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Certain series related to the triple sine function

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Certain Series Related to the Triple Sine Function

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Abstract. We compute special values of Dirichlet series whose coefficients are given by the inverse of certain binomial coefficients via the triple sine function.

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1 Introduction

In the famous proof of the irrationality of $\zeta(3)$ due to Apéry [1], he used the following expression:

$$\zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

We refer to van der Poorten [2] and Koecher [3] for explanations and backgrounds. Also Apéry gave a proof of the irrationality of $\zeta(2)$ by using

$$\zeta(2) = 3\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}}.$$

We can find in [2] and [3] that

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}} = \frac{1}{3} + \frac{2\sqrt{3}\pi}{27},$$

$$\sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}} = \frac{\sqrt{3}\pi}{9},$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}} = \frac{\pi^2}{18},$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4\binom{2n}{n}} = \frac{\pi^4}{3240}.$$

Unfortunately, it remains open to find the expression for

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}}.$$

The purpose of this paper is to investigate this problem via the triple sine function $\mathcal{S}_3(x)$ studied in the previous papers [4, 5, 6, 7, 8]. We refer to the excellent survey of Manin [9]. Here the triple sine function is defined as

$$\mathcal{S}_3(x) = e^{\frac{x^2}{2}} \prod_{n=1}^{\infty} \left(\left(1 - \frac{x^2}{n^2}\right)^{n^2} e^{x^2} \right), \tag{1.1}$$

which is an entire function of order 3. This reminds us of the usual sine function, and actually we define the first sine function $\mathcal{S}_1(x)$ as

$$\mathcal{S}_1(x) = 2\sin(\pi x) = 2\pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right).$$

Our result is

Theorem 1

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}} = 4\pi^2 \log \mathcal{S}_3\left(\frac{1}{6}\right).$$

This is obtained as a special case of

Theorem 2

$$\sum_{n=1}^{\infty} \frac{(2\sin \pi x)^{2n}}{n^3 \binom{2n}{n}} = 4\pi^2 \log \mathcal{S}_3(x)$$

for $-1/2 \leqq x \leqq 1/2$.

From Theorem 1 we moreover show the following identity:

Theorem 3 *It holds that*

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}} = -\frac{4}{3}\zeta(3) + \frac{\sqrt{3}\pi}{3}\left(2L(2,\chi_6) - L(2,\chi_3)\right),$$

where χ_6 *and* χ_3 *are the nontrivial characters modulo* 6 *and* 3*, respectively, and* $L(s,\chi)$ *is the Dirichlet* L*-function.*

2 The Triple Sine Function

In this section we first recall basic properties of the triple sine function (1.1). We find from the definition that

$$\log \mathcal{S}_3(x) = \frac{x^2}{2} + \sum_{n=1}^{\infty} \left(n^2 \log\left(1 - \frac{x^2}{n^2}\right) + x^2 \right)$$

and thus

$$\frac{\mathcal{S}_3'}{\mathcal{S}_3}(x) = \pi x^2 \cot(\pi x), \tag{2.1}$$

where we used the identity

$$\cot(\pi x) = \frac{x}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{x^2 - n^2}.$$

Hence we have

$$\log \mathcal{S}_3(x) = \pi \int_0^x t^2 \cot(\pi t) dt \tag{2.2}$$

as the both sides vanish when $x = 0$. By

$$\cot(\pi t) = i\frac{1 + e^{-2i\pi t}}{1 - e^{-2i\pi t}} = i \left(1 + 2\sum_{m=1}^{\infty} e^{-2\pi imt} \right), \quad (\mathrm{Im}(t) < 0)$$

it holds in $\mathrm{Im}(z) < 0$ that

$$\log \mathcal{S}_3(z) = i \int_0^z \pi t^2 \left(1 + 2\sum_{m=1}^{\infty} e^{-2\pi imt} \right) dt,$$

where the contour is taken in $\mathrm{Im}(t) < 0$. By integrating by parts, we compute

$$\int_0^z t^2 e^{\alpha t} dt = \frac{z^2 e^{\alpha z}}{\alpha} - \frac{2ze^{\alpha z}}{\alpha^2} + \frac{2(e^{\alpha z} - 1)}{\alpha^3}.$$

Therefore the following expression holds for $\mathrm{Im}(z) < 0$.

$$\log \mathcal{S}_3(z) = -\frac{2}{(2\pi i)^2} \sum_{n=1}^{\infty} \left(\frac{e^{-2\pi izn} - 1}{n^3} + 2\pi iz \frac{e^{-2\pi izn}}{n^2} + \frac{(2\pi iz)^2}{2} \frac{e^{-2\pi izn}}{n} \right) + \frac{\pi i}{3} z^3.$$

By taking the real part and by continuity, we have for $x \in \mathbb{R}$ $(0 < x < 1)$

$$\log \mathcal{S}_3(x) = \frac{2}{(2\pi)^2} \sum_{n=1}^{\infty} \left(\frac{\cos(2\pi nx) - 1}{n^3} + \frac{2\pi x \sin(2\pi nx)}{n^2} - \frac{(2\pi x)^2 \cos(2\pi nx)}{2n} \right).$$

We appeal to the formula

$$\log(2\sin\pi x) = -\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n}$$

to get

$$\log\mathcal{S}_3(x) = x^2\log(2\sin\pi x) + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^3} + \frac{x}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n^2} - \frac{1}{2\pi^2}\zeta(3). \qquad (2.3)$$

(We can prove this formula also by showing that both sides are 0 at $x = 1$ and that differentiations of both sides are equal to $\pi x^2\cot(\pi x)$.)

Now, letting $x = \frac{1}{6}$ we obtain,

$$\log\mathcal{S}_3\left(\frac{1}{6}\right) = \frac{1}{2\pi^2}\left(\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi}{3}}{n^3} - \zeta(3)\right) + \frac{1}{6\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{3}}{n^2}.$$

The Dirichlet series with coefficients $\sin\frac{n\pi}{3}$ and $\cos\frac{n\pi}{3}$ are calculated as follows:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{3}}{n^2} &= \frac{\sqrt{3}}{2}\left(\sum_{n\equiv 1,2\,(\mathrm{mod}\,6)}\frac{1}{n^2} - \sum_{n\equiv 4,5\,(\mathrm{mod}\,6)}\frac{1}{n^2}\right)\\
&= \frac{\sqrt{3}}{2}\left(2\left(\sum_{n\equiv 1\,(\mathrm{mod}\,6)}\frac{1}{n^2} - \sum_{n\equiv 5\,(\mathrm{mod}\,6)}\frac{1}{n^2}\right) - \left(\sum_{n\equiv 1,4\,(\mathrm{mod}\,6)}\frac{1}{n^2} - \sum_{n\equiv 2,5\,(\mathrm{mod}\,6)}\frac{1}{n^2}\right)\right)\\
&= \frac{\sqrt{3}}{2}\left(2L(2,\chi_6) - L(2,\chi_3)\right),
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi}{3}}{n^3} &= \frac{1}{2}\left(\sum_{n\equiv 1,5\,(\mathrm{mod}\,6)}\frac{1}{n^3} - \sum_{n\equiv 2,4\,(\mathrm{mod}\,6)}\frac{1}{n^3}\right) + \sum_{n\equiv 0\,(\mathrm{mod}\,6)}\frac{1}{n^3} - \sum_{n\equiv 3\,(\mathrm{mod}\,6)}\frac{1}{n^3}\\
&= \sum_{n\equiv 1,5\,(\mathrm{mod}\,6)}\frac{1}{n^3} - \frac{1}{2}\left(\sum_{n\equiv 1,2,4,5\,(\mathrm{mod}\,6)}\frac{1}{n^3}\right) + \sum_{n\equiv 0\,(\mathrm{mod}\,6)}\frac{1}{n^3} - \sum_{n\equiv 3\,(\mathrm{mod}\,6)}\frac{1}{n^3}\\
&= L(3,\mathbf{1}_6) - \frac{1}{2}L(3,\mathbf{1}_3) + \sum_{n\equiv 0\,(\mathrm{mod}\,6)}\frac{1}{n^3} - \sum_{n\equiv 3\,(\mathrm{mod}\,6)}\frac{1}{n^3}\\
&= (1-2^{-3})(1-3^{-3})\zeta(3) - \frac{1}{2}(1-3^{-3})\zeta(3) + 6^{-3}\zeta(3) - 3^{-3}(1-2^{-3})\zeta(3)\\
&= \frac{1}{3}\zeta(3),
\end{aligned}$$

where $\mathbf{1}_m$ denotes the trivial Dirichlet character modulo m. Thus we have

Theorem 4

$$\log \mathcal{S}_3\left(\frac{1}{6}\right) = \frac{1}{2\pi^2}\left(-\frac{2}{3}\zeta(3) + \frac{\sqrt{3}\pi}{6}\left(2L(2,\chi_6) - L(2,\chi_3)\right)\right).$$

Remark 5 When $x = \frac{1}{2}$ in (2.3), we have

$$\begin{aligned}\log \mathcal{S}_3\left(\frac{1}{2}\right) &= \frac{1}{4}\log 2 + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^3} \\ &= \frac{1}{4}\log 2 - \frac{1}{\pi^2}\sum_{n:odd}\frac{1}{n^3} \\ &= \frac{1}{4}\log 2 - \frac{7}{8\pi^2}\zeta(3).\end{aligned}$$

Hence

$$\zeta(3) = \frac{8\pi^2}{7}\log\left(\mathcal{S}_3\left(\frac{1}{2}\right)^{-1} 2^{\frac{1}{4}}\right)$$

as in [4], [7], [8].

3 Proofs of Theorems

Theorem 1 is a special case of Theorem 2. Theorem 3 is obtained from Theorems 1 and 4. Hence it suffices to prove Theorem 2. It is known by Euler ([2, p.203], [3, p.62]) that

$$\sum_{n=1}^{\infty}\frac{(2\sin(\pi x))^{2n}}{n^2\binom{2n}{n}} = 2\pi^2 x^2$$

for $-1/2 \leqq x \leqq 1/2$. Therefore it follows that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\binom{2n}{n}^{-1} 2^{2n}(\sin(\pi x))^{2n-1}\cos(\pi x) = 2\pi^2 x^2 \cot(\pi x).$$

Integrating both sides, we have

$$\begin{aligned}\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^3}\binom{2n}{n}^{-1} 2^{2n-1}(\sin(\pi x))^{2n} &= 2\pi^2\int_0^x t^2\cot(\pi t)dt \\ &= 2\pi\log \mathcal{S}_3(x)\end{aligned}$$

by the equation (2.2). This completes the proof of Theorem 2.

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