}$ are dual Lagrangians.

Now let (M, ω, s) be an elementary solvable symplectic symmetric space with associated exact triple $(\mathfrak{h}(\mathfrak{g}), \sigma, \Omega = \delta\xi)$ as above. In a neighborhood U of the origin, the map

$$\mathfrak{p} = \mathfrak{a} \times \mathfrak{l} \to M : (a, l) \mapsto \exp(a)\exp(l).o \tag{6}$$

turns out to be a Darboux chart when $U \subset \mathfrak{p}$ has the symplectic structure $\Omega = \delta\xi$. Moreover, there exists a unique immersion $\phi : U \cap \mathfrak{a} \to \mathfrak{a}$ such that in the local coordinate system (6), one has the following linearization property:

$$\xi(\sinh(a)l) = \xi[\phi(a), l]; \tag{7}$$

where, for $a \in \mathfrak{a}$ we set $\sinh(a) := \frac{1}{2}(\exp(\rho(a)) - \exp(-\rho(a))) \in \mathrm{End}(\mathfrak{b})$. This immersion is called the **twisting map**.

Proposition 3.1 *An elementary solvable symplectic symmetric space is strictly geodesically convex if and only if its associated twisting map extends to $\mathfrak{a}$ as a global diffeomorphism of $\mathfrak{a}$. In this case, the Darboux chart (6) extends as a global symplectomorphism $(\mathfrak{p}, \Omega) \to (M, \omega)$.*

Associated to the twisting map one has a three-point function $S \in C^\infty(M \times M \times M, \mathbb{R})$ called the **WKB-phase** of the elementary solvable symplectic symmetric space:

$$S(x_0, x_1, x_2) := \xi\left(\oint_{0,1,2} \sinh(a_0 - a_1)l_2\right); \tag{8}$$

where $\oint_{0,1,2}$ stands for cyclic summation and where $x_i = (a_i, l_i) \quad (i = 0,1,2)$. The phase S turns out to be invariant under the (diagonal) action of the symmetries $\{s_x\}_{x \in M}$ on $M \times M \times M$. This will be the essential constituent of the associative oscillatory kernel defining a symmetry-invariant strict quantization on every elementary solvable symplectic symmetric space. We now recall this construction as in [Bie00].

Definition 3.2 *For a compactly supported function $u \in C^\infty_c(\mathfrak{p})$, identifying $\mathfrak{l}^\star$ with $\mathfrak{a}$, we denote by $\tilde{u} \in C^\infty(\mathfrak{a} \times \mathfrak{a})$ its partial Fourier transform:*

$$\tilde{u}(a, \alpha) := \int_{\mathfrak{l}} e^{i\Omega(\alpha, l)} u(a, l) \mathrm{d}l. \tag{9}$$

We also denote by $\phi_\hbar : \mathfrak{a} \longrightarrow \mathfrak{a}$ the one-parameter family of twisting maps:

$$\phi_\hbar(a) := \frac{2}{\hbar}\phi(\frac{\hbar}{2}a). \tag{10}$$

For $u, v \in C^\infty_c(\mathfrak{p})$, we set

$$< u \mid v >_\hbar := \int_{\mathfrak{a}\times\mathfrak{a}} \tilde{u}(a,\alpha)\overline{\tilde{v}(a,\alpha)}\, |Jac_{\phi^{-1}}(\alpha)|\, \mathrm{d}a\, \mathrm{d}\alpha. \tag{11}$$

The pair $(C^\infty(\mathfrak{p}), <\,,\,>_\hbar)$ is a pre-Hilbert space, and we denote by $\mathcal{H}_\hbar$ its Hilbert completion.

The Hilbert product $<\,,\,>_\hbar$ turns out to be symmetry-invariant on $C^\infty_c(M)$. The action of the transvection group then extends by continuity to an isometric action on $\mathcal{H}_\hbar$.

Theorem 3.1 *[Bie00] Let (M, ω, s) be a strictly geodesically convex elementary solvable symplectic symmetric space. Realize it symplectically as $(\mathfrak{p} = \mathfrak{a} \times \mathfrak{l}, \Omega)$, and define the two-point function $A \in C^\infty(M \times M)$ by:*

$$A(x_1, x_2) := |Jac_\phi(a_1 - a_2.)| \tag{12}$$

This function is called the **WKB-amplitude** *and turns out to be symmetry-invariant. In this notation, one has the following.*

(i) For all $\hbar \in \mathbb{R}\backslash\{0\}$ and $u, v \in C^\infty_c(M)$, the formula:

$$u \star_\hbar v(x_0) := \int_{M\times M} u(x_1)\, v(x_2)\, A(x_1, x_2)\, e^{\frac{i}{\hbar}S(x_0,x_1,x_2)}\, \mathrm{d}x_1\, \mathrm{d}x_2 \tag{13}$$

extends as an associative product on $\mathcal{H}_\hbar$ ($\mathrm{d}x$ denotes some normalization of the symplectic volume on (M,ω)). Moreover, (for suitable u, v and x_0) the stationary phase method yields a power series expansion of the form

$$u \star_\hbar v(x_0) \sim uv(x_0) + \frac{\hbar}{2i}\{u, v\}(x_0) + o(\hbar^2); \tag{14}$$

where $\{\,,\,\}$ denotes the symplectic Poisson bracket on (M, ω).

(ii) The pair $(\mathcal{H}_\hbar, \star_\hbar)$ is a topological Hilbert algebra which the transvection group of (M, ω, s) acts on by automorphisms.

A classical procedure then produces a similar result in the $C^\star$-context, see [Bie00] for details.

Remark 3.1 Wether a symmetric space is strictly geodesically convex is of course entirely encoded in the spectral content of the splitting endomorphism $\rho : \mathfrak{a} \longrightarrow \mathbf{End}(\mathfrak{b})$. This is discussed in detail in [Bie00].

4 Symplectic Lie groups associated to a class of symmetric spaces

Definition 4.1 *A* **symplectic** *Lie algebra is a pair $(\mathfrak{s}, \omega)$ where $\mathfrak{s}$ is a Lie algebra and $\omega \in \bigwedge^2(\mathfrak{s}^\star)$ is a non-degenerate Chevalley two-cocycle w.r.t. the trivial representation of $\mathfrak{s}$.*

In this section, we associate symplectic Lie algebras to a class of symplectic symmetric spaces.

4.1 Holonomy isotropic symplectic symmetric spaces

Definition 4.2 *A symplectic triple* $t = (\mathfrak{g}, \sigma, \Omega)$ *is called* **holonomy isotropic** *(***HI***) if* $[\mathfrak{k}, \mathfrak{p}]$ *is an isotropic subspace of* $(\mathfrak{p}, \Omega)$.

Proposition 4.1 *[Bie98] A symplectic triple* $t = (\mathfrak{g}, \sigma, \Omega)$ *is holonomy isotropic if and only if* $[\mathfrak{g}, \mathfrak{g}]$ *is Abelian.*

Definition 4.3 *Let* $t = (\mathfrak{g}, \sigma, \Omega)$ *be HI and consider the extension sequence*

$$0 \to [\mathfrak{g}, \mathfrak{g}] \to \mathfrak{g} \to \mathfrak{a} := \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to 0. \tag{15}$$

The HI triple t is called **split** *if this extension is split.*

Lemma 4.1 *Let* $t = (\mathfrak{g}, \sigma, \Omega)$ *be HI split. Set* $\mathfrak{b} = [\mathfrak{g}, \mathfrak{g}]$ *and denote by* $\rho : \mathfrak{a} \to \mathrm{End}(\mathfrak{b})$ *the splitting homomorphism. Then, realizing* $\mathfrak{g}$ *as the semi-direct product* $\mathfrak{g} = \mathfrak{b} \times_\rho \mathfrak{a}$, *one can assume that* $\mathfrak{a}$ *is stable under* σ.

Proof. For $a \in \mathfrak{a} \subset \mathfrak{g}$, write $a = a_\mathfrak{k} + a_\mathfrak{p}$ according to the decomposition w.r.t. σ. Then for all $a, a' \in \mathfrak{a}$, one has $0 = [a, a'] = [a_\mathfrak{p}, a'_\mathfrak{p}] + b \quad b \in [\mathfrak{k}, \mathfrak{p}]$ since $\mathfrak{k}$ is Abelian. This yields $[a_\mathfrak{p}, a'_\mathfrak{p}] = 0$.

Therefore, for $\mathrm{pr}_\mathfrak{p} : \mathfrak{g} \to \mathfrak{p}$ the projection parallel to $\mathfrak{k}$, the $\mathfrak{p}$-component $\mathrm{pr}_\mathfrak{p}(\mathfrak{a})$ is an Abelian subalgebra of $\mathfrak{g}$ supplementary to $\mathfrak{b}$. A dimension count then yields the lemma. ∎

Lemma 4.2 *Assume that* $t = (\mathfrak{g}, \sigma, \Omega)$ *is HI split, indecomposable and non-flat. Set* $0 \to \mathfrak{b} \to \mathfrak{g} \to \mathfrak{a} \to 0$ *as in Lemma 4.1. Then* $\mathfrak{a}$ *and* $\mathfrak{l} = [\mathfrak{k}, \mathfrak{p}]$ *are in duality. In particular, there exists a* $\mathfrak{k}$*-invariant symplectic structure on* $\mathfrak{p}$ *for which* $\mathfrak{a}$ *is Lagrangian.*

Proof. Set $V := \mathfrak{l}^\perp \cap \mathfrak{a}$ and choose a subspace W of $\mathfrak{a}$ in duality with $\mathfrak{l}$. Then counting dimensions yields $\mathfrak{a} = W \oplus V$. Moreover, in the decomposition $\mathfrak{p} = \mathfrak{l} \oplus W \oplus V$, the matrix of Ω is of the form

$$[\Omega] = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & B \\ 0 & -B' & A \end{pmatrix}.$$

Since $\det[\Omega] \neq 0$, one gets $\det \begin{pmatrix} -I & B \\ 0 & A \end{pmatrix} \neq 0$; hence $\det A \neq 0$ and V is symplectic. Now, $\Omega([\mathfrak{k}, V], \mathfrak{p}) = \Omega(V, \mathfrak{l}) = 0$, hence $[\mathfrak{k}, V] = 0$. Also $[V, \mathfrak{l}] = [V, [\mathfrak{k}, \mathfrak{p}]] = 0$ by Jacobi. Thus V is central, and therefore trivial by indecomposability. ∎

We now assume that $(\mathfrak{g}^1, \sigma^1)$ is the involutive Lie algebra underlying a split HI symplectic triple which is indecomposable and non-flat. We fix Ω^1 such that the HI symplectic triple $t^1 = (\mathfrak{g}^1, \sigma^1, \Omega^1)$ with $0 \to \mathfrak{b}^1 \to \mathfrak{g}^1 \to \mathfrak{a}^1 \to 0$ has $\mathfrak{a}^1$ and $\mathfrak{l}^1 = [\mathfrak{k}^1, \mathfrak{p}^1]$ dual Lagrangian subspaces. We then consider the associated exact triple that we denote by $t = (\mathfrak{g}, \sigma, \Omega)$ (if t^1 is already exact we set $t = t^1$). Observe that, since $\mathfrak{a}^1$ is isotropic, the triple t is elementary solvable with

$$0 \to \mathfrak{b} := [\mathfrak{g}, \mathfrak{g}] \to \mathfrak{g} \to \mathfrak{a} := \mathfrak{a}^1 \to 0.$$

We now follow the same procedure in [Bie00]. The map $\rho : \mathfrak{a} \to \mathrm{End}(\mathfrak{b})$ is injective (because Ω is nondegenerate), so we may identify $\mathfrak{a}$ with its image : $\mathfrak{a} = \rho(\mathfrak{a})$. Let $\Sigma : \mathrm{End}(\mathfrak{b}) \to \mathrm{End}(\mathfrak{b})$ be the automorphism induced by the conjugation with respect to the involution $\sigma|_\mathfrak{b} \in GL(\mathfrak{b})$, i.e. $\Sigma = Ad(\sigma|_\mathfrak{b})$. The automorphism Σ is involutive and preserves the canonical Levi decomposition $\mathrm{End}(\mathfrak{b}) = \mathcal{Z} \oplus sl(\mathfrak{b})$, where $\mathcal{Z}$ denotes the center of $\mathrm{End}(\mathfrak{b})$. Writing the element $a = \rho(a) \in \mathfrak{a}$ as $a = a_Z + a_0$ with respect to this decomposition, one has : $\Sigma(a) = a_Z + \Sigma(a_0) = -a = -a_Z - a_0$, because the endomorphisms a and $\sigma|_\mathfrak{b}$ anticommute. Hence $\Sigma(a_0) = -2a_Z - a_0$ and therefore $a_Z = 0$. So, $\mathfrak{a}$ actually lies in the semisimple part $sl(\mathfrak{b})$. For any $x \in sl(\mathfrak{b})$, we denote by $x = x^S + x^N, \quad x^S, x^N \in sl(\mathfrak{b})$, its abstract Jordan-Chevalley decomposition. Observe that, for $sl(\mathfrak{b}) = sl_+ \oplus sl_-$, the decomposition in (± 1)-Σ- eigenspaces, one has $\mathfrak{a} \subset sl_-$. Also, $\mathfrak{a}_N := \{a^N\}_{a \in \mathfrak{a}}$ is an Abelian subalgebra in sl_- commuting with $\mathfrak{a}$. Set $\mathfrak{a}_S := \{a^S\}_{a \in \mathfrak{a}}$.

Consider the complexification $\mathfrak{b}^c := \mathfrak{b} \otimes \mathbb{C}$ and $\mathbb{C}$-linearly extend the endomorphisms $\{\rho(a)\}_{a\in\mathfrak{a}}$ and σ. Also consider the complex Lie algebra $sl(\mathfrak{b}^c) = sl(\mathfrak{b}) \otimes \mathbb{C}$ and $\mathbb{C}$-linearly extend to $sl(\mathfrak{b}^c)$ the involution Σ.
Let

$$\mathfrak{b}^c =: \bigoplus_{\alpha\in\Phi} \mathfrak{b}_\alpha \tag{16}$$

be the weight space decomposition w.r.t. the action of $\mathfrak{a}_S$. Note that for all α, one has $\mathfrak{a}_N.\mathfrak{b}_\alpha \subset \mathfrak{b}_\alpha$. Moreover, for all $X_\alpha \in \mathfrak{b}_\alpha$ and $a^S \in \mathfrak{a}_S$, one has

$$\sigma(a^S.X_\alpha) = \alpha(a^S)\sigma(X_\alpha) = \sigma a^S \sigma^{-1} \sigma X_\alpha = \Sigma(a^S).\sigma(X_\alpha) = -a^S.\sigma(X_\alpha).$$

Therefore, $-\alpha \in \Phi$ and $\sigma\mathfrak{b}_\alpha = \mathfrak{b}_{-\alpha}$. Note in particular that $\sigma\mathfrak{b}_0 = \mathfrak{b}_0$.

Lemma 4.3 *If the triple t^1 is assumed indecomposable and non-flat, then*

$$\mathfrak{b}_0 = 0.$$

Proof. Assume $0 \in \Phi$. For all $\alpha \in \Phi$, the subspace

$$V_\alpha := \mathfrak{b}_\alpha \oplus \mathfrak{b}_{-\alpha}$$

of $\mathfrak{b}^c$ is stable under σ. In particular, the complexified involutive Lie algebra $(\mathfrak{g}^c := \mathfrak{g}\otimes\mathbb{C}, \sigma)$ can be expressed as $\mathfrak{g}^c = \mathfrak{a}^c \times_\rho \mathfrak{b}^c$ with

$$\mathfrak{b}^c = \bigoplus_{\alpha\in\Phi^+} \mathfrak{b}_\alpha \oplus \mathfrak{b}_0,$$

where the 'positive system' of weights Φ^+ is chosen so that

$$\Phi = \{0\} \cup \Phi^+ \cup (-\Phi^+) \text{ (disjoint union.)}$$

One therefore has the decomposition

$$V_\alpha = \mathfrak{k}_\alpha \oplus \mathfrak{l}_\alpha$$

into $(\pm)$-eigenspaces for σ. Moreover, since $\mathfrak{g}^c = [\mathfrak{g}^c, \mathfrak{g}^c]$ and $[\mathfrak{a}, \mathfrak{b}_\alpha] \subset \mathfrak{b}_\alpha$, one has:

$$\mathfrak{l}_\alpha = [\mathfrak{a}, \mathfrak{k}_\alpha] \text{ and}$$
$$\mathfrak{k}_\alpha = [\mathfrak{a}, \mathfrak{l}_\alpha],$$

for all $\alpha \in \Phi^+ \cup \{0\}$. This implies $\mathfrak{l}_0 = [\mathfrak{a}_N, \mathfrak{k}_0]$ and $\mathfrak{k}_0 = [\mathfrak{a}_N, \mathfrak{l}_0]$. Hence $\mathfrak{l}_0 = [\mathfrak{a}_N, [\mathfrak{a}_N, \mathfrak{l}_0]]$ and an induction yields $\mathfrak{l}_0 = 0$. ■

Corollary 4.1 *A nilpotent HI split symplectic symmetric space is flat.*

Proposition 4.2 *Let $t = (\mathfrak{g} = \mathfrak{b} \times_\rho \mathfrak{a}, \sigma, \Omega = \delta\xi)$ be the exact triple associated with a non-flat indecomposable split symplectic triple t^1. Let Φ be the set of weights associated with the (complex) action of $\mathfrak{a}_S$ on $\mathfrak{b}^c$. Fix a positive system Φ^+ and set*

$$\mathfrak{b}^+ := \bigoplus_{\alpha\in\Phi^+} \mathfrak{b}_\alpha.$$

Then the pair $(\mathfrak{s}^c := \mathfrak{a}^c \times_\rho \mathfrak{b}^+, \Omega|_{\mathfrak{s}^c})$ is a (complex) symplectic Lie algebra.

Proof. By the proof of Lemma 4.3, the restricted projection $\mathfrak{b}^+ \xrightarrow{p} \mathfrak{l}^c : X \mapsto \frac{1}{2}(X - \sigma(X))$ is a linear isomorphism. Moreover, for all $X \in \mathfrak{b}^+, a \in \mathfrak{a}^c$, one has $\Omega(X, a) = \xi[p(c), a]$. The proposition follows from the non-degeneracy of the pairing $\mathfrak{a}^c \times \mathfrak{l}^c \to \mathbb{C}$. ■

Definition 4.4 *Let t be a HI split symplectic triple. Decompose t into a direct sum of indecomposables and a flat factor. Proposition 4.2 then canonically associates to t a (complex) symplectic Lie algebra $\mathfrak{s}^c(t)$, the* **complex symplectic Lie algebra associated with** t.

Combining the results of the present section with Section 3, one gets

Theorem 4.1 *Let $\mathfrak{s}$ be a symplectic Lie algebra associated with the exact triple of a strictly geodesically convex HI split symplectic symmetric space M. Then M is the manifold underlying the connected simply connected Lie group whose Lie algebra is $\mathfrak{s}$. Moreover, Theorem 3.1 defines a left-invariant strict deformation quantization of this symplectic Lie group.*

Remark 4.1
1. Passing to a formal star product by the a stationary phase method, one gets a left-invariant star product on every (group) direct factor of this symplectic Lie group.
2. In the case of the non-Abelian two-dimensional Lie algebra, strict universal deformation formulae associated with non-oscillatory kernels were studied in [BiMae02].

5 Elementary solvable pre-symplectic Lie groups and symmetric spaces

5.1 A class of solvable Lie groups

Definition 5.1 *A Lie group is called* **pre-symplectic** *if it carries a left-invariant Poisson structure.*

One then observes

Proposition 5.1 *Let $(G, \mathbf{w})$ be a pre-symplectic Lie group with neutral element e. Then,*

(i) the orthodual $\mathfrak{s}$ of the radical of $\mathbf{w}_e$,

$$\mathfrak{s} := (rad\ \mathbf{w}_e)^{\perp},$$

is a Lie subalgebra of the Lie algebra $\mathfrak{g}$ of G;

(ii) the symplectic leaves of $\mathbf{w}$ are the left classes of the analytic subgroup S of G whose Lie algebra is $\mathfrak{s}$.

In particular, the Lie group S is symplectic in the sense of Lichnérowicz. Symplectic Lie groups often tend to be solvable [LiMe88].

Definition 5.2 *A symplectic Lie algebra $(\mathfrak{s}, \omega)$ is called* **elementary solvable** *if*

(i) it is a split extension of Abelian Lie algebras $\mathfrak{a}$ and $\mathfrak{d}$:

$$0 \longrightarrow \mathfrak{d} \longrightarrow \mathfrak{s} \longrightarrow \mathfrak{a} \longrightarrow 0; \tag{17}$$

(ii) The cocycle ω is exact.

Proposition 5.2 *Every elementary solvable symplectic Lie algebra is associated with a split HI symplectic symmetric space.*

Proof. Denote by $\rho : \mathfrak{a} \longrightarrow \mathrm{End}(\mathfrak{d})$ the splitting homomorphism and by $\overline{\rho} : \mathfrak{a} \longrightarrow \mathrm{End}(\mathfrak{d})$ the opposite representation: $\overline{\rho}(a)(X) := -\rho(a)(X), \quad X \in \mathfrak{d}$. Set

$$\mathfrak{b} := \mathfrak{d} \oplus \mathfrak{d}$$

and let $\mathfrak{a}$ act on $\mathfrak{b}$ via $\rho \oplus \overline{\rho}$. Define the involution $\sigma_{\mathfrak{b}}$ of $\mathfrak{b}$ by

$$\sigma_{\mathfrak{b}}(X, Y) = (Y, X), \qquad X, Y \in \mathfrak{d}.$$

Set

$$\mathfrak{g} := \mathfrak{b} \times_{\rho \oplus \overline{\rho}} \mathfrak{a}$$

and define the involution σ of $\mathfrak{g}$ as

$$\sigma := \sigma_{\mathfrak{b}} \oplus (-\mathrm{id}_{\mathfrak{a}}).$$

One then observes that $(\mathfrak{g}, \sigma)$ is an involutive Lie algebra. Note that $\mathfrak{k} = \{(X, X)\}_{X \in \mathfrak{d}}$ while $\mathfrak{p} = \{(X, -X)\}_{X \in \mathfrak{d}}$. Let $\eta \in \mathfrak{d}^\star$ be such that $\delta\eta = \omega$ and define $\xi \in \mathfrak{k}^\star$ by

$$\xi(X, X) := \eta(X), \qquad X \in \mathfrak{d}.$$

Extending ξ to $\mathfrak{g}$ by 0 on $\mathfrak{p}$, one defines a symplectic coboundary on $\mathfrak{g}$:

$$\Omega := \delta\xi.$$

The triple $(\mathfrak{g}, \sigma, \Omega)$ then defines the desired elementary solvable symplectic symmetric space. ■

Definition 5.3 *Let $(\mathfrak{s}, \omega)$ be an elementary solvable symplectic Lie algebra. Consider a split Abelian extension of $\mathfrak{s}$:*

$$0 \longrightarrow \mathfrak{q} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{s} \longrightarrow 0. \tag{18}$$

Then $\mathfrak{g}$ has a canonical a Poisson structure $\mathbf{w}_e$ whose associated symplectic Lie algebra is $(\mathfrak{s}, \omega)$ (for $q_i^\star \in \mathfrak{q}^\star$ and $s_i^\star \in \mathfrak{s}^\star$ $(i = 1, 2)$, one has $\mathbf{w}_e((q_1^\star, s_1^\star), (q_2^\star, s_2^\star)) := \omega(^\sharp s_1^\star, {}^\sharp s_2^\star)$ where $\sharp : \mathfrak{s}^\star \to \mathfrak{s}$ denotes the musical isomorphism).
Such a (pre-symplectic) split Abelian extension of an elementary symplectic Lie algebra is called an **elementary solvable pre-symplectic** *Lie algebra.*

6 Universal Deformation Formulae

Let $(\mathfrak{g} = \mathfrak{q} \times \mathfrak{s}, \mathbf{w}_e)$ be an elementary solvable pre-symplectic Lie algebra with associated symplectic Lie algebra $\mathfrak{s}$. Consider the associated connected simply connected Lie groups G, Q and S, and define the chart:

$$Q \times S \to G : (q, s) \mapsto qs. \tag{19}$$

We assume this is a global diffeomorphism.

Theorem 6.1 *Let $(\mathbf{H}^S, \star^S)$ be an associative algebra of functions on S such that $\mathbf{H}^S \subset Fun(S)$ is an invariant subspace w.r.t. the left regular representation of S on $Fun(S)$ and $\star^S$ is a S-left-invariant product on $\mathbf{H}^S$. Then the function space $\mathbf{H} := \{u \in Fun(G) \mid \forall p \in Q : u(q, .) \in \mathbf{H}^S\}$ is an invariant subspace of $Fun(G)$ w.r.t. the left regular representation of G and the formula*

$$u \star v(q, s) := (u(q, .) \star^S v(q, .))(s) \tag{20}$$

defines a G-left-invariant associative product on $\mathbf{H}$.

Proof. The only thing to check is the left-invariance. We begin by writing $\mathfrak{s}$ as $\mathfrak{s} = \mathfrak{b} \times \mathfrak{a}$ as in Section 4. Let A and B be the corresponding subgroups in S. Within the chart (19), the group multiplication reads as follows. First, for $q, q' \in Q, s' \in S$, one has $q.(q', s') = (qq', s')$. Moreover for $s = ab \in S$, one has $s.(q', s') = sq's' = abq's' = aq'a^{-1}abs'$ because $[\mathfrak{b}, \mathfrak{q}] = 0$. Hence $s.(q', s') = (aq'a^{-1}, ss')$. This immediately implies the first assertion . Moreover, one has

$$\begin{aligned}
(L_q^\star u \star L_q^\star v)(q', s') &= ((L_q^\star u(q', .) \star^S (L_q^\star v(q', .))(s') = (u(qq', .) \star^S v(qq', .))(s') = L_q^\star(u \star v)(q', s') \text{ and} \\
(L_s^\star u \star L_s^\star v)(q', s') &= (((L_s^\star u)(q', .) \star^S ((L_s^\star v)(q', .))(s') = L_s^\star(u(aq'a^{-1}, .)) \star^S L_s^\star(v(aq'a^{-1}, .))(s') \\
&= L_s^\star(u(aq'a^{-1}, .) \star^S v(aq'a^{-1}, .))(s') = u \star v(aq'a^{-1}, ss') = L_s^\star(u \star v)(q', s').
\end{aligned}$$

■

Of course, this also holds at the formal level i.e. in the case where $\mathbf{H}^S = C^\infty(S)[[\nu]]$ and $\star^S = \star^S_\nu$ is a left-invariant formal star product. Moreover, one has

Corollary 6.1 *In the formal case, the G-invariant star product on G defined in Theorem 6.1 deforms the usual pointwise product of functions in the direction of the left-invariant Poisson structure $\tilde{w}$, provided $\star^S_\nu$ does so on S.*

7 Crossed, Smash and Co-products

Every algebra, coalgebra, bialgebra, Hopf algebra and vector space is taken over the field $k = \mathbb{R}$ or $\mathbb{C}$. For classical definitions and facts on these subjects, we refer to [Swe69], [Abe80] or more fundamentally to [MiMo65].
To calculate with a coproduct Δ, we use the Sweedler notation [Swe69]:

$$\Delta(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)}. \tag{21}$$

With this notation, we can write coassociativity $\Delta^{(2)}(b) := (\Delta \otimes \mathtt{Id}) \circ \Delta(b) = (\mathtt{Id} \otimes \Delta) \circ \Delta(b)$ in the following way:

$$\Delta^{(2)}(b) := \sum_{(b)} b_{(1)} \otimes b_{(2)} \otimes b_{(3)} = \sum_{(b)(b_1)} b_{(1)(1)} \otimes b_{(1)(2)} \otimes b_{(2)} = \sum_{(b)(b_2)} b_{(1)} \otimes b_{(2)(1)} \otimes b_{(2)(2)}. \tag{22}$$

By associativity one can define $\Delta^{(n)}(b) := \sum_{(b)} b_{(1)} \otimes b_{(2)} \otimes \cdots \otimes b_{(n+1)}$, $n \in \mathbb{N}$. Cocommutativity means

$$\Delta^{(n)}(b) = \sum_{(b)} b_{(\sigma(1))} \otimes b_{(\sigma(2))} \otimes \cdots \otimes b_{(\sigma(n+1))} \ , \ \forall \sigma \in \mathfrak{S}_n. \tag{23}$$

We need two classical definitions [Abe80]:

Definition 7.1 *Let $(B, ., \Delta_B)$ be a bialgebra and $(C, .)$ an algebra. C is a (left-) B-**module algebra** if C is a B-module and if the product on C is a B-module map w.r.t. the canonical B-module structure on $C \otimes C$ defined by the coproduct of B. This means, for all $a, b \in B$ and $f, g \in C$:*

1. $(ab) \rightharpoonup f = a \rightharpoonup (b \rightharpoonup f)$
2. $a \rightharpoonup (f.g) = \sum_{(a)} (a_{(1)} \rightharpoonup f).(a_{(2)} \rightharpoonup g) = \mu_C(\Delta_B(a) \rightharpoonup (f \otimes g))$ *where μ_C is the product on C.*

Definition 7.2 *Let $(B, ., \Delta_B)$ be a bialgebra and (D, Δ_D) a coalgebra. D is a (left-) B-**module coalgebra** if D is a B-module and if the coproduct Δ_D on D is a B-module map w.r.t. the canonical B-module structure on $D \otimes D$ defined by the coproduct of B. This means, for all $a, b \in B$ and $f \in D$:*

1. $(ab) \rightharpoonup f = a \rightharpoonup (b \rightharpoonup f)$
2. $\Delta_D(a \rightharpoonup f) = \Delta_B(a) \rightharpoonup \Delta_D(f) = \sum_{(a)(f)} (a_{(1)} \rightharpoonup f_{(1)}) \otimes (a_{(2)} \rightharpoonup f_{(2)}).$

Combining these two definitions, we get

Definition 7.3 *Let $(B, ., \Delta_B)$ and $(C, ., \Delta_C)$ be two bialgebras. C is a (left-) B-**module bialgebra** if C is both a B-module algebra and a B-module coalgebra.*

Finally we recall the following classical definition [ML75]:

Definition 7.4 *Let $(B, .)$ be a algebra and C a vector space. C is a B-**bimodule** (or left-right B-module) if C is a left B-module and a right B-module with the following compatibility condition:*

$$(a \rightharpoonup f) \leftharpoonup b = a \rightharpoonup (f \leftharpoonup b) \ , \ \forall a, b \in B, \forall f \in C. \tag{24}$$

7.1 Crossed products

The literature on Hopf algebras contains a large collection of what we can call generically semi-direct products, or crossed products. Let us describe some of them. The simplest example of these crossed products is usually called the smash product (see [Swe68, Mol77]):

Definition 7.5 *Let B be a bialgebra and C a B-module algebra. The smash product $C\sharp B$ is the algebra constructed on the vector space $C \otimes B$ where the multiplication is defined by*

$$(f \otimes a) \vec{\star} (g \otimes b) = \sum_{(a)} f\ (a_{(1)} \rightharpoonup g) \otimes a_{(2)} b \tag{25}$$

for $f, g \in C$ and $a, b \in B$.

Remark 7.1

1. Verifying associativity is a direct calculation (cf. 7.6).
2. The smash product can be seen as the algebraic version of what is called "crossed product" in the C^*-algebra literature [DVDZ99, Ped79].
3. (a) Let H and K be groups and let $\tau : K \longrightarrow \mathtt{Aut}(H)$ be an action of K on H. This induces a $k[K]$-module algebra structure on $k[H]$. Then $k[H]\sharp k[K] \cong k[H \rtimes K]$, $H \rtimes K$ denoting the semi-direct product of H by K.

   (b) Similarly, for Lie algebras $\mathfrak{h}$ and $\mathfrak{k}$, a Lie algebra homomorphism $\sigma : \mathfrak{k} \longrightarrow \mathtt{Der}(\mathfrak{h})$ induces a $\mathsf{U}\mathfrak{k}$-module algebra structure on $\mathsf{U}\mathfrak{h}$ ($\mathsf{U}\mathfrak{h}$, resp. $\mathfrak{k}$, denoting the universal envelopping algebra of $\mathfrak{h}$, resp. $\mathfrak{g}$). Then $\mathsf{U}\mathfrak{h}\sharp\mathsf{U}\mathfrak{k} \cong \mathsf{U}(\mathfrak{h} \rtimes \mathfrak{k})$.
4. This product can be seen in the cohomological interpretation of Sweedler [Swe68] as a representative of the trivial class of a theory of extensions. The formula of the smash product can be "twisted" a little more by some 2-cocycle from $B \otimes B$ to C and is called a crossed product.
5. If B and C are bialgebras, C a B-module algebra and B a C-module algebra, with some compatibilities between the two actions, one can write some kind of more "symmetric" formula. S. Majid has called double crossproduct the resulting algebra [Maj90].
6. If C is a bialgebra and B is cocommutative, the natural tensor coproduct on $C \otimes B$ yields a bialgebra structure on $C\sharp B$. If everything is Hopf, $C\sharp B$ can be made Hopf as well [Mol77].
7. By dualizing Definition 7.5, one gets a coalgebra called the cosmash product. Combining smash and cosmash in order to form a bialgebra leads to the notion of bicrossproduct [Maj90].

Assuming B is cocommutative we now introduce a generalization of the smash product.

Definition 7.6 *Let B be a cocommutative bialgebra and C a B-bimodule algebra (i.e. a B-module algebra for both, left and right, B-module structures). The* **L-R-smash product** *$C\natural B$ is the algebra constructed on the vector space $C \otimes B$ where the multiplication is defined by*

$$(f \otimes a) \star (g \otimes b) = \sum_{(a)(b)} (f \leftharpoonup b_{(1)})(a_{(1)} \rightharpoonup g) \otimes a_{(2)} b_{(2)} \tag{26}$$

for $f, g \in C$ and $a, b \in B$.

Remark 7.2 The L-R-smash product $C\natural B$ is an associative algebra. Indeed, this is an easy adaptation of the proof of the associativity for the smash product: we first compute

$$((f \otimes a) \star (g \otimes b)) \star (h \otimes c) = \left(\sum_{(a)(b)} (f \leftharpoonup b_{(1)})(a_{(1)} \rightharpoonup g) \otimes a_{(2)} b_{(2)} \right) \star (h \star c)$$

$$= \sum_{(a)(b)(c)(a_{(2)}b_{(2)})} (((f \leftharpoondown b_{(1)})(a_{(1)} \rightharpoonup g)) \leftharpoondown c_{(1)}) \left((a_{(2)}b_{(2)})_{(1)} \rightharpoonup h\right) \otimes (a_{(2)}b_{(2)})_{(2)}c_{(2)}$$
$$= \sum_{(a)(b)(c)(c_{(1)})(a_{(2)})(b_{(2)})} ((f \leftharpoondown b_{(1)}) \leftharpoondown c_{(1)(1)}) \left((a_{(1)} \rightharpoonup g) \leftharpoondown c_{(1)(2)}\right) \left(a_{(2)(1)}b_{(2)(1)} \rightharpoonup h\right) \otimes a_{(2)(2)}b_{(2)(2)})c_{(2)}$$
$$= \sum_{(a)(b)(c)} (f \leftharpoondown b_{(1)}c_{(1)}) \left((a_{(1)} \rightharpoonup g) \leftharpoondown c_{(2)}\right) \left(a_{(2)}b_{(2)} \rightharpoonup h\right) \otimes a_{(3)}b_{(3)}c_{(3)}.$$

Now we compute

$$(f \otimes a) \star ((g \otimes b) \star (h \otimes c)) = (f \star a) \star \left(\sum_{(b)(c)} (g \leftharpoondown c_{(1)})(b_{(1)} \rightharpoonup h) \otimes b_{(2)}c_{(2)} \right)$$
$$= \sum_{(a)(b)(c)(b_{(2)}c_{(2)})} (f \leftharpoondown (b_{(2)}c_{(2)})_{(1)}) \left(a_{(1)} \rightharpoonup ((g \leftharpoondown c_{(1)})(b_{(1)} \rightharpoonup h))\right) \otimes a_{(2)}(b_{(2)}c_{(2)})_{(2)}$$
$$= \sum_{(a)(a_{(1)})(b)(c)(b_{(2)})(c_{(2)})} (f \leftharpoondown b_{(2)(1)}c_{(2)(1)}) \left(a_{(1)(1)} \rightharpoonup (g \leftharpoondown c_{(1)})\right) \left(a_{(1)(2)} \rightharpoonup (b_{(1)} \rightharpoonup h)\right) \otimes a_{(2)}b_{(2)(2)}c_{(2)(2)}$$
$$= \sum_{(a)(b)(c)} (f \leftharpoondown b_{(2)}c_{(2)}) \left((a_{(1)} \rightharpoonup g) \leftharpoondown c_{(1)}\right) \left(a_{(2)}b_{(1)} \rightharpoonup h\right) \otimes a_{(3)}b_{(3)}c_{(3)}.$$

Since B is cocommutative we have the result.

In the same spirit, one has

Lemma 7.1 *If C is a B-bimodule bialgebra, the natural tensor product coalgebra structure on $C \otimes B$ defines a bialgebra structure to $C \natural B$.*

If C and B are Hopf algebras, $C \natural B$ is a Hopf algebra as well, defining the antipode by

$$\begin{aligned} J_\star(f \otimes a) &= \sum_{(a)} J_B(a_{(1)}) \rightharpoonup J_C(f) \leftharpoondown J_B(a_{(2)}) \otimes J_B(a_{(3)}) \\ &= \sum_{(a)} (1_C \otimes J_B(a_{(1)})) \star (J_C(f) \otimes 1_B) \star (1_C \otimes J_B(a_{(2)})). \end{aligned} \tag{27}$$

Proof. The same type of calculation as above. ∎

Under certain conditions, the L-R-smash product can be decomposed in "smash product like" terms:

Lemma 7.2 *(i) If B is commutative, one has*

$$\begin{aligned} (f \otimes a) \star (g \otimes b) &= \left((f \otimes 1) \overleftarrow{\star} (1 \otimes b)\right) . \left((1 \otimes a) \overrightarrow{\star} (g \otimes 1)\right) \\ &= \left(\sum_{(b)} (f \leftharpoondown b_{(1)}) \otimes b_{(2)} \right) . \left(\sum_{(a)} (a_{(1)} \rightharpoonup g) \otimes a_{(2)} \right). \end{aligned}$$

(ii) If C is commutative, one has

$$\begin{aligned} (f \otimes a) \star (g \otimes b) &= \left((1 \otimes a) \overrightarrow{\star} (g \otimes 1)\right) . \left((f \otimes 1) \overleftarrow{\star} (1 \otimes b)\right) \\ &= \left(\sum_{(a)} (a_{(1)} \rightharpoonup g) \otimes a_{(2)} \right) . \left(\sum_{(b)} (f \leftharpoondown b_{(1)}) \otimes b_{(2)} \right). \end{aligned}$$

The proof is straightforward.
Now by a careful computation, one proves

Proposition 7.1 *Let B be a cocommutative bialgebra, C a B-bimodule algebra and $(C \natural B, \star)$ their L-R-smash product.*
Let S be a linear automorphism of C (as a vector space). We define:

(i) the product $\bullet^S$ *on* C *by*

$$f \bullet^S g = S^{-1}(S(f).S(g))\,; \tag{28}$$

(ii) the left and right B*-module structures,* $\overset{S}{\rightharpoonup}$ *and* $\overset{S}{\leftharpoonup}$*, by*

$$a \overset{S}{\rightharpoonup} f := S^{-1}(a \rightharpoonup S(f)) \quad \text{and} \quad f \overset{S}{\leftharpoonup} a := S^{-1}(S(f) \leftharpoonup a)\,; \tag{29}$$

(iii) the product, $\star^S$*, on* $C \otimes B$ *by*

$$(f \otimes a) \star^S (g \otimes b) = T^{-1}(T(f \otimes a) \star T(g \otimes b)) \tag{30}$$

where $T := S \otimes \mathtt{Id}$.

Then $(C, \bullet^S)$ *is a* B*-bimodule algebra for* $\overset{S}{\rightharpoonup}$ *and* $\overset{S}{\leftharpoonup}$ *and* $\star^S$ *is the L-R-smash product defined by these structures.*
Moreover, if $(C, ., \Delta_C, J_C, \rightharpoonup, \leftharpoonup)$ *is a Hopf algebra and a* B*-module bialgebra, then*

$$C_S := (C, \bullet^S, \Delta_C^S := (S^{-1} \otimes S^{-1}) \circ \Delta_C \circ S, J_C^S := S^{-1} \circ J_C \circ S, \overset{S}{\rightharpoonup}, \overset{S}{\leftharpoonup})$$

is also a Hopf algebra and a B*-module bialgebra. Therefore, by Lemma 7.1,*

$$(C_S \natural B, \star^S, \Delta^S = (23) \circ (\Delta_C^S \otimes \Delta_B), J_\star^S),$$

is a Hopf algebra for Δ^S *the natural tensor product coalgebra structure on* $C_S \natural B$ *(with* $(23) : C \otimes C \otimes B \otimes B \rightarrow C \otimes B \otimes C \otimes B$, $c_1 \otimes c_2 \otimes b_1 \otimes b_2 \mapsto c_1 \otimes b_1 \otimes c_2 \otimes b_2$*) and* $J_\star^S$ *the antipode given on* $C_S \natural B$ *by Lemma 7.1.*
Also, one has

$$\Delta^S = (T^{-1} \otimes T^{-1}) \circ (23) \circ (\Delta_C \otimes \Delta_B) \circ T \quad \text{and} \quad J_\star^S = T^{-1} \circ J_\star \circ T$$

with $T = S \otimes \mathtt{Id}$.

8 Examples in deformation quantization

8.1 A construction on $T^\star(G)$

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $T^\star(G)$ its cotangent bundle. We denote by $\mathsf{U}\mathfrak{g}$, $\mathsf{T}\mathfrak{g}$ and $\mathsf{S}\mathfrak{g}$ respectively the enveloping, tensor and symmetric algebras of $\mathfrak{g}$. Let $\mathsf{Pol}(\mathfrak{g}^\star)$ be the algebra of polynomial functions on $\mathfrak{g}^\star$. We have the usual identifications:

$$\mathcal{C}^\infty(T^\star G) \simeq \mathcal{C}^\infty(G \times \mathfrak{g}^\star) \simeq \mathcal{C}^\infty(G) \hat{\otimes} \mathcal{C}^\infty(\mathfrak{g}^\star) \supset \mathcal{C}^\infty(G) \otimes \mathsf{Pol}(\mathfrak{g}^\star) \simeq \mathcal{C}^\infty(G) \otimes \mathsf{S}\mathfrak{g}.$$

First we deform $\mathsf{S}\mathfrak{g}$ via the "parametrized version", $\mathsf{U}_t\mathfrak{g}$, of $\mathsf{U}\mathfrak{g}$ defined by

$$\mathsf{U}_t\mathfrak{g} = \frac{\mathsf{T}\mathfrak{g}[[t]]}{< XY - YX - t[X,Y]; X, Y \in \mathfrak{g} >}.$$

$\mathsf{U}_t\mathfrak{g}$ is naturally a Hopf algebra with $\Delta(X) = 1 \otimes X + X \otimes 1$, $\epsilon(X) = 0$ and $S(X) = -X$ for $X \in \mathfrak{g}$. For $X \in \mathfrak{g}$, we denote by $\widetilde{X}$ (resp. $\overline{X}$) the left- (resp. right-) invariant vector field on G such that $\widetilde{X}_e = \overline{X}_e = X$. We consider the following $k[[t]]$-bilinear actions of $B = \mathsf{U}_t\mathfrak{g}$ on $C = \mathcal{C}^\infty(G)[[t]]$, for $f \in C$ and $\lambda \in [0,1]$:

(i) $(X \rightharpoonup f)(x) = t(\lambda - 1)\;(\widetilde{X}.f)(x)$,

(ii) $(f \leftharpoonup X)(x) = t\lambda\;(\overline{X}.f)(x)$.

One then has

Lemma 8.1 *C is a B-bimodule algebra w.r.t. the above left and right actions (i) and (ii).*

Definition 8.1 *We denote by $\star_\lambda$ the star product on $(\mathcal{C}^\infty(G) \otimes \mathsf{Pol}(\mathfrak{g}^*))\,[[t]]$ given by the L-R-smash product on $\mathcal{C}^\infty(G)[[t]] \otimes \mathsf{U}_t\mathfrak{g}$ constructed from the bimodule structure of the preceding lemma.*

Proposition 8.1 *For $G = \mathbb{R}^n$, $\star_{\frac{1}{2}}$ is the Moyal star product (Weyl ordered), $\star_0$ is the standard ordered star product and $\star_1$ the anti-standard ordered one. In general $\star_\lambda$ yields the λ-ordered quantization, within the notation of M. Pflaum [Pfl99].*

Proof. Let $\{q_1, q_2, \ldots, q_n\}$ be coordinates on $\mathbb{R}^n$ and $\{p_1, p_2, \ldots, p_n\}$ dual coordinates. For $\mathbf{l} = (l_1, l_2, \ldots, l_n)$ and $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ in $\mathbb{N}^n$, set $|\mathbf{l}| = l_1 + l_2 + \ldots + l_n$ and $\mathbf{p}^\mathbf{r} = p_1^{r_1} p_2^{r_2} \ldots p_n^{r_n}$. Define

$$\frac{\partial^{|\mathbf{l}|+|\mathbf{m}|} u}{\partial \mathbf{q}^\mathbf{m} \partial \mathbf{p}^\mathbf{l}} = \frac{\partial^{|\mathbf{l}|+|\mathbf{m}|} u}{\partial q_1^{m_1} \partial q_2^{m_2} \cdots \partial q_n^{m_n} \partial p_1^{l_1} \partial p_2^{l_2} \cdots \partial p_n^{l_n}}, \quad u \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^{n*}).$$

For $\mathbf{r}! := r_1! r_2! \ldots r_n!$ and $\begin{pmatrix} \mathbf{r} \\ \mathbf{l} \end{pmatrix} := \frac{\mathbf{r}!}{\mathbf{l}! \mathbf{r}-\mathbf{l}!} = \begin{pmatrix} r_1 \\ l_1 \end{pmatrix} \begin{pmatrix} r_2 \\ l_2 \end{pmatrix} \cdots \begin{pmatrix} r_n \\ l_n \end{pmatrix}$, we obtain $\Delta(\mathbf{p}^\mathbf{r}) = \sum_{\mathbf{l}=\mathbf{0}}^{\mathbf{r}} \begin{pmatrix} \mathbf{r} \\ \mathbf{l} \end{pmatrix} \mathbf{p}^\mathbf{l} \otimes \mathbf{p}^{\mathbf{r}-\mathbf{l}}$, for the coproduct in $\mathsf{Pol}(\mathbb{R}^{n*}) = \mathsf{S}\mathbb{R}^n$.

Now $(f \otimes \mathbf{p}^\mathbf{r}) \star_\lambda (g \otimes \mathbf{p}^\mathbf{s}) = \sum_{(\mathbf{p}^\mathbf{r})} \sum_{(\mathbf{p}^\mathbf{s})} \left(f \leftharpoonup \mathbf{p}^\mathbf{s}_{(1)}\right) \left(\mathbf{p}^\mathbf{r}_{(1)} \rightharpoonup g\right) \otimes \mathbf{p}^\mathbf{r}_{(2)} \mathbf{p}^\mathbf{s}_{(2)}$

$$\begin{aligned}
&= \sum_{(\mathbf{p}^\mathbf{r})} \sum_{\mathbf{m}=\mathbf{0}}^{\mathbf{s}} \begin{pmatrix} \mathbf{s} \\ \mathbf{m} \end{pmatrix} (f \leftharpoonup \mathbf{p}^\mathbf{m}) \left(\mathbf{p}^\mathbf{r}_{(1)} \rightharpoonup g\right) \otimes \mathbf{p}^\mathbf{r}_{(2)} \mathbf{p}^{\mathbf{s}-\mathbf{m}} \\
&= \sum_{\mathbf{l}=\mathbf{0}}^{\mathbf{r}} \sum_{\mathbf{m}=\mathbf{0}}^{\mathbf{s}} \begin{pmatrix} \mathbf{r} \\ \mathbf{l} \end{pmatrix} \begin{pmatrix} \mathbf{s} \\ \mathbf{m} \end{pmatrix} (f \leftharpoonup \mathbf{p}^\mathbf{m}) (\mathbf{p}^\mathbf{l} \rightharpoonup g) \otimes \mathbf{p}^{\mathbf{r}-\mathbf{l}} \mathbf{p}^{\mathbf{s}-\mathbf{m}} \\
&= \sum_{\mathbf{l}=\mathbf{0}}^{\mathbf{r}} \sum_{\mathbf{m}=\mathbf{0}}^{\mathbf{s}} t^{|\mathbf{l}|+|\mathbf{m}|} \begin{pmatrix} \mathbf{r} \\ \mathbf{l} \end{pmatrix} \begin{pmatrix} \mathbf{s} \\ \mathbf{m} \end{pmatrix} \lambda^{|\mathbf{m}|} (\lambda - 1)^{|\mathbf{l}|} \frac{\partial^{|\mathbf{m}|} f}{\partial \mathbf{q}^\mathbf{m}} \frac{\partial^{|\mathbf{l}|} g}{\partial \mathbf{q}^\mathbf{l}} \otimes \mathbf{p}^{\mathbf{r}+\mathbf{s}-(\mathbf{l}+\mathbf{m})}
\end{aligned}$$

The formula for the λ-ordered star product $\star_\lambda$ [Pfl99] is

$$u *_\lambda v = \sum_{\mathbf{l}+\mathbf{m} \geqslant \mathbf{0}} \frac{t^{|\mathbf{l}|+|\mathbf{m}|}}{\mathbf{l}! \mathbf{m}!} \lambda^{|\mathbf{m}|} (\lambda - 1)^{|\mathbf{l}|} \frac{\partial^{|\mathbf{l}|+|\mathbf{m}|} u}{\partial \mathbf{q}^\mathbf{m} \partial \mathbf{p}^\mathbf{l}} \frac{\partial^{|\mathbf{l}|+|\mathbf{m}|} v}{\partial \mathbf{q}^\mathbf{l} \partial \mathbf{p}^\mathbf{m}}.$$

Hence $(f(\mathbf{q})\mathbf{p}^\mathbf{r}) \star_\lambda (g(\mathbf{q})\mathbf{p}^\mathbf{s}) = \sum_{\mathbf{l}+\mathbf{m} \geqslant \mathbf{0}} \frac{t^{|\mathbf{l}|+|\mathbf{m}|}}{\mathbf{l}! \mathbf{m}!} \lambda^{|\mathbf{m}|} (\lambda - 1)^{|\mathbf{l}|} \frac{\partial^{|\mathbf{l}|+|\mathbf{m}|} f(\mathbf{q})\mathbf{p}^\mathbf{r}}{\partial \mathbf{q}^\mathbf{m} \partial \mathbf{p}^\mathbf{l}} \frac{\partial^{|\mathbf{l}|+|\mathbf{m}|} g(\mathbf{q})\mathbf{p}^\mathbf{s}}{\partial \mathbf{q}^\mathbf{l} \partial \mathbf{p}^\mathbf{m}}$

$$\begin{aligned}
&= \sum_{\mathbf{l}+\mathbf{m} \geqslant \mathbf{0}} \frac{t^{|\mathbf{l}|+|\mathbf{m}|}}{\mathbf{l}! \mathbf{m}!} \lambda^{|\mathbf{m}|} (\lambda - 1)^{|\mathbf{l}|} \frac{\partial^{|\mathbf{l}|} \mathbf{p}^\mathbf{r}}{\partial \mathbf{p}^\mathbf{l}} \frac{\partial^{|\mathbf{m}|} \mathbf{p}^\mathbf{s}}{\partial \mathbf{p}^\mathbf{m}} \frac{\partial^{|\mathbf{m}|} f}{\partial \mathbf{q}^\mathbf{m}} \frac{\partial^{|\mathbf{l}|} g}{\partial \mathbf{q}^\mathbf{l}} \\
&= \sum_{\mathbf{l}=\mathbf{0}}^{\mathbf{r}} \sum_{\mathbf{m}=\mathbf{0}}^{\mathbf{s}} \frac{t^{|\mathbf{l}|+|\mathbf{m}|}}{\mathbf{l}! \mathbf{m}!} \frac{\mathbf{r}!}{(\mathbf{r}-\mathbf{l})!} \mathbf{p}^{\mathbf{r}-\mathbf{l}} \frac{\mathbf{s}!}{(\mathbf{s}-\mathbf{m})!} \mathbf{p}^{\mathbf{s}-\mathbf{m}} \lambda^{|\mathbf{m}|} (\lambda - 1)^{|\mathbf{l}|} \frac{\partial^{|\mathbf{m}|} f}{\partial \mathbf{q}^\mathbf{m}} \frac{\partial^{|\mathbf{l}|} g}{\partial \mathbf{q}^\mathbf{l}} \\
&= \sum_{\mathbf{l}=\mathbf{0}}^{\mathbf{r}} \sum_{\mathbf{m}=\mathbf{0}}^{\mathbf{s}} t^{|\mathbf{l}|+|\mathbf{m}|} \begin{pmatrix} \mathbf{r} \\ \mathbf{l} \end{pmatrix} \begin{pmatrix} \mathbf{s} \\ \mathbf{m} \end{pmatrix} \lambda^{|\mathbf{m}|} (\lambda - 1)^{|\mathbf{l}|} \frac{\partial^{|\mathbf{m}|} f}{\partial \mathbf{q}^\mathbf{m}} \frac{\partial^{|\mathbf{l}|} g}{\partial \mathbf{q}^\mathbf{l}} \mathbf{p}^{\mathbf{r}+\mathbf{s}-(\mathbf{l}+\mathbf{m})}.
\end{aligned}$$

∎

Remark 8.1 In the general case, it would be interesting to compare our λ-ordered L-R smash product with classical constructions of star products on $T^*(G)$ with Gutt's product as one example [Gut83].

8.2 Hopf structures

We have discussed (see Lemma 7.1) the possibility of having a Hopf structure on $C\natural B$. Let us consider the particular case of $\mathcal{C}^\infty(\mathbb{R}^n)[[t]]\natural \mathrm{U}_t\mathbb{R}^n = \mathcal{C}^\infty(\mathbb{R}^n)[[t]]\natural \mathrm{S}\mathbb{R}^n$ ($\mathbb{R}^n$ is commutative). $\mathrm{S}\mathbb{R}^n$ is endowed with its natural Hopf structure but we also need a Hopf structure on $\mathcal{C}^\infty(\mathbb{R}^n)[[t]] = \mathcal{C}^\infty(\mathbb{R}^n)\otimes\mathbb{R}[[t]]$. We will not use the usual one. Our alternative structure is defined as follows.

Definition 8.2 *We endow $\mathbb{R}[[t]]$ with the usual product, the co-product $\Delta(P)(t_1,t_2) := P(t_1+t_2)$, the co-unit $\epsilon(P) = P(0)$ and the antipode $J(t) = -t$. We consider the Hopf algebra $(\mathcal{C}^\infty(\mathbb{R}^n), ., \mathbf{1}, \Delta_C, \epsilon_C, J_C)$, with pointwise multiplication, the unit $\mathbf{1}$ (the constant function of value 1), the coproduct $\Delta_C(f)(x,y) = f(x+y)$, the co-unit $\epsilon(f) = f(0)$ and the antipode $J_C(f)(x) = f(-x)$. The tensor product of these two Hopf algebras then yields a Hopf algebra denoted by*

$$(\mathcal{C}^\infty(\mathbb{R}^n)[[t]], ., \mathbf{1}, \Delta_t, \epsilon_t, J_t).$$

Note that Δ_t and J_t are not linear in t. We then define, on the L-R smash $\mathcal{C}^\infty(\mathbb{R}^n)[[t]]\natural \mathrm{S}\mathbb{R}^n$,

$$\Delta_\star := (23)\circ(\Delta_t\otimes\Delta_B), \quad \epsilon_\star := \epsilon_t\otimes\epsilon_B \quad \textit{and} \quad J_\star \textit{ as in Lemma 7.1 .}$$

Proposition 8.2 *$(\mathcal{C}^\infty(\mathbb{R}^n)[[t]]\natural \mathrm{S}\mathbb{R}^n, \star_\lambda, \mathbf{1}\otimes 1, \Delta_\star, \epsilon_\star, J_\star)$ is a Hopf algebra.*

Proof. According to Lemma 7.1, the only thing to show is that the actions $\rightharpoonup$ and $\leftharpoonup$ are coalgebra maps. Let $\{X_i; i = 1, ..., n\}$ be a basis of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is commutative, left and right invariant vector fields coincide so it is enough to show that $\rightharpoonup$ is a coalgebra map, that is, for $a\in \mathrm{S}\mathbb{R}^n$ and $\tilde{f}\in\mathcal{C}^\infty(\mathbb{R}^n)[[t]]$,

$$\Delta_t(a\rightharpoonup\tilde{f}) = \Delta_B(a)\rightharpoonup\Delta_t(\tilde{f}).$$

By additivity it suffices to check this for $\tilde{f} = t^n f,\ n\in\mathbb{N}, f\in\mathcal{C}^\infty(\mathbb{R}^n)$, and since

$$\begin{aligned}\Delta_t(X_ia\rightharpoonup\tilde{f}) &= \Delta_t(X_i\rightharpoonup(a\rightharpoonup\tilde{f})) = \Delta_B(X_i)\rightharpoonup\Delta_t(a\rightharpoonup\tilde{f}) = \Delta_B(X_i)\rightharpoonup(\Delta_B(a)\rightharpoonup\Delta_t(\tilde{f}))\\ &= (\Delta(X_i)(\Delta_B(a))\rightharpoonup\Delta_t(\tilde{f}) = \Delta(X_ia)\rightharpoonup\Delta_t(\tilde{f}),\end{aligned}$$

it suffices to check this on the X_i's. We have

$$\begin{aligned}&\Delta_t(X_i\rightharpoonup t^nf)(X,Y) = \Delta_t(t^{n+1}\frac{\partial f}{\partial x_i})(X,Y) = \Delta_t(t^{n+1})\Delta_t(\frac{\partial f}{\partial x_i})(X,Y) = (t_1+t_2)^{n+1}\frac{\partial f}{\partial x_i}(X+Y)\\ &\text{and } (\Delta_b(X_i)\rightharpoonup\Delta_t(t^nf))(X,Y) = (X_i\otimes 1 + 1\otimes X_i)\rightharpoonup((t_1+t_2)^nf(X+Y))\\ =\ & t_1(t_1+t_2)^n(\frac{\partial}{\partial x_i}\otimes 1).f(X+Y) + t_2(t_1+t_2)^n(1\otimes\frac{\partial}{\partial x_i}).f(X+Y)\\ =\ & t_1(t_1+t_2)^n\frac{\partial f}{\partial x_i}(X+Y) + t_2(t_1+t_2)^n\frac{\partial f}{\partial x_i}(X+Y) = (t_1+t_2)^{n+1}\frac{\partial f}{\partial x_i})(X+Y)\end{aligned}$$

■

Remark 8.2 The case $\lambda = \frac{1}{2}$ yields the usual Hopf structure on the enveloping algebra of the Heisenberg Lie algebra.

8.3 Symplectic symmetric spaces

On an elementary solvable symplectic symmetric space we have seen that there exists a global Darboux chart such that $(M,\omega)\simeq(\mathfrak{p} = \mathfrak{l}\oplus\mathfrak{a}, \Omega)$. Thus one has

$$\mathcal{C}^\infty(M)\simeq\mathcal{C}^\infty(\mathfrak{p})\simeq\mathcal{C}^\infty(\mathfrak{l})\hat{\otimes}\mathcal{C}^\infty(\mathfrak{a}) \underset{\mathfrak{a}\simeq\mathfrak{l}^\star}{\simeq} \mathcal{C}^\infty(\mathfrak{l})\hat{\otimes}\mathcal{C}^\infty(\mathfrak{l}^\star)\supset\mathcal{C}^\infty(\mathfrak{l})\otimes Pol(\mathfrak{l}^\star) \underset{\mathfrak{l}\text{ Abelian}}{\simeq} \mathcal{C}^\infty(\mathfrak{l})\otimes\mathrm{U}\mathfrak{l}.$$

WKB-quantization on such a space turns out to be a L-R-smash product. Namely, one has

Proposition 8.3 *The formal version of the invariant WKB-quantization of an elementary solvable symplectic symmetric spaces defined in Theorem 3.1 is a L-R-smash product of the form $\star^S$ (cf. Proposition 7.1).*

Proof. Let $\mathcal{S}(\mathfrak{l})$ denote the Schwartz space of the vector space $\mathfrak{l}$. It is shown in [Bie00] that the map

$$\begin{array}{ccc} \mathcal{S}(\mathfrak{l}) & \xrightarrow{S} & \mathcal{S}(\mathfrak{l}) \\ u & \mapsto & S(u) := F^{-1}\phi^\star_\hbar F(u) \end{array}$$

is a linear injection for all $\hbar \in \mathbb{R}$ (F denotes the partial Fourier transform (9)). An asymptotic expansion in a power series in $\hbar$ then yields a formal equivalence, again denoted by S:

$$S := \mathtt{Id} + o(\hbar) : C^\infty(\mathfrak{l})[[\hbar]] \to C^\infty(\mathfrak{l})[[\hbar]].$$

Carrying the Moyal star product on $(\mathfrak{p} = \mathfrak{l} \times \mathfrak{a}, \Omega)$ by $T := S \otimes \mathtt{Id}$ yields a star product on $M \simeq \mathfrak{p}$ which coincides with the asymptotic expansion of the invariant WKB-product (13). ■

Subsection 8.2 then yields

Corollary 8.1 *The UDF's for elementary pre-symplectic Lie groups constructed in Section 6 admit compatible co-products and antipodes.*

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