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Convergent star product algebras on “ax+b”

by

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1. Introduction

The notion of convergent star product is generally understood as the data of a one parameter family $\{E_t\}_{t\in I} \subset C^\infty(M)$ of function algebras on a Poisson manifold $(M, \{\ ,\ \})$. On each of them one is given an associative algebra structure $\star_t$ which respect to which the function space E_t is closed. The family of products $\{\star_t\}$ should moreover define in some sense a deformation of the commutative pointwise product of functions in the direction of the Poisson structure $\{\ ,\ \}$.

Stable function algebras for the Weyl-Moyal product have been studied in various contexts. For instance, see:

- [7] for such a study in the framework of tempered distributions on a symplectic vector space;
- [9] for a $C^\star$-algebraic study on $\mathbb{R}^d$-manifolds;
- [8] for non-tempered stable function spaces on $\mathbb{C}^n$.

A special feature of the Weyl-Moyal star product—independently of the functional framework— is its maximal invariance under the group of affine transformations with respect to a *flat* affine connection. This can be rephrased by saying that the Weyl-Moyal quantization is universal with respect to actions of $\mathbb{R}^d$ [9]. A natural question is then the one of defining universal (convergent) deformations for non-Abelian Lie group actions. In the formal framework, this has been investigated in [5]. In, [1, 6], such formulae in the case of $ax + b$ have been studied within the context of Wigner formalism and signal analysis. However, the question of defining an adapted functional framework has not been investigated. In [2, 3], one finds a functional analytic study for solvable Lie group actions and symmetric spaces in the tempered distributions and/or $C^\star$-context.

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It therefore appears quite challenging to investigate the problem of defining non-tempered (e.g. exponential growth) function spaces on such a non-Abelian Lie group which are stable under some left-invariant (convergent) star product. In other words, studying a situation where non-temperedness and non-linear invariance mix. This is what is done in this paper for the particular case of the group $ax+b$. More precisely, we first start by giving a construction of a left-invariant star product on the (symplectic) group manifold underlying the Lie group $ax+b$. This star product is obtained via an equivalence transformation T performed on Moyal's product (which is not left-invariant). The equivalence T involves two ingredients: a partial Laplace transform and a family $\{\phi_{\nu,\gamma} : \mathbb{C} \to \mathbb{C}\}_{\nu,\gamma\in\mathbb{C}}$ of holomorphic maps. For special values of of the parameters ν and γ one refinds the functional calculus studied in [2, 3]. But for other values, one can define stable function algebras constituted by type-S functions. Using a holomorphic presentation of these spaces, one gets non-commutative algebra structures on spaces of entire functions on $\mathbb{C}^2$. Such a space contains functions of exponential growth, one therefore has a non-tempered invariant calculus.
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2. Formal star products on "$ax+b$"

In this section, we briefly recall results appearing in [2, 3]. Let $\mathcal{G}$ denote the Lie algebra of the group of affine transformations of the real line. Formally, one has $\mathcal{G} = \text{span}_{\mathbb{R}}\{A, E\}$ with table $[A, E] = 2E$. Consider the linear map $\lambda : \mathcal{G} \to C^\infty(\mathbb{R}^2) : X \mapsto \lambda_X$ defined by $\lambda_A(a,l) = 2l;\quad \lambda_E(a,l) = e^{-2a}$, where $\mathbb{R}^2 = \{(a,l)\}$. One then checks that the map λ is a homomorphism of Lie algebras when $C^\infty(\mathbb{R}^2)$ is endowed with the symplectic Poisson bracket $\{\, , \,\} := \partial_a \wedge \partial_l$. Moreover, if $\star^M_\nu$ denotes the formal Moyal star product on $C^\infty(\mathbb{R}^2)[[\nu]]$ (i.e $u \star^M_\nu v = u \exp(\nu \overleftarrow{\partial_a} \wedge \overrightarrow{\partial_l}) v \quad u, v \in C^\infty(\mathbb{R}^2)[[\nu]]$), one has $[\lambda_A, \lambda_E]_\nu = 2\nu\{\lambda_A, \lambda_E\}$ (where $[u,v]_\nu := u \star^M_\nu v - v \star^M_\nu u$). In particular, the formula

$$\rho_\nu(X)u := \frac{1}{2\nu}[\lambda_X, u]_\nu \qquad X \in \mathcal{G}, u \in C^\infty(\mathbb{R}^2)[[\nu]]$$

defines a homomorphism of Lie algebras

$$\rho_\nu : \mathcal{G} \to \mathcal{D}er(C^\infty(\mathbb{R}^2)[[\nu]], \star^M_\nu).$$

Explicitly, one has $\rho_\nu(A)u = -\partial_a u;\quad \rho_\nu(E)u = -\frac{e^{-2a}}{\nu}\sinh(\nu\partial_l)u$. Intertwining the representation ρ_ν by a transformation of the type

$$\mathcal{L}(u)(a,z) := \int_{\mathbb{R}} e^{-zl} u(a,l)\, dl,$$

one gets

$$\hat{\rho}_\nu(A)\mathcal{L}(u) := \mathcal{L}(\rho_\nu(A)u) = -\partial_a\mathcal{L}(u);$$
$$\hat{\rho}_\nu(E)\mathcal{L}(u) := \mathcal{L}(\rho_\nu(E)u) = -\tfrac{e^{-2a}}{\nu}\sinh(\nu z)\mathcal{L}(u),$$

where we assumed $u(a,\pm\infty)=0$. Now, set formally

$$\mathcal{Z}_\nu(u)(a,z) := \int_{\mathbb{R}} \exp\left(\gamma\frac{1}{\nu}\sinh(\nu z)l\right) u(a,l)\,dl,$$

and

$$f \bullet_\nu g := \mathcal{Z}_\nu({\mathcal{Z}_\nu}^{-1}f.{\mathcal{Z}_\nu}^{-1}g) \qquad (\gamma\in\mathbb{C}_0).$$

PROPOSITION 2.1. *For all $X\in\mathcal{G}$, $\hat{\rho}_\nu(X)$ is a derivation of the commutative product $\bullet_\nu$.*

In other words, the associative formal product $u\star_\nu v := T^{-1}(Tu\star^M_\nu Tv)$ where $T=\mathcal{L}^{-1}\mathcal{Z}_\nu$ is invariant under the infinitesimal action $\mathcal{G}\to\Gamma T(\mathbb{R}^2): X\mapsto \sharp(d\lambda_X)$ where $\sharp(d\lambda_X)$ denotes the Hamiltonian vector field associated (via the symplectic structure) to the function λ_X. This action of $\mathcal{G}$ turns out to exponentiate as a global simply transitive symplectic action of the group $G=$ "$ax+b$" on $\mathbb{R}^2$, providing an identification of the group manifold underlying G with $\mathbb{R}^2$.

The integral form of the transformation $\mathcal{Z}_\nu$ allows to define specific functions algebras on $\mathbb{R}^2$ (as opposed to power series algebras) stable under the product $\star_\nu$ where the formal Moyal product $\star^M_\nu$ is replaced by its "convergent" version: the Weyl product. The case where $\nu\in i\mathbb{R},\quad \gamma=1,\quad z\in i\mathbb{R}$ has been studied in [3, 2]. In what follows, we are concerned with the general case where $\nu\in i\mathbb{R},\quad \gamma\in U(1),\quad z\in\mathbb{C}$. We end up this section by observing that the intertwiner $T=\mathcal{L}^{-1}\mathcal{Z}_\nu$ can be expressed as

$$T=\mathcal{L}^{-1}\circ(\phi_{\nu,\gamma})^\star\circ\mathcal{L}$$

where we set $\phi_{\nu,\gamma}(z) := \frac{\gamma}{\nu}\sinh(\nu z)$. The map $\phi_{\nu,\gamma}$ will be referred in the sequel as the "twisting map" (see Section 5).

REMARK 2.2. An alternative simple way for obtaining an explicit formula of an invariant star product on $ax+b$ is based on the following observation [1, 6]. The symplectic group manifold underlying $ax+b$ can be seen as an open coadjoint orbit $\mathcal{O}$ in $\mathcal{G}^\star$. Quantizing the Poisson manifold $\mathcal{G}^\star$ via the universal enveloping algebra product and then restricting to $\mathcal{O}$ yields an invariant star product on $\mathcal{O}$ hence a left-invariant one on $ax+b$. It is classical that an oscillatory integral formula for this product can be written down in terms of the Campbell-Baker-Haussdorff function (see [1] for explicit computation: Formulae 5.8 and 5.12). Our product $\star_\nu$ described above is different from the universal enveloping algebra product. Indeed, their invariance diffeomorphism groups do not coincide [2].

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3. Fundamental spaces of type $\mathcal{S}$

In this section, we follow Chapter IV of I.M.Guelfand's book [4]. We will denote by $\mathcal{O}(\mathbb{C}^m)$ the space of holomorphic (entire) functions on $\mathbb{C}^m$.

DEFINITION 3.1. *Let $\alpha, \beta \in \mathbb{R}^m$. The fundamental space $\mathcal{S}_\alpha^\beta(m)$ is defined as the space of holomorphic functions $\varphi \in \mathcal{O}(\mathbb{C}^m)$ such that there exists $a, b \in (\mathbb{R}^+)^m$ and $C > 0$ with*

$$|\varphi(x+iy)| \leq C \exp\left(-a|x|^{\frac{1}{\alpha}} + b|y|^{\frac{1}{1-\beta}}\right),$$

where we adopt the usual notations : $a|x|^e = \sum_j a_j |x_j|^{e_j} \quad (a, x, e \in \mathbb{R}^m)$; $\frac{1}{\alpha} = (\frac{1}{\alpha_1}, ..., \frac{1}{\alpha_m})$; $1 - \beta = (1-\beta_1, ..., 1-\beta_m)$.

Every element $\varphi \in \mathcal{S}_\alpha^\beta(m)$ is entirely determined by its restriction the "real axis" $\varphi(x) \quad x \in \mathbb{R}^m$. We will often identify the space $\mathcal{S}_\alpha^\beta(m)$ with the subspace $\left(\mathcal{S}_\alpha^\beta(m)\right)_x$ of $C^\infty(\mathbb{R}^m)$ constituted by the restrictions. In order to consider only non-trivial spaces, we will assume $\alpha + \beta \geq 1; \quad \alpha > 0; \quad \beta > 0$. We will denote by $\mathcal{F}(u)(\xi)$ the Fourier transform of the function $u \in L^1(\mathbb{R}^m)$:

$$\mathcal{F}(u)(\xi) := \int_{\mathbb{R}^m} e^{i\xi.x} u(x)\, dx,$$

where $\xi.x$ denotes the canonical dot product on $\mathbb{R}^m$. For even $m = 2n$, we will denote by J the endomorphism of $\mathbb{R}^{2n}$ defined by the matrix

$$[J] := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. We denote by ω the bilinear symplectic structure on $\mathbb{R}^{2n}$ defined by $\omega(x, y) := x.Jy$.

DEFINITION 3.2. *We define the symplectic Fourier transform of the function $u \in L^1(\mathbb{R}^{2n})$ as*

$$S\mathcal{F}(u)(y) := \int_{\mathbb{R}^{2n}} e^{i\omega(x,y)} u(x)\, dx \qquad (y \in \mathbb{R}^{2n}).$$

Equivalently, one has $S\mathcal{F} = J^* \circ \mathcal{F}$, which yields (see [4])

LEMMA 3.3. *One has*

$$S\mathcal{F}(\mathcal{S}_\alpha^\beta(2n)) = \mathcal{S}_{\sigma(\beta)}^{\sigma(\alpha)}(2n),$$

where for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^{2n}$ $(\alpha_i \in \mathbb{R}^n)$, we set $\sigma(\alpha) := (\alpha_2, \alpha_1)$.

DEFINITION 3.4. *For $u, v \in L^1(\mathbb{R}^{2n})$, one defines their twisted convolution by*

$$u \times_q v(x) := \int_{\mathbb{R}^{2n}} e^{iq\omega(x,y)} u(y)\, v(x-y)\, dy \qquad (q \in \mathbb{R}_0).$$

LEMMA 3.5. *One has*

$$\mathcal{S}_\alpha^\beta(m).\mathcal{S}_{\alpha'}^{\beta'}(m) \subset \mathcal{S}_{\min(\alpha,\alpha')}^{\max(\beta,\beta')}(m);$$

where we set $\max(\beta,\beta') := (\max(\beta_1,\beta_1'),..., \max(\beta_m,\beta_m'))$.

Proof. For simplicity, we assume $m = 1$. Let $\varphi \in \mathcal{S}_\alpha^\beta$ and $\varphi' \in \mathcal{S}_{\alpha'}^{\beta'}$. Then, for $z = x + iy$, one has

$$|\varphi\varphi'(z)| \leq CC' \exp\left(-a|x|^{\frac{1}{\alpha}} + b|y|^{\frac{1}{1-\beta}} - a'|x|^{\frac{1}{\alpha'}} + b'|y|^{\frac{1}{1-\beta'}}\right)$$

which is lower than $C" \exp\left(-a"|x|^{\max(\frac{1}{\alpha},\frac{1}{\alpha'})} + b"|y|^{\max(\frac{1}{1-\beta},\frac{1}{1-\beta'})}\right)$ for some $C", a", b"$. But, $\max(\frac{1}{\alpha},\frac{1}{\alpha'}) = \frac{1}{\min(\alpha,\alpha')}$ and $\max(\frac{1}{1-\beta},\frac{1}{1-\beta'}) = \frac{1}{1-\max(\beta,\beta')}$. ■

LEMMA 3.6. *Let $u, v \in \mathcal{S}_\alpha^\beta(2n)$. Then $u \times_q v \in \mathcal{S}_{\sigma\beta}^{\sigma\alpha}(2n)$.*

Proof. Changing the variables following $y \mapsto -y$, one gets $u \times_q v = \left[d_q^* \mathcal{SF}(\tilde{u}\alpha_x v)\right]^{-}$ where $\tilde{u}(x) := u(-x)$, $(\alpha_x v)(y) := v(y + x)$ and where d_q denotes the dilation in $\mathbb{R}^m : x \mapsto qx \quad (q \in \mathbb{R})$. Hence $u \times_q v \in \mathcal{SF}\left(\mathcal{S}_{\min(\alpha,\alpha)}^{\max(\beta,\beta)}(2n)\right) = \mathcal{S}_{\sigma(\beta)}^{\sigma(\alpha)}(2n)$. ■

DEFINITION 3.7. *(see e.g. [7]) The Weyl product between u and v in $L^1(\mathbb{R}^{2n})$ is defined by*

$$u \star_q^W v := \mathcal{SF}\left[\mathcal{SF}(u) \times_q \mathcal{SF}(v)\right].$$

PROPOSITION 3.8. *[8] Let $u, v \in \mathcal{S}_\alpha^\beta(2n)$, Then $u \star_q^W v \in \mathcal{S}_{\sigma(\beta)}^{\sigma(\alpha)}(2n)$. In particular, the space $\mathcal{S}_\alpha^{\sigma(\alpha)}(2n)$ is stable under the Weyl product.*

REMARK 3.9. The space $\mathcal{S}_\alpha^{\sigma(\alpha)}(2n)$ is stable under the pointwise multiplication as well.

4. Laplace Transformation

In this section we follow L. Schwartz' book [10]. We adopt the following notations. We denote by $\mathcal{D}$ the space of compactly supported smooth functions on $\mathbb{R}$ endowed with the topology of test functions. We denote by $\mathcal{D}'$ the space of distributions on $\mathbb{R}$. Also, if Ω is an open domain in $\mathbb{C}$, we set $\mathcal{O}(\Omega)$ for the space of holomorphic functions on Ω.

DEFINITION 4.1. *Let Γ be an open interval in $\mathbb{R}$. The fundamental space $\mathcal{S}'_\Gamma$ is defined as the space of distributions $T \in \mathcal{D}'$ such that for all $\xi \in \Gamma$, the distribution $\exp(-\xi.x)T_x$ is tempered. We denote by $\mathcal{O}_\Gamma$ the space of holomorphic functions $F \in \mathcal{O}(\Gamma + i\mathbb{R})$ such that for all compact set $K \subset \Gamma$, the restriction $F|_{K+i\mathbb{R}}(\xi + i\eta)$ is bounded by a polynomial in η.*

PROPOSITION 4.2. *One defines the Laplace transform of an element $T \in \mathcal{S}'_\Gamma$ as the Fourier transform of $e^{-\xi.x}T_x$*

$$\mathcal{L}(T)(\xi + i\eta) := (\mathcal{F}_x [\exp(-\xi.x)T_x]) (\eta).$$

Then, setting $z = \xi + i\eta \in \Gamma + i\mathbb{R}$, one has a linear isomorphism

$$\mathcal{L} : \mathcal{S}'_\Gamma \to \mathcal{O}_\Gamma.$$

REMARK 4.3. Provided the following integrals make sense, one has

$$\mathcal{L}(T)(z) = \int_{\mathbb{R}} e^{-zx}T_x \, dx \text{ and } \mathcal{L}^{-1}F(x) = \int_{c+i\mathbb{R}} e^{zx}F(z)dz,$$

where c is any element of Γ. Indeed, if $F = \mathcal{L}T$ one has for all $c \in \Gamma$:

$$\int_{c+i\mathbb{R}} e^{zx}F(z)dz = e^{xc}\int_{\mathbb{R}} F(c+it)e^{ixt}dt =$$
$$= e^{xc}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{it(x-y)}\left(T_y e^{-yc}\right) dy\, dt = e^{xc}\delta_x \left(T_y e^{-yc}\right) = T_x.$$

Now we assume $m = 1$ and set $\mathcal{S}^\beta_\alpha(1) =: \mathcal{S}^\beta_\alpha$.

LEMMA 4.4. *For all $\xi \in \mathbb{R}$ the function $e^{-\xi x}$ is a multiplier in $\mathcal{S}^\beta_\alpha$ ($\alpha < 1$).*

Proof. Let $\varphi \in \mathcal{S}^\beta_\alpha$. Then the function $f(z) := e^{-\xi z}\varphi(z)$ is entire and one has

$$|f(z)| \leq C \exp\left(-a|x|^{\frac{1}{\alpha}} + b|y|^{\frac{1}{1-\beta}}\right) e^{|\xi||x|}$$

which is lower than $C' \exp\left(-a'|x|^{\frac{1}{\alpha}} + b|y|^{\frac{1}{1-\beta}}\right)$ as soon as $\alpha < 1$. ∎

In particular, one has $\mathcal{S}^\beta_\alpha \subset \mathcal{S}'_\mathbb{R} = \cap_\Gamma \mathcal{S}'_\Gamma$ as soon as $\alpha < 1$.

PROPOSITION 4.5. *The Laplace transformation yields a linear isomorphism*

$$\mathcal{L} : \left(\mathcal{S}^\beta_\alpha\right)_x \to J^*\mathcal{S}^\alpha_\beta$$

where $J : \mathbb{C} \to \mathbb{C}$ *denotes the multiplication by* $i = \sqrt{-1}$.

Proof. Let $\varphi \in \mathcal{S}^\beta_\alpha$. Then $(J^*\mathcal{L}(\varphi))(x+iy) = \int_\mathbb{R} e^{ixt-yt}\varphi(t)\,dt$. Hence $(J^*\mathcal{L}(\varphi))|_\mathbb{R}(x) = (\mathcal{F}(\varphi))(x) \in \mathcal{S}^\alpha_\beta$ ([4]). Thus $\mathcal{L}(\varphi) \in J^*\mathcal{S}^\alpha_\beta$. Now let $f \in J^*\mathcal{S}^\alpha_\beta$. One has $|f(x+iy)| \leq C\exp\left(-a|y|^{\frac{1}{\alpha}} + b|x|^{\frac{1}{1-\beta}}\right)$ which guarantees that $f \in \mathcal{O}_\mathbb{R}$. Therefore $\mathcal{L}^{-1}f(x) = e^{xc}\int_\mathbb{R} e^{ixt}f(c+it)\,dt \quad (c \in \mathbb{R})$. Choosing $c = 0$, one gets $\mathcal{L}^{-1}f = \mathcal{F}J^*f \in \mathcal{S}^\beta_\alpha$. One therefore has an isomorphism $\mathcal{L}^{-1} : J^*\mathcal{S}^\alpha_\beta \to \mathcal{S}^\beta_\alpha$. ∎

5. Twisting maps

DEFINITION 5.1. *Let* $q \in \mathbb{R}$ *and* $\theta \in [0, 2\pi[$. *We define the twisting map* $\phi_{q,\theta} : \mathbb{C} \to \mathbb{C}$ *by*

$$\begin{cases} \phi_{q,\theta}(z) = \frac{e^{i\theta}}{q}\sinh(iqz) & \text{if } q \neq 0 \\ \phi_{0,\theta}(z) = z. \end{cases}$$

LEMMA 5.2. *The twisting map* $\phi_{q,0}$ $(q \neq 0)$ *establishes a biholomorphic diffeomorphism*

$$\phi_{q,0} : S_q := \{z \in \mathbb{C} \mid |Re(z)| < \frac{\pi}{2q}\} \to \mathbb{C} - \{\pm i[\frac{1}{q}, \infty[\}.$$

Proof. In coordinates $z = x + iy$, the twisting map is

$$\phi_{q,0}(x,y) = \left(\frac{1}{q}\sinh(qy)\cos(qx), \frac{1}{q}\cosh(qy)\sin(qx)\right).$$

In particular, the imaginary axis $x = 0$ is sent onto the real axis $y = 0$, while the image of the vertical line $x = \pm\frac{\pi}{2q}$ under $\phi_{q,0}$ is the half imaginary line $\pm i[\frac{1}{q}, \infty[$. At last, for $c \in]0, \frac{\pi}{2}[$ the image of the vertical line $x = \pm\frac{c}{q}$ is the branch of hyperbola $\{-(\frac{qx}{\cos c})^2 + (\frac{qy}{\sin c})^2 = 1\} \cap \{\pm y > 0\}$. For $\epsilon \in]0, \frac{\pi}{2}[$, one therefore gets a holomorphic diffeomorphism between the strip $]-\frac{\epsilon}{q}, \frac{\epsilon}{q}[+i\mathbb{R}$ and the region $\{-(\frac{qx}{\cos\epsilon})^2 + (\frac{qy}{\sin\epsilon})^2 < 1\}$. ∎

As explained in Section 2, we are interested in considering transformations of the type $\mathcal{L}^{-1}\circ\phi^*_{q,\theta}\circ\mathcal{L}$ or $\mathcal{L}^{-1}\circ(\phi^{-1}_{q,\theta})^*\circ\mathcal{L}$. Let F be a function defined on some domain Ω of $\mathbb{C}$. In order to define $\mathcal{L}^{-1}(\phi^{-1}_{q,\theta})^*F$, we want $(\phi^{-1}_{q,\theta})^*F\in\mathcal{O}_\Gamma$. In particular we want $(\phi^{-1}_{q,\theta})^*F$ to be defined on vertical lines. This imposes to θ to be an integral multiple of $\frac{\pi}{2}$.

PROPOSITION 5.3. *Let $\alpha,\beta\in]0,1[$ be such that $\alpha+\beta\geq 1$.*

(i) For all open interval I of positive numbers ($I\subset\mathbb{R}^+_0$), one has the injection $(\phi^{-1}_{q,0})^*: J^*\mathcal{S}^\alpha_\beta\to\mathcal{O}_I\cap\mathcal{O}_{-I}$.

(ii) For all open interval $I\subset]-\frac{\pi}{2q},\frac{\pi}{2q}[$, one has the injection $(\phi^{-1}_{q,\frac{\pi}{2}})^*: J^*\mathcal{S}^\alpha_\beta\to\mathcal{O}_I$.

Proof. Let $F\in J^*\mathcal{S}^\alpha_\beta$ and set for simplicity $\phi=\sinh(iz)$. Let $I\subset]0,\infty[$ and K be a compact set in I. Then ϕ^{-1} is well defined on the strip $K+i\mathbb{R}$ and $\phi^{-1}(K+i\mathbb{R})\subset]-\frac{\pi}{2},\frac{\pi}{2}[+i\mathbb{R}$ (cf. Lemma 5.2). Now, consider the integral $I_K:=\int_{K+i\mathbb{R}}|(\phi^{-1})^*F(z)|\,|dz\wedge\overline{dz}|$. If $I_K<\infty$ for all K, then $(\phi^{-1})^*F\in\mathcal{O}_I$. Changing the variables $z=\phi(w)$, one gets $I_K=\int_{\phi^{-1}(K+i\mathbb{R})}|F(w)|\,|\mathrm{Jac}_\phi(w)|\,|dw\wedge\overline{dw}|$ where Jac_ϕ is the Jacobian determinant of ϕ. A computation yields $\mathrm{Jac}_\phi(w)=\cos^2(x)+\sinh^2(y)$ $(w=x+iy)$ and, since $\phi^{-1}(K+i\mathbb{R})\subset]-\frac{\pi}{2},\frac{\pi}{2}[+i\mathbb{R}$, one gets $I_K\leq\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\infty}^{\infty}\exp\left(-a|y|^{\frac{1}{\beta}}+b|x|^{\frac{1}{1-\alpha}}\right)\left(\cos^2(x)+\cosh^2(y)\right)dy\,dx$ which is lower than $2b\exp\left[\left(\frac{\pi}{2}\right)^{\frac{1}{1-\alpha}}\right]\int_{-\infty}^{\infty}\exp\left(-a|y|^{\frac{1}{\beta}}\right)(1+\cosh^2(y))\,dy$, thus finite as soon as $\beta<1$. The exact same argument works for $I\subset]-\frac{\pi}{2},0[$, hence $(\phi^{-1})^*F\in\mathcal{O}_I\cap\mathcal{O}_{-I}$. A similar argument yields item (ii). ■

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6. Twisted Weyl product

6.1. The product formula

Let us consider the fundamental space $\mathcal{S}^{\sigma(\alpha_1,\alpha_2)}_{(\alpha_1,\alpha_2)}(2) \quad (\alpha_1,\alpha_2)\in\mathbb{R}^2$ (cf. Section 3). Let $\varphi\in\mathcal{S}^{\sigma(\alpha_1,\alpha_2)}_{(\alpha_1,\alpha_2)}(2)$ and consider the partial function $\varphi_{x_1}: x\mapsto\varphi(x_1,x)$. For all $x_1\in\mathbb{R}$, the function φ_{x_1} belongs to $\mathcal{S}^{\alpha_1}_{\alpha_2}(1) =: \mathcal{S}^{\alpha_1}_{\alpha_2}$. Therefore provided some restrictions on (α_1,α_2), the function $\mathcal{L}^{-1}(\phi^{-1}_{q,k\frac{\pi}{2}})^*\mathcal{L}(\varphi_{x_1})$ $(k=0,1)$ is well defined as an element of $\mathcal{S}'_I$ (cf. Proposition 5.3).

DEFINITION 6.1. *We define the linear map*

$$\mathcal{S}^{\sigma(\alpha_1,\alpha_2)}_{(\alpha_1,\alpha_2)}(2) \xrightarrow{\tau^{(k)}_q} C^\infty(\mathbb{R}^2) \quad (k=0,1)$$

by

$$\tau^{(k)}_q := \mathrm{id}_{x_1}\otimes\left(\mathcal{L}^{-1}\circ(\phi^{-1}_{q,k\frac{\pi}{2}})^*\circ\mathcal{L}\right)_{x_2} \qquad (x_1,x_2)\in\mathbb{R}^2.$$

We denote by $E^{(k)}_{(\alpha_1,\alpha_2)}$ its range in $C^\infty(\mathbb{R}^2)$. The inverse map $\mathrm{id}_{x_1}\otimes\left(\mathcal{L}^{-1}\circ\phi^*_{q,k\frac{\pi}{2}}\circ\mathcal{L}\right)_{x_2}|_{E^{(k)}_{(\alpha_1,\alpha_2)}}$ will be denoted by $T^{(k)}_q$. It yields a linear isomorphism $T^{(k)}_q: E^{(k)}_{(\alpha_1,\alpha_2)}\to\mathcal{S}^{\sigma(\alpha_1,\alpha_2)}_{(\alpha_1,\alpha_2)}(2)$.

PROPOSITION 6.2. *The formula*

$$u\star^{(k)}_q v := \tau^{(k)}_q\left(T^{(k)}_q u\star^W_q T^{(k)}_q v\right)$$

defines an associative $\mathbb{R}$-algebra structure $\star^{(k)}_q$ on $E^{(k)}_{(\alpha_1,\alpha_2)}$.

Proof. One has $T^{(k)}_q u, T^{(k)}_q v\in\mathcal{S}^{\sigma(\alpha_1,\alpha_2)}_{(\alpha_1,\alpha_2)}(2)$ provided $u,v\in E^{(k)}_{(\alpha_1,\alpha_2)}$. We know that $\mathcal{S}^{\sigma(\alpha_1,\alpha_2)}_{(\alpha_1,\alpha_2)}(2)$ is stable under Weyl's product $\star^W_q$ (cf. Proposition 3.8). Hence $u\star^{(k)}_q v$ is well defined as an element of $E^{(k)}_{(\alpha_1,\alpha_2)}$. The associativity is obvious since $\star^{(k)}_q$ is nothing else than the transportation of Weyl's product via the isomorphism $T^{(k)}_q: E^{(k)}_{(\alpha_1,\alpha_2)}\to\mathcal{S}^{\sigma(\alpha_1,\alpha_2)}_{(\alpha_1,\alpha_2)}(2)$. ■

DEFINITION 6.3. *The product $\star^{(k)}_q$ on $E^{(k)}_{(\alpha_1,\alpha_2)}$ will be referred as the twisted Weyl product.*

6.2. OBSERVABLES OF EXPONENTIAL TYPE

Given a convergent star product, an important question is the one of existence of non-tempered observables in the domain of the star product. This question has been studied for the case of the Moyal-Weyl product in [8].

Let $A = (\alpha_1, \alpha_2) \in]0, 1[^2$ with $\alpha_1 + \alpha_2 \geq 1$. Set $\mathcal{S}_A := \mathcal{S}^{\sigma(\alpha_1,\alpha_2)}_{(\alpha_1,\alpha_2)}(2)$. Viewing $\mathcal{S}_A = (\mathcal{S}_A)_x \quad (x \in \mathbb{R}^2)$ as a subspace of $C^\infty(\mathbb{R}^2)$, we consider the following alternate presentation of $\mathcal{S}_A$. Consider the sequence of maps

$$\begin{array}{rcccl} (\mathcal{S}_A)_x & \to & \mathcal{O}(\mathbb{C}^2) & \to & C^\infty(\mathbb{R}^2) \\ f(x_1, x_2) & \mapsto & f(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2) & \mapsto & f(iy_1, x_2). \end{array}$$

The function $\hat{f}(y_1, x_2) := f(iy_1, x_2)$ determines completely f. So that we have an injection $(\mathcal{S}_A)_x \to C^\infty(\mathbb{R}^2) : f \mapsto \hat{f}$. Remark that the space $\hat{\mathcal{S}_A}$ contains elements of exponential growth. For example, one has $f = e^{-(z_1^2 + z_2^2)} \in \mathcal{S}_{(\frac{1}{2},\frac{1}{2})}$, which yields $|\hat{f}(y_1, x_2)| = e^{y_1^2 - x_2^2}$.

We denote by $\widehat{\star^W_q}$ the product on $\hat{\mathcal{S}_A}$ obtained by transporting Weyl's product on $(\mathcal{S}_A)_x$ via $f \mapsto \hat{f}$. Observe that $\hat{\mathcal{S}_A}$ is still stable under the pointwise multiplication whose $\widehat{\star^W_q}$ is a non-commutative deformation of. Observe also that for every $\psi \in \hat{\mathcal{S}_A}$ and $y_1 \in \mathbb{R}$ the partial function $x_2 \mapsto \psi(y_1, x_2)$ is in $\mathcal{S}^{\alpha_1}_{\alpha_2}$. Therefore the transformations $\tau^{(k)}_q$ and $T^{(k)}_q$ are well defined on $\hat{\mathcal{S}_A}$.

PROPOSITION 6.4. *Set* $\widehat{E^{(k)}_{(\alpha_1,\alpha_2)}} := \tau^{(k)}_q(\hat{\mathcal{S}_A})$. *Then for all* $a, b \in \widehat{E^{(k)}_{(\alpha_1,\alpha_2)}}$, *the formula*

$$a \,\widehat{\star^{(k)}_q}\, b := \tau^{(k)}_q \left(T^{(k)}_q a \,\widehat{\star^W_q}\, T^{(k)}_q b \right)$$

defines an associative $\mathbb{R}$*-algebra structure on* $\widehat{E^{(k)}_{(\alpha_1,\alpha_2)}}$. *The space* $\widehat{E^{(k)}_{(\alpha_1,\alpha_2)}}$ *contains elements of exponential growth.*

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