SEMI-LÉVY PROCESSES, SEMI-SELFSIMILAR ADDITIVE PROCESSES, AND SEMI-STATIONARY ORNSTEIN–UHLENBECK TYPE PROCESSES

MAKOTO MAEJIMA AND KEN-ITI SATO

Abstract. For any $d \times d$ matrix Q of which all eigenvalues have positive real parts, the correspondence of Q-semi-selfsimilar additive processes on $\mathbb{R}^d$, periodic $\mathbb{R}^d$-valued independently scattered random measures over $\mathbb{R}$ (semi-Lévy processes), and semi-stationary Ornstein–Uhlenbeck type processes on $\mathbb{R}^d$ with drift coefficient $-Qx$ is established via stochastic integrals and Lamperti transformations. This gives representations of Q-semi-selfdecomposable distributions on $\mathbb{R}^d$. Results related to Q-selfdecomposable distributions are derived as consequences. Applications and examples in semi-stable Lévy processes are given.

1. Introduction

The works of Wolfe [26], Jurek and Vervaat [6], Sato and Yamazato [20], [21], Sato [16], and Jeanblanc, Pitman, and Yor [4] combined show that the following three classes have one-one correspondence with each other — the class of selfsimilar additive processes, the class of stationary Ornstein–Uhlenbeck type processes, and the class of Lévy processes with finite log-moment. The last one can be considered as the class of homogeneous independently scattered random measures with finite log-moment. The correspondence is given by Lamperti transformations and stochastic integrals. At the same time each of these classes yields a representation of a selfdecomposable distribution. The aim of this paper is to give an extension of this correspondence to certain wider classes and to discuss Ornstein–Uhlenbeck type processes in a wide sense.

Before going to statement of main results, let us give some definitions.

Let $\mathbf{M}_d$ be the class of $d \times d$ real matrices and $\mathbf{M}_d^+$ the class of $Q \in \mathbf{M}_d$ all of whose eigenvalues have positive (> 0) real parts. Let I be the identity matrix and $a^Q = \sum_{n=0}^{\infty} (n!)^{-1} (\log a)^n Q^n \in \mathbf{M}_d$ for $a > 0$ and $Q \in \mathbf{M}_d$. Sometimes we also use the class $\mathbf{M}_{l \times d}$ of $l \times d$ real matrices. Denote the transpose of $F \in \mathbf{M}_{l \times d}$ by F'. Let $\mathcal{L}(X)$ be the distribution of a random element X. When $\mathcal{L}(X) = \mathcal{L}(Y)$ for two random

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elements X and Y, we write $X \stackrel{\mathrm{d}}{=} Y$. For two stochastic processes $\{X_t\}$ and $\{Y_t\}$, $\{X_t\} \stackrel{\mathrm{d}}{=} \{Y_t\}$ means that they have an identical distribution as infinite-dimensional random elements, that is, have an identical system of finite-dimensional distributions, while $X_t \stackrel{\mathrm{d}}{=} Y_t$ means that X_t and Y_t are identically distributed for a fixed t. The characteristic function of a distribution μ on $\mathbb{R}^d$ is denoted by $\widehat{\mu}(z)$, $z \in \mathbb{R}^d$. For an interval J, $\mathcal{B}_J$ is the class of Borel sets in J and $\mathcal{B}_J^0$ is the class of Borel sets whose closures in the relative topology on J are compact.

A process $\{X_t\colon t \geqslant 0\}$ on $\mathbb{R}^d$ continuous in probability with independent increments, with cadlag paths a. s., and $X_0 = 0$ a. s. is called an *additive process* (see [17]). It is called a *Lévy process* if, in addition, $X_{t+u} - X_{s+u} \stackrel{\mathrm{d}}{=} X_t - X_s$ for all nonnegative t, s, u. We call an additive process satisfying the relation that $X_{t+p} - X_{s+p} \stackrel{\mathrm{d}}{=} X_t - X_s$ with a fixed $p > 0$ a *semi-Lévy process* with *period* p. An additive process is said to have finite log-moment if $E \log^+ |X_t| < \infty$ for all t. Here $\log^+ a = 0 \vee \log a$ for $0 \leqslant a < \infty$. An additive process is said to be *natural* if the location parameter γ_t in the generating triplets (A_t, ν_t, γ_t) is locally of bounded variation in t (see [18]). An additive process is natural if and only if it is a semimartingale. All Lévy processes are natural.

Let $Q \in \mathbf{M}_d^+$. A process $\{X_t\colon t \geqslant 0\}$ on $\mathbb{R}^d$ is called *Q-selfsimilar* if $\{X_{at}\} \stackrel{\mathrm{d}}{=} \{a^Q X_t\}$ for all $a > 0$. Note that the value of X_t (an element of $\mathbb{R}^d$) is always considered as a column vector. If the assumption that $\{X_{at}\} \stackrel{\mathrm{d}}{=} \{a^Q X_t\}$ is made only for a fixed $a > 1$, the process is called *Q-semi-selfsimilar* with *epoch* a. Especially cI-selfsimilar and cI-semi-selfsimilar processes with $c > 0$ are called c-selfsimilar (see [15], [17]) and c-semi-selfsimilar (see [10], [17]), respectively. In this case, H is usually used instead of c.

Let $Q \in \mathbf{M}_d^+$. A distribution μ on $\mathbb{R}^d$ satisfying

$$\widehat{\mu}(z) = \widehat{\mu}(b^{Q'} z)\widehat{\rho}_b(z) \tag{1.1}$$

with some (automatically infinitely divisible) distribution ρ_b for every $b \in (0,1)$ is called *Q-selfdecomposable*. Thus, for any $c > 0$, the Q-selfdecomposability and the cQ-selfdecomposability are equivalent. Following [11], we introduce, with $b \in (0,1)$ fixed, the class $L_0(b, Q)$ of distributions μ on $\mathbb{R}^d$ satisfying (1.1) with some infinitely divisible distributions ρ_b. Distributions in $L_0(b, Q)$ are called *(b, Q)-decomposable*. Distributions (b, Q)-decomposable with some b are called Q-semi-selfdecomposable. Usually I-selfdecomposable distributions are called *selfdecomposable* and I-semi-selfdecomposable distributions are called *semi-selfdecomposable* (see [9], [17]).

We will extend the correspondence mentioned at the beginning to the "semi" case and simultaneously to the Q-case from the usual non-"semi" I-case. We use the notion of $\mathbb{R}^d$-valued independently scattered random measure (i. s. r. m.) over an interval J, $\{M(B)\colon B \in \mathcal{B}_J^0\}$, introduced in the case $d = 1$ by Urbanik and Woyczynski [25] and Rajput and Rosinski [14]. Precise definition of this notion will be given in Section 3. For a class of $\mathbf{M}_{l\times d}$-valued functions $F(s)$ including all locally bounded measurable functions, we can define $\int_B F(s)M(ds)$ for $B \in \mathcal{B}_J^0$. A natural additive process $\{X_t\colon t \geqslant 0\}$ on $\mathbb{R}^d$ induces a unique $\mathbb{R}^d$-valued i. s. r. m. over $[0,\infty)$, $\{M(B)\colon B \in \mathcal{B}_{[0,\infty)}^0\}$, satisfying $M((s,t]) = X_t - X_s$ a. s. for $0 \leqslant s < t < \infty$. Any $\mathbb{R}^d$-valued i. s. r. m. over $[0,\infty)$ is obtained in this way. In this case $\int_B F(s)M(ds)$ is written as $\int_B F(s)dX_s$. When J is an interval infinite to the left, we define $\int_{-\infty}^t F(u)M(du)$ to be the limit in probability of $\int_{(s,t]} F(u)M(du)$ as $s \downarrow -\infty$ whenever this limit exists.

Given an $\mathbb{R}^d$-valued nonrandom cadlag function Y_s of $s \in \mathbb{R}$ and a matrix $Q \in \mathbf{M}_d$, consider the equation

$$(1.2) \qquad Z_{s_2} - Z_{s_1} = Y_{s_2} - Y_{s_1} - Q\int_{s_1}^{s_2} Z_u du \quad \text{for } s_1 < s_2$$

for an unknown nonrandom cadlag function Z_s of $s \in \mathbb{R}$. When the condition $Z_{s_0} = \xi$ is imposed, (1.2) has a unique solution. When $\{\Lambda(B)\colon B \in \mathcal{B}_{\mathbb{R}}^0\}$ is an $\mathbb{R}^d$-valued i. s. r. m. over $\mathbb{R}$, we call the equation

$$(1.3) \qquad Z_{s_2} - Z_{s_1} = \Lambda((s_1, s_2]) - Q\int_{s_1}^{s_2} Z_u du$$

Langevin equation based on Λ and Q. By a solution $\{Z_s\colon s \in \mathbb{R}\}$ of (1.3) we mean a cadlag process which satisfies (1.3) a. s. for every s_1, s_2 with $s_1 < s_2$. Any solution of (1.3) is called an *Ornstein–Uhlenbeck type* (OU type) process generated by Λ and Q. If we introduce a cadlag process $\{Y_s\colon s \in \mathbb{R}\}$ such that $Y_{s_2} - Y_{s_1} = \Lambda((s_1, s_2])$, then (1.3) is a random version of (1.2), and thus (1.3) is solved pathwise. A process $\{Z_s\}$ satisfying $\{Z_{s+u}\} \stackrel{\mathrm{d}}{=} \{Z_s\}$ for all u is called *stationary*. A process $\{Z_s\}$ satisfying $\{Z_{s+p}\} \stackrel{\mathrm{d}}{=} \{Z_s\}$ for a fixed $p > 0$ is called *semi-stationary* (or periodically stationary) with *period* p. We say that $\{\Lambda(B)\colon B \in \mathcal{B}_{\mathbb{R}}^0\}$ has finite log-moment if $E\log^+|\Lambda(B)| < \infty$ for all $B \in \mathcal{B}_{\mathbb{R}}^0$.

The following three theorems are our main results.

Theorem 1.1. *Let $Q \in \mathbf{M}_d^+$, $a > 1$, and $p = \log a$. Let $\{X_t\colon t \geqslant 0\}$ be an arbitrary Q-semi-selfsimilar natural additive process on $\mathbb{R}^d$ with epoch a. Define*

$$(1.4) \qquad Z_s = e^{-sQ}X_{e^s} \quad \textit{for } s \in \mathbb{R}$$

and

$$\Lambda(B) = \int_{\exp B} t^{-Q} dX_t \quad \textit{for } B \in \mathcal{B}^0_{\mathbb{R}}\ , \tag{1.5}$$

where $\exp B = \{t = e^s \colon s \in B\}$. *Then* $\{\Lambda(B) \colon B \in \mathcal{B}^0_{\mathbb{R}}\}$ *is an* $\mathbb{R}^d$*-valued i. s. r. m. periodic with period* p *and having finite log-moment. The process* $\{X_t \colon t \geqslant 0\}$ *is expressed by* Λ *as*

$$X_t = \int_{-\infty}^{\log t} e^{sQ} \Lambda(ds) \quad \textit{for all } t > 0,\ \textit{a. s.} \tag{1.6}$$

The process $\{Z_s \colon s \in \mathbb{R}\}$ *is the unique semi-stationary OU type process with period* p *generated by* Λ *and* Q. *It is expressible as*

$$Z_s = e^{-sQ} \int_{-\infty}^{s} e^{uQ} \Lambda(du) \quad \textit{for all } s \in \mathbb{R},\ \textit{a. s.} \tag{1.7}$$

Theorem 1.2. *Let* $Q \in \mathbf{M}_d^+$, $p > 0$, *and* $a = e^p$. *Let* $\{\Lambda(B) \colon B \in \mathcal{B}^0_{\mathbb{R}}\}$ *be an arbitrary* $\mathbb{R}^d$*-valued i. s. r. m. periodic with period* p *and having finite log-moment. A semi-stationary OU type process with period* p, $\{Z_s \colon s \in \mathbb{R}\}$, *generated by* Λ *and* Q *exists and is unique. Define*

$$\begin{cases} X_t = t^Q Z_{\log t} & \textit{for } t > 0 \\ X_0 = 0. \end{cases} \tag{1.8}$$

Then $\{X_t \colon t \geqslant 0\}$ *is a natural* Q*-semi-selfsimilar additive process on* $\mathbb{R}^d$ *with epoch* a *and* $\{Z_s\}$ *and* $\{\Lambda(B)\}$ *are recovered from* $\{X_t\}$ *in the form of* (1.4) *and* (1.5).

Theorem 1.3. *Let* $Q \in \mathbf{M}_d^+$ *and* $a > 1$. *A distribution* μ *on* $\mathbb{R}^d$ *is expressible as* $\mu = \mathcal{L}(X_1) = \mathcal{L}(Z_0)$ *by the processes* $\{X_t \colon t \geqslant 0\}$ *and* $\{Z_s \colon s \in \mathbb{R}\}$ *in Theorem* 1.1 *or* 1.2 *if and only if it is* (a^{-1}, Q)*-decomposable.*

The associated filtrations of the processes and the random measure in Theorem 1.1 or 1.2 satisfy the following:

$$\sigma(X_t \colon t \in [0, e^s]) = \sigma(Z_u \colon u \in (-\infty, s]) = \sigma(\Lambda(B) \colon B \in \mathcal{B}^0_{(-\infty, s]})\ .$$

Relations (1.4) and (1.8) between $\{X_t \colon t \geqslant 0\}$ and $\{Z_s \colon s \in \mathbb{R}\}$ are generalization of the Lamperti transformation between selfsimilar processes and stationary processes introduced by Lamperti [7]. In the case of symmetric stable processes on $\mathbb{R}$, this transformation was already recognized by Doob [3] p. 368. Between semi-selfsimilar and semi-stationary processes it was given in [10].

By Theorems 1.1–1.3, not only selfdecomposable but also semi-selfdecomposable and (b, Q)-decomposable distributions have now been connected with the three classes

— the class of $\{X_t\}$, the class of $\{Z_s\}$, and the class of $\{\Lambda(B)\}$. For the forms of the Lévy measures of these distributions, see [11] and [17]. Semi-selfdecomposable distributions are expected to have wider flexibility in modeling such as in [1] than stable, semi-stable, and selfdecomposable distributions.

Applications of Theorems 1.1–1.3 to further analysis of selfsimilar additive processes will be given in a forthcoming paper [19].

Organization of this paper is as follows. Section 2 gives basic facts on semi-Lévy processes. We need some results on random measures and stochastic integrals, which are introduced in Section 3. We study in Section 4 solutions of the Langevin equations on $\mathbb{R}^d$ based on $\mathbb{R}^d$-valued i. s. r. m. and matrices Q. The notion of mildness at $-\infty$ of solutions of the Langevin equations is introduced and the existence condition for solutions mild at $-\infty$ is given. Stationary and semi-stationary solutions are mild at $-\infty$. The existence condition is more analyzed in the case of periodic i. s. r. m. Using these results, we give in Section 5 proofs of Theorems 1.1–1.3. Formulation of results in the non-"semi" Q-case is given in Section 6 as consequences of Theorems 1.1–1.3. Finally Section 7 contains some results related to (b, Q, a)-semi-stable distributions and (b, Q, a)-semi-stable Lévy processes and some examples appearing in the study of diffusion processes in random environments.

Our notation and definitions follow [17]. But, in addition to the notation introduced above, we use the following: $ID = ID(\mathbb{R}^d)$ is the class of all infinitely divisible distributions on $\mathbb{R}^d$; $\mathcal{B}_0(\mathbb{R}^d)$ is the class of all Borel sets B on $\mathbb{R}^d$ satisfying $\inf_{x\in B} |x| > 0$; δ_a is the distribution concentrated at a point a ; p-lim stands for limit in probability; the norm of $Q \in \mathbf{M}_d$ is $\|Q\| = \sup_{|x|\leqslant 1} |Qx|$; $\mathrm{tr}A$ is the trace of a symmetric nonnegative-definite matrix A. A set or a function is called measurable if it is Borel measurable. For a distribution μ, μ^n is the n-fold convolution of μ. If the characteristic function $\widehat{\mu}(z)$ of a distribution μ on $\mathbb{R}^d$ vanishes nowhere, then there is a unique continuous function $f(z)$ on $\mathbb{R}^d$ such that $f(0) = 0$ and $\widehat{\mu}(z) = e^{f(z)}$. This $f(z)$ is called the distinguished logarithm of $\widehat{\mu}(z)$ and written as $f(z) = \log \widehat{\mu}(z)$ ([17] p. 33).

Let $c(x)$ be a real-valued bounded measurable function satisfying

$$c(x) = \begin{cases} 1 + o(|x|) & \text{as } |x| \to 0, \\ O(|x|^{-1}) & \text{as } |x| \to \infty. \end{cases} \tag{1.9}$$

The generating triplet $(A, \nu, \gamma)_c$ of an infinitely divisible distribution μ on $\mathbb{R}^d$ is defined by the formula

$$\log \widehat{\mu}(z) = -\frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^d} g_c(z,x)\nu(dx) + i\langle z, \gamma\rangle \ ,$$

where $g_c(z,x) = e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle c(x)$; A is the Gaussian covariance matrix and ν is the Lévy measure of μ ; γ is the location parameter, which depends on the choice of $c(x)$. Standard choice of $c(x)$ is $1_{\{|x|\leqslant 1\}}(x)$ or $(1+|x|^2)^{-1}$. In this paper we use

$$c(x) = (1+|x|^2)^{-1}, \tag{1.10}$$

unless otherwise indicated. Thus we write (A, ν, γ) for $(A, \nu, \gamma)_c$ with $c(x)$ of (1.10).

2. Semi-Lévy processes

We will consider periodic independently scattered random measures. Semi-Lévy processes are their counterparts in stochastic processes. We gather basic properties and examples of semi-Lévy processes.

Proposition 2.1. *Let $\{X_t \colon t \geqslant 0\}$ be an additive process on $\mathbb{R}^d$ and let $\mathcal{L}(X_t) = \mu_t$. If it is a semi-Lévy process with period p, then*

$$\mu_{np+t} = \mu_p^n * \mu_t \tag{2.1}$$

for any $n \in \mathbb{Z}_+$ and $t \geqslant 0$. If (2.1) holds for all $n \in \mathbb{Z}_+$ and $t \in [0,p)$, then $\{X_t\}$ is a semi-Lévy process with period p.

Proof. Let $\mu_{s,t} = \mathcal{L}(X_t - X_s)$ for $0 \leqslant s \leqslant t$. Then $\mu_{0,t} = \mu_t$ and $\mu_{s,t} * \mu_{t,u} = \mu_{s,u}$ for $s \leqslant t \leqslant u$, If $\{X_t\}$ is semi-Lévy with period p, then $\mu_{s,t} = \mu_{s+p,t+p}$, $\mu_{2p} = \mu_p * \mu_{p,2p} = \mu_p^2$, and by induction $\mu_{np} = \mu_p^n$ for any $n \in \mathbb{Z}_+$, which implies $\mu_{np+t} = \mu_{np} * \mu_{np,np+t} = \mu_p^n * \mu_{0,t} = \mu_p^n * \mu_t$.

Conversely, assume that (2.1) holds for all $n \in \mathbb{Z}_+$ and $t \in [0,p)$. Then (2.1) holds for any $n \in \mathbb{Z}_+$ and $t \geqslant 0$. Indeed, if $kp \leqslant t < (k+1)p$, then $\mu_{np+t} = \mu_p^{n+k} * \mu_{t-kp} = \mu_p^n * \mu_p^k * \mu_{t-kp} = \mu_p^n * \mu_t$. Hence for $0 \leqslant s \leqslant t$, $\mu_{p+s} * \mu_{p+s,p+t} = \mu_{p+t} = \mu_p * \mu_t = \mu_p * \mu_s * \mu_{s,t} = \mu_{p+s} * \mu_{s,t}$. Since $\widehat{\mu}_{p+s}(z) \neq 0$, we get $\mu_{p+s,p+t} = \mu_{s,t}$. Hence $\{X_t\}$ is semi-Lévy with period p. $\square$

Proposition 2.2. *If $\{X_t \colon t \geqslant 0\}$ is a semi-Lévy process with period p, then $\mu_t = \mathcal{L}(X_t)$ satisfies the following: $\mu_0 = \delta_0$, $\mu_t \in ID(\mathbb{R}^d)$, μ_t is continuous as a function of t, and, for any choice of $0 \leqslant s \leqslant t$, there is $\mu_{s,t} \in ID(\mathbb{R}^d)$ such that $\mu_t = \mu_s * \mu_{s,t}$.*

In the converse direction, if a class of probability measures $\{\mu_t\colon t \in [0,p]\}$ on $\mathbb{R}^d$ satisfying these conditions for $t \in [0,p]$ is given, then there exists, uniquely in law, a semi-Lévy process $\{X_t\colon t \geqslant 0\}$ with period p such that $\mu_t = \mathcal{L}(X_t)$ for $t \in [0,p]$.

Proof. In order to see the first half, it is enough to choose $\mu_{s,t} = \mathcal{L}(X_t - X_s)$. Let us prove the second half. If $t > p$, then choose an integer k such that $kp \leqslant t < (k+1)p$ and define $\mu_t = \mu_p^k * \mu_{t-kp}$. Then $\mu_t \in ID$ and, for any $0 \leqslant s \leqslant t$, there is $\mu_{s,t} \in ID$ such that $\mu_t = \mu_s * \mu_{s,t}$. We can prove that $\mu_{s,t} * \mu_{t,u} = \mu_{s,u}$ for $0 \leqslant s \leqslant t \leqslant u$, using $\widehat{\mu}_s(z) \neq 0$. Further, $\mu_0 = \delta_0$, $\mu_{s,t} \to \delta_0$ as $s \uparrow t$, and $\mu_{s,t} \to \delta_0$ as $t \downarrow s$. Thus, by [17] Theorem 9.7, there is an additive process in law $\{X_t\colon t \geqslant 0\}$ such that $\mathcal{L}(X_t) = \mu_t$ and $\mathcal{L}(X_t - X_s) = \mu_{s,t}$. Then, by [17] Theorem 11.5, there is an additive process modification. It is a semi-Lévy process by Proposition 2.1. The uniqueness in law is obvious. □

Proposition 2.3. *Let $\{X_t\colon t \geqslant 0\}$ be a semi-Lévy process with period p. Then, $E\log^+|X_t| < \infty$ for all $t \geqslant 0$ if and only if $E\log^+|X_p| < \infty$.*

Proof. The Lévy measure ν_t of X_t is increasing in t and $\nu_{np} = n\nu_p$. By Theorem 25.3 of [17], $E\log^+|X_t|$ is finite if and only if $\int \log^+|x|\nu_t(dx) < \infty$. Hence the assertion follows. □

Remark. There is a semi-Lévy process $\{X_t\}$ with period p such that $E\log^+|X_t|$ is finite for $t < p$ but infinite for $t = p$. For example, let $d = 1$, $p = 1$, and

$$\nu_t(dx) = 1_{(0,t/(1-t))}(x)\, x^{-1}(\log(2+x))^{-2}dx \quad \text{for } 0 < t \leqslant 1$$

and construct $\{X_t\}$, using Proposition 2.2.

Example 2.4. Let $\{X_t\colon t \geqslant 0\}$ be a semi-Lévy process on $\mathbb{R}^d$ with period p. Denote

$$\widetilde{\mu}_t = \mathcal{L}(X_p - X_{p-t}) \quad \text{for } 0 \leqslant t \leqslant p. \tag{2.2}$$

Then there exists, uniquely in law, a semi-Lévy process $\{\widetilde{X}_t\colon t \geqslant 0\}$ with period p such that $\mathcal{L}(\widetilde{X}_t) = \widetilde{\mu}_t$ for $0 \leqslant t \leqslant p$. Indeed, we can apply Proposition 2.2.

We give a new characterization of strictly stable Lévy processes.

Proposition 2.5. *If $\{X_t\colon t \geqslant 0\}$ is a selfsimilar, semi-Lévy process, then it is a strictly stable Lévy process.* (*The converse is trivial.*)

Proof. Suppose that $\{X_t\colon t \geqslant 0\}$ is a c-selfsimilar, semi-Lévy process with period $p > 0$, where $c > 0$. Let $\mu_t = \mathcal{L}(X_t)$. Then $\widehat{\mu}_{np}(z) = \widehat{\mu}_p(z)^n$ for $n \in \mathbb{Z}_+$. On the other hand, by c-selfsimilarity, $\widehat{\mu}_{np}(z) = \widehat{\mu}_p(n^c z)$. Hence $\widehat{\mu}_p(z)^n = \widehat{\mu}_p(n^c z)$. This means that μ_p is strictly stable with index $\alpha = 1/c$. Again by c-selfsimilarity, we have $\widehat{\mu}_1(z) = \widehat{\mu}_{(1/p)p}(z) = \widehat{\mu}_p((1/p)^c z)$, and thus μ_1 is also strictly stable with index $\alpha = 1/c$. Therefore $\widehat{\mu}_1(z)^a = \widehat{\mu}_1(a^{1/\alpha} z)$ for any $a > 0$. Once again by c-selfsimilarity, $\widehat{\mu}_a(z) = \widehat{\mu}_1(a^c z)$. These imply

$$\widehat{\mu}_a(z) = \widehat{\mu}_1(z)^a \quad \text{for all } a > 0. \tag{2.3}$$

Let $\mu_{t,t+h} = \mathcal{L}(X_{t+h} - X_t)$. Since $\widehat{\mu}_t(z) \neq 0$, we have

$$\widehat{\mu}_{t,t+h}(z) = \widehat{\mu}_{t+h}(z)\widehat{\mu}_t(z)^{-1} = \widehat{\mu}_1(z)^{t+h}\widehat{\mu}_1(z)^{-t} = \widehat{\mu}_1(z)^h = \widehat{\mu}_h(z)$$

by using (2.3). Thus $\{X_t\}$ has stationary increments, and hence $\{X_t\}$ is a Lévy process such that $\mathcal{L}(X_t)$ is strictly stable. □

3. Independently scattered random measures and stochastic integrals

We define $\mathbb{R}^d$-valued independently scattered random measures.

Definition 3.1. Let J be an interval in $\mathbb{R}$. A family $\{M(B) \colon B \in \mathcal{B}_J^0\}$ of $\mathbb{R}^d$-valued random variables is called $\mathbb{R}^d$-valued *independently scattered random measure* (i. s. r. m.) over J, if the following three conditions are satisfied:

(1) for any sequence $B_1, B_2, \ldots$ of disjoint sets in $\mathcal{B}_J^0$ with $B = \bigcup_{n=1}^{\infty} B_n \in \mathcal{B}_J^0$, $M(B) = \sum_{n=1}^{\infty} M(B_n)$ a. s., where the series is convergent a. s.,

(2) for any finite sequence $B_1, \ldots, B_n$ of disjoint sets in $\mathcal{B}_J^0$, $M(B_1), \ldots, M(B_n)$ are independent,

(3) $M(\{a\}) = 0$ a. s. for every one-point set $\{a\} \subset J$.

If, in addition,

(4) $M(B) \stackrel{\mathrm{d}}{=} M(B+a)$ for every $B \in \mathcal{B}_J^0$ and $a \in \mathbb{R}$ satisfying $B + a \in \mathcal{B}_J^0$,

then $\{M(B) \colon B \in \mathcal{B}_J^0\}$ is called *homogeneous* i. s. r. m. Let $p > 0$. If $\{M(B) \colon B \in \mathcal{B}_J^0\}$ satisfies (1), (2), (3), and

(5) $M(B) \stackrel{\mathrm{d}}{=} M(B+p)$ for every $B \in \mathcal{B}_J^0$ satisfying $B + p \in \mathcal{B}_J^0$,

then it is called a *periodic* i. s. r. m. with period p or, for short, p-periodic i. s. r. m.

The definitions of additive, Lévy, and semi-Lévy processes and those in law are extended to the case where the parameter set is $J = [0, t_0)$ or $[0, t_0]$. Under these names we always retain the condition that $X_0 = 0$ a. s.

The notions and the results in the rest of this section are extensions of a part of Sections 2–4 of [18], where only the case $J = [0, \infty)$ is studied. We omit most of proofs of our assertions, but they can be given either in a way similar to [18] or by reduction to the case $J = [0, \infty)$.

Definition 3.2. Let $J = [0, t_0)$, $[0, t_0]$, or $[0, \infty)$ and let $\{X_t : t \in J\}$ be a J-parameter additive process in law on $\mathbb{R}^d$. As $\mu_t = \mathcal{L}(X_t) \in ID$, the triplet of μ_t is denoted by (A_t, ν_t, γ_t). We say that $\{X_t : t \in J\}$ is *natural* if γ_t is locally of bounded variation on J, that is, of bounded variation on each $[t_1, t_2]$ satisfying $[t_1, t_2] \subset J$.

Remark. The definition above does not depend on the choice of $c(x)$ satisfying (1.9). Any J-parameter Lévy process in law on $\mathbb{R}^d$ is natural, since $\gamma_t = (t/t_1)\gamma_{t_1}$, where t_1 is positive and fixed in J. When $\{X_t : t \in J\}$ is a J-parameter semi-Lévy process on $\mathbb{R}^d$ with period p, it is natural if and only if γ_t is of bounded variation on $[0, p]$. Thus, using Proposition 2.2 or its analogue for $J = [0, t_0)$ or $[0, t_0]$, it is easy to see that there exist non-natural J-parameter semi-Lévy processes on $\mathbb{R}^d$. We are assuming $p < t_0$ if $J = [0, t_0)$ or $[0, t_0]$.

The connection between i. s. r. m. and additive processes in law is described in the following two propositions.

Proposition 3.3. *Let $J = [0, t_0)$, $[0, t_0]$, or $[0, \infty)$. If $\{M(B) : B \in \mathcal{B}_J^0\}$ is an $\mathbb{R}^d$-valued i. s. r. m. over J, then $\{X_t : t \in J\}$ defined by*

$$X_t = M([0, t]) \quad a.\,s. \quad \text{for } t \in J \tag{3.1}$$

is a J-parameter natural additive process in law on $\mathbb{R}^d$. Conversely, if $\{X_t : t \in J\}$ is a J-parameter natural additive process in law on $\mathbb{R}^d$, then there is a unique (in the a. s. sense) $\mathbb{R}^d$-valued i. s. r. m. $\{M(B) : B \in \mathcal{B}_J^0\}$ over J such that (3.1) holds. In this correspondence, $\{X_t\}$ is a Lévy process in law if and only if $\{M(B)\}$ is homogeneous; $\{X_t\}$ is a natural semi-Lévy process in law with period p if and only if $\{M(B)\}$ is p-periodic.

Proposition 3.4. *Let J be an interval in $\mathbb{R}$.*

(i) *Suppose that $\{M(B) : B \in \mathcal{B}_J^0\}$ is an $\mathbb{R}^d$-valued i. s. r. m. over J. Define, for each $s \in J$ and $t \geqslant 0$ with $s + t \in J$,*

$$X_t^{(s)} = M((s, s+t]) \quad a.\,s., \tag{3.2}$$

where we understand that $(s, s] = \emptyset$. Then,

(1) *for each* $s \in J$, $\{X_t^{(s)} \colon t \in (J-s) \cap [0,\infty)\}$ *is a* $((J-s)\cap[0,\infty))$*-parameter natural additive process in law on* $\mathbb{R}^d$,

(2) $X_{t_1}^{(s_1)} + X_{t_2}^{(s_1+t_1)} = X_{t_1+t_2}^{(s_1)}$ *a. s. if* $s_1, s_1+t_1, s_1+t_1+t_2 \in J$.

(ii) *Suppose that* $\Big\{\{X_t^{(s)} \colon t \in (J-s)\cap[0,\infty)\} \colon s \in J\Big\}$ *is a family of processes satisfying* (1) *and* (2) *above. Then there is a unique (in the a. s. sense)* $\mathbb{R}^d$*-valued i. s. r. m. over* J, $\{M(B) \colon B \in \mathcal{B}_J^0\}$, *such that* (3.2) *holds for all* $s \in J$ *and* $t \geqslant 0$ *with* $s+t \in J$.

Proof. The assertion (i) is easy to see; (ii) is proved by reduction to Proposition 3.3. □

Example 3.5. Let $\{X_t \colon t \geqslant 0\}$ and $\{Y_t \colon t \geqslant 0\}$ be independent additive processes in law on $\mathbb{R}^d$. Then there exists a unique $\mathbb{R}^d$-valued i. s. r. m. $\{M(B) \colon B \in \mathcal{B}_{\mathbb{R}}^0\}$ over $\mathbb{R}$ such that

$$M((s,t]) = \begin{cases} X_t - X_s & \text{for } 0 \leqslant s < t \\ X_t + Y_{-s} & \text{for } s < 0 \leqslant t \\ -Y_{-t} + Y_{-s} & \text{for } s < t \leqslant 0, \end{cases} \tag{3.3}$$

because we can apply Proposition 3.4. If $\{X_t\}$ is a Lévy process in law and $\{Y_t\} \stackrel{\mathrm{d}}{=} \{X_t\}$, then $\{M(B)\}$ is homogeneous. If $\{X_t\}$ is a semi-Lévy process in law with period p and $\{Y_t\} \stackrel{\mathrm{d}}{=} \{\widetilde{X}_t\}$, where $\{\widetilde{X}_t\}$ is constructed from $\{X_t\}$ as in Example 2.4, then $\{M(B)\}$ is p-periodic.

In the rest of this section, J is an arbitrary interval in $\mathbb{R}$.

Proposition 3.6. *Let* $\{M(B) \colon B \in \mathcal{B}_J^0\}$ *be an* $\mathbb{R}^d$*-valued i. s. r. m. over* J. *Then,* $\mathcal{L}(M(B)) \in ID(\mathbb{R}^d)$ *for each* B. *Let* (A_B, ν_B, γ_B) *be the triplet of* $\mu_B = \mathcal{L}(M(B))$. *Then,* A_B, γ_B, *and* $\nu_B(C)$ *for each* $C \in \mathcal{B}_0(\mathbb{R}^d)$ *are countably additive in* $B \in \mathcal{B}_J^0$.

We use the notation μ_B, A_B, ν_B, and γ_B as in the proposition above. The variation measure of γ_B is denoted by $|\gamma|_B$.

Definition 3.7. Let $\{M(B) \colon B \in \mathcal{B}_J^0\}$ be an $\mathbb{R}^d$-valued i. s. r. m. over J. A pair $(\{\rho_s \colon s \in J\}, \sigma)$ is called a *differential representation* of $\{M(B)\}$ if the following conditions are satisfied:

(1) σ is a locally finite measure on J, that is, a measure on J such that $\sigma(B) < \infty$ for all $B \in \mathcal{B}_J^0$,

(2) σ is continuous (that is, atomless),

(3) $\rho_s \in ID(\mathbb{R}^d)$ for $s \in J$,

(4) $\log \widehat{\rho}_s(z)$ is measurable in $s \in J$ for each $z \in \mathbb{R}^d$,

(5) $\int_B |\log \widehat{\rho}_s(z)|\sigma(ds) < \infty$ for all $B \in \mathcal{B}_J^0$ and $z \in \mathbb{R}^d$,

(6) we have

$$Ee^{i\langle z, M(B)\rangle} = \exp \int_B \log \widehat{\rho}_s(z)\sigma(ds) \quad \text{for all } B \in \mathcal{B}_J^0 \text{ and } z \in \mathbb{R}^d. \tag{3.4}$$

The measure σ on J such that

$$\sigma(B) = \mathrm{tr}(A_B) + \int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu_B(dx) + |\gamma|_B \quad \text{for } B \in \mathcal{B}_J^0 \tag{3.5}$$

is called the *canonical measure* of $\{M(B)\}$. A pair $(\{\rho_s\}, \sigma)$ is called a *canonical differential representation* of $\{M(B)\}$ if it is a differential representation and σ is the canonical measure of $\{M(B)\}$. When $J = [0, t_0)$, $[0, t_0]$, or $[0, \infty)$ and $\{M(B)\}$ corresponds to the J-parameter additive process in law $\{X_t \colon t \in J\}$ by (3.1), then these notions of $\{M(B)\}$ are sometimes considered as those of $\{X_t\}$.

Proposition 3.8. *Let $\{M(B) \colon B \in \mathcal{B}_J^0\}$ be an $\mathbb{R}^d$-valued i. s. r. m. over J.*

(i) *Let $(\{\rho_s\}, \sigma)$ be a differential representation of $\{M(B)\}$ and let $(A_s^\rho, \nu_s^\rho, \gamma_s^\rho)$ be the triplet of ρ_s. Then A_s^ρ, γ_s^ρ, and $\nu_s^\rho(C)$ for any $C \in \mathcal{B}_0(\mathbb{R}^d)$ are measurable in s, and*

$$\int_B \left(\mathrm{tr}(A_s^\rho) + \int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu_s^\rho(dx) + |\gamma_s^\rho| \right) \sigma(ds) < \infty \ ,$$

$$A_B = \int_B A_s^\rho \sigma(ds), \quad \nu_B(C) = \int_B \nu_s^\rho(C)\sigma(ds), \quad \gamma_B = \int_B \gamma_s^\rho \sigma(ds)$$

for $B \in \mathcal{B}_J^0$ and $C \in \mathcal{B}_0(\mathbb{R}^d)$, and

$$\log \widehat{\mu}_B(z) = \int_B \log \widehat{\rho}_s(z)\, \sigma(ds) \ .$$

(ii) *A canonical differential representation $(\{\rho_s\}, \sigma)$ of $\{M(B)\}$ exists and satisfies*

$$\operatorname*{esssup}_{s \in J} \left(\mathrm{tr}(A_s^\rho) + \int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu_s^\rho(dx) + |\gamma_s^\rho| \right) < \infty \ ,$$

where the essential supremum is with respect to σ. If $(\{\rho_s^1\}, \sigma)$ and $(\{\rho_s^2\}, \sigma)$ are canonical differential representations of $\{M(B)\}$, then $\rho_s^1 = \rho_s^2$ for σ-a. e. $s \in J$.

Thus, when $J = [0, t_0)$, $[0, t_0]$, or $[0, \infty)$ and $\{X_t\}$ is a J-parameter additive process in law on $\mathbb{R}^d$, then a differential representation of $\{X_t\}$ exists if and only if $\{X_t\}$ is natural. For example, the canonical measure of a J-parameter Lévy process in law on $\mathbb{R}^d$ is a constant multiple of the Lebesgue measure restricted to J.

Let $\{M(B)\colon B \in \mathcal{B}_J^0\}$ be an $\mathbb{R}^d$-valued i. s. r. m. over J. We define stochastic integrals of nonrandom functions by $\{M(B)\}$.

Definition 3.9. If $F(s)$ is a function on J such that

$$F(s) = \sum_{j=1}^{n} 1_{B_j}(s) R_j \ . \tag{3.6}$$

where $B_1 \ldots, B_n$ are disjoint Borel sets in J and $R_1, \ldots, R_n$ are in $\mathbf{M}_{l\times d}$, then we say that $F(s)$ is an $\mathbf{M}_{l\times d}$-valued *simple function* and define, for $B \in \mathcal{B}_J^0$,

$$\int_B F(s) M(ds) = \sum_{j=1}^{n} R_j \, M(B \cap B_j) \ . \tag{3.7}$$

An $\mathbf{M}_{l\times d}$-valued function $F(s)$ on J is said to be M*-integrable* if it is measurable and if there is a sequence of simple functions $F_1(s), F_2(s) \ldots$ on J such that

(1) $F_n(s) \to F(s)$ for σ-a. e. $s \in J$, where σ is the canonical measure of $\{M(B)\}$,

(2) for every $B \in \mathcal{B}_J^0$, the sequence $\int_B F_n(s) M(ds)$ is convergent in probability as $n \to \infty$.

The limit in probability in (2) is denoted by $\int_B F(s)M(ds)$ and called the (stochastic) integral of F over B by M. When $J = [0, t_0)$, $[0, t_0]$, or $[0, \infty)$, then, using the J-parameter natural additive process in law $\{X_t\colon t \in J\}$ satisfying (3.1), we sometimes write $\int_B F(s) dX_s$ for $\int_B F(s)M(ds)$.

Obviously the definition (3.7) of the integral of a simple function does not depend (in the a. s. sense) on the choice of the representation (3.6) of F. But the following fact, which guarantees that the integral is well-defined in M-integrable case, is nontrivial.

Proposition 3.10. *If $F(s)$ is a measurable $\mathbf{M}_{l\times d}$-valued function on J and if $F_n^1(s)$ and $F_n^2(s)$, $n = 1, 2, \ldots$, are sequences of simple functions satisfying* (1) *and* (2) *of Definition* 3.9 *with $F_n(s)$ replaced by $F_n^1(s)$ and $F_n^2(s)$, then, for every $B \in \mathcal{B}_J^0$,*

$$\operatorname*{p-lim}_{n\to\infty} \int_B F_n^1(s) M(ds) = \operatorname*{p-lim}_{n\to\infty} \int_B F_n^2(s) M(ds) \quad a.\,s.$$

Proposition 3.11. *Let $F(s)$ be an $\mathbf{M}_{l\times d}$-valued measurable function bounded on each $B \in \mathcal{B}_J^0$. Then $F(s)$ is M-integrable. Moreover, if $F_n(s)$ is a sequence of simple functions on J such that $F_n(s) \to F(s)$ σ-a. e., where σ is the canonical measure, and if $\sup_n \sup_{s\in B} \|F_n(s)\| < \infty$ for every $B \in \mathcal{B}_J^0$, then*

$$\operatorname*{p-lim}_{n\to\infty} \int_B F_n(s) M(ds) = \int_B F(s) M(ds) \quad \text{for } B \in \mathcal{B}_J^0.$$

Proposition 3.12. *If $F_1(s)$ and $F_2(s)$ are M-integrable $\mathbf{M}_{l\times d}$-valued functions on J, then, for any a_1 and a_2 in $\mathbb{R}$, $a_1F_1(s)+a_2F_2(s)$ is M-integrable and*

$$\int_B (a_1F_1(s)+a_2F_2(s))M(ds) = a_1\int_B F_1(s)M(ds) + a_2\int_B F_2(s)M(ds) \quad a.\,s.$$

for any $B\in\mathcal{B}_J^0$.

Proposition 3.13. *Let $F(s)$ be an M-integrable $\mathbf{M}_{l\times d}$-valued function on J. Let $\Lambda(B)=\int_B F(s)M(ds)$ and $\lambda_B=\mathcal{L}(\Lambda(B))$ for $B\in\mathcal{B}_J^0$. Then $\{\Lambda(B)\colon B\in\mathcal{B}_J^0\}$ is an $\mathbb{R}^l$-valued i. s. r. m. over J. If $(\{\rho_s\},\sigma)$ is a differential representation of $\{M(B)\}$, then, for $B\in\mathcal{B}_J^0$ and $z\in\mathbb{R}^l$,*

$$\int_B |\log\widehat{\rho}_s(F(s)'z)|\sigma(ds)<\infty \quad \text{and} \quad \log\widehat{\lambda}_B(z)=\int_B \log\widehat{\rho}_s(F(s)'z)\sigma(ds).$$

Here $\log\widehat{\rho}_s(F(s)'z)$ means $(\log\widehat{\rho}_s(w))_{w=F(s)'z}$.

Even if $F(s)$ is M-integrable, we cannot always define $\int_B F(s)M(ds)$ for $B\in\mathcal{B}_J\setminus\mathcal{B}_J^0$.

Definition 3.14. Let $F(s)$ be an M-integrable $\mathbf{M}_{l\times d}$-valued function on J. If J is infinite to the right and if, for $t\in J$, $\int_{(t,u]}F(s)M(ds)$ is convergent in probability as $u\to\infty$, then we say that $\int_t^\infty F(s)M(ds)$ is *definable* and define

$$\int_t^\infty F(s)M(ds) = \operatorname*{p-lim}_{u\to\infty}\int_{(t,u]}F(s)M(ds)\ .$$

If J is infinite to the left, then the notion of definability and the definition are given similarly to $\int_{-\infty}^t F(s)M(ds)$. When $J=[0,\infty)$, then, using the natural additive process in law $\{X_t\colon t\geqslant 0\}$ satisfying (3.1), write $\int_t^\infty F(s)dX_s$ for $\int_t^\infty F(s)M(ds)$.

Remark. If J is infinite to the right (resp. to the left) and $\int_t^\infty F(s)M(ds)$ (resp. $\int_{-\infty}^t F(s)M(ds)$) is definable, then it is a J-parameter stochastic process continuous in probability with independent increments. Hence it has a cadlag modification by the argument in Theorem 11.5 of [17]. Henceforth $\int_t^\infty F(s)M(ds)$ (resp. $\int_{-\infty}^t F(s)M(ds)$) denotes this modification. We also use, for a fixed $t_0\in J$, the notation

$$\int_{t_0}^t F(s)M(ds) = \begin{cases} \int_{(t_0,t]}F(s)M(ds) & \text{for } t\in J\cap(t_0,\infty) \\ 0 & \text{for } t=t_0 \\ -\int_{(t,t_0]}F(s)M(ds) & \text{for } t\in J\cap(-\infty,t_0) \end{cases}$$

and mean a cadlag version over J.

4. Ornstein–Uhlenbeck type processes generated by independently scattered random measures

In the first half of the following theorem, we notice that the nonrandom equation (1.2) is always solvable. This is an $\mathbb{R}^d$-version of a result of Cheridito, Kawaguchi, and Maejima [2], who consider a more general class of functions when $d = 1$. The second half specializes it to the case of independently scattered random measures, that is, the case of Langevin equation. There are many related papers such as Doob [3], Mikosch and Norvaiša [13], and Surgailis et al. [22].

Theorem 4.1. *Let $Q \in \mathbf{M}_d$ and $s_0 \in \mathbb{R}$. Given a nonrandom cadlag function Y_s of $s \in \mathbb{R}$ and a point $\xi \in \mathbb{R}^d$, there exists unique cadlag function Z_s of $s \in \mathbb{R}$ satisfying equation* (1.2) *and condition $Z_{s_0} = \xi$. Let $\{\Lambda(B) \colon B \in \mathcal{B}^0_{\mathbb{R}}\}$ be an $\mathbb{R}^d$-valued i. s. r. m. over $\mathbb{R}$ and let Ξ be an $\mathbb{R}^d$-valued random variable. Then, there exists a unique* (*in the a. s. sense*) *cadlag process $\{Z_s \colon s \in \mathbb{R}\}$ such that Langevin equation* (1.3) *is satisfied a. s. for every s_1 and s_2 with $s_1 < s_2$ and $Z_{s_0} = \Xi$ a. s. This $\{Z_s \colon s \in \mathbb{R}\}$ is represented as*

$$Z_s = e^{(s_0 - s)Q}\Xi + e^{-sQ}\int_{s_0}^{s} e^{uQ}\Lambda(du) \quad \textit{for } s \in \mathbb{R},\ \textit{a. s.} \tag{4.1}$$

Thus we get the Ornstein-Uhlenbeck type process generated by Λ and Q satisfying $Z_{s_0} = \Xi$.

Proof of Theorem 4.1. Define

$$Z_s = e^{(s_0 - s)Q}\xi + Y_s - e^{(s_0 - s)Q}Y_{s_0} - \int_{s_0}^{s} Qe^{(u-s)Q}Y_u du \quad \text{for } s \in \mathbb{R}. \tag{4.2}$$

Then Z_s is a cadlag function with $Z_{s_0} = \xi$. By a straightforward calculation we can prove that Z_s satisfies (1.2). In order to see the uniqueness, suppose that $Z_s^{(1)}$ and $Z_s^{(2)}$ are cadlag solutions of (1.2) with $Z_{s_0}^{(j)} = \xi$ for $j = 1, 2$. Then

$$Z_s^{(1)} - Z_s^{(2)} = -Q\int_{s_0}^{s} (Z_u^{(1)} - Z_u^{(2)}) du. \tag{4.3}$$

Let $V_s = Z_s^{(1)} - Z_s^{(2)}$. Then we get

$$V_s = \frac{(-Q)^n}{(n-1)!}\int_{s_0}^{s} (s-u)^{n-1} V_u du \quad \text{for } n = 1, 2, \dots .$$

Since $((n-1)!)^{-1}(s-u)^{n-1}(-Q)^n \to 0$ uniformly in $u \in [0, s]$ as $n \to \infty$, we get $V_s = 0$. That is, $Z_s^{(1)} = Z_s^{(2)}$.

Let $\{\Lambda(B)\}$ be an $\mathbb{R}^d$-valued i. s. r. m. over $\mathbb{R}$. Define Y_t^0 to be $\Lambda((0,t])$ for $t>0$, zero for $t=0$, and $-\Lambda((t,0])$ for $t<0$. Then, $\{Y_t^0\colon t\in\mathbb{R}\}$ has a cadlag modification $\{Y_t\colon t\in\mathbb{R}\}$ as in Theorem 11.5 of [17]. We have

$$Y_t - Y_s = \Lambda((s,t]) \quad \text{a. s. for every } s,t \text{ with } s<t. \tag{4.4}$$

With this $\{Y_t\colon t\in\mathbb{R}\}$ we can uniquely solve (1.2) pathwise under the condition $Z_{s_0}=\Xi$. The resulting $\{Z_s\colon s\in\mathbb{R}\}$ satisfies (1.3) a. s. for every s_1,s_2 with $s_1<s_2$. The uniqueness is proved in the same way as in the nonrandom case, since (4.3) holds for all s, a. s. The expression (4.1) is obtained from (4.2). Here we use the integration-by-parts formula that

$$\int_{s_1}^{s_2} e^{uQ}dY_u = e^{s_2Q}Y_{s_2} - e^{s_1Q}Y_{s_1} - \int_{s_1}^{s_2} Qe^{uQ}Y_u du \quad \text{a.s. for } 0\leqslant s_1<s_2$$

and its analogue for $s_1<s_2\leqslant 0$. This is a special case of Corollary 4.9 of [18]. □

Definition 4.2. An OU type process $\{Z_s\colon s\in\mathbb{R}\}$ generated by Λ and Q is said to be *mild at* $-\infty$ if $\text{p-lim}_{s\to-\infty}e^{sQ}Z_s=0$.

Theorem 4.3. *Let $\{\Lambda(B)\colon B\in\mathcal{B}^0_{\mathbb{R}}\}$ be an $\mathbb{R}^d$-valued i. s. r. m. over $\mathbb{R}$ and $Q\in\mathbf{M}_d$. Then the following are equivalent:*

(1) $\int_{-\infty}^0 e^{sQ}\Lambda(ds)$ *is definable,*
(2) $\text{p-lim}_{s\to-\infty}e^{sQ}Z_s$ *exists for every OU type process $\{Z_s\colon s\in\mathbb{R}\}$ generated by Λ and Q,*
(3) *an OU type process $\{Z_s\colon s\in\mathbb{R}\}$ mild at $-\infty$ generated by Λ and Q exists.*

In this case, an OU type process mild at $-\infty$ generated by Λ and Q is unique a. s. and expressed as

$$Z_s = e^{-sQ}\int_{-\infty}^{s} e^{uQ}\Lambda(du) \quad \textit{for } s\in\mathbb{R},\ \textit{a. s.} \tag{4.5}$$

Proof. If $\{Z_s\}$ is an OU type process generated by Λ and Q, then, by Theorem 4.1,

$$Z_s = e^{(s_0-s)Q}Z_{s_0} + e^{-sQ}\int_{s_0}^{s} e^{uQ}\Lambda(du) \quad \text{for } s_0,s\in\mathbb{R}, \quad \text{a. s.} \tag{4.6}$$

That is,

$$e^{sQ}Z_s - e^{s_0Q}Z_{s_0} = \int_{s_0}^{s} e^{uQ}\Lambda(du), \quad \text{a. s.} \tag{4.7}$$

Letting $s=0$ and $s_0\to-\infty$, we get the equivalence of (1) and (2). If (3) holds, then, letting $s_0\to-\infty$ in (4.7), we see that (1) and (4.5) are true. This shows the uniqueness of a solution mild at $-\infty$. If (1) holds, then by Theorem 4.1, the solution $\{Z_s\}$ of Langevin equation with $Z_0 = \Xi = \int_{-\infty}^0 e^{sQ}\Lambda(ds)$ a. s. satisfies

$$Z_s = e^{-sQ}\int_{-\infty}^0 e^{uQ}\Lambda(du) + e^{-sQ}\int_0^s e^{uQ}\Lambda(du) = e^{-sQ}\int_{-\infty}^s e^{uQ}\Lambda(du) \quad \text{a. s.},$$

which shows that $\text{p-lim}_{s\to-\infty} e^{sQ}Z_s = 0$. □

Remark. Let $(\{\rho_s\},\sigma)$ be a differential representation of $\{\Lambda(B)\colon B \in \mathcal{B}^0_{\mathbb{R}}\}$. In the case of (1)–(3) of Theorem 4.3, $\lim_{s_0\to-\infty}\int_{s_0}^s \widehat{\rho}_u(e^{uQ'}z)\sigma(du)$ exists and equals the distinguished logarithm of the characteristic function of $\int_{-\infty}^s e^{uQ}\Lambda(du)$. This follows from Proposition 3.13 and [17] Lemma 7.7.

We apply Theorem 4.3 to periodic i. s. r. m.

Theorem 4.4. *Let $\{\Lambda(B)\colon B \in \mathcal{B}^0_{\mathbb{R}}\}$ be an $\mathbb{R}^d$-valued p-periodic i. s. r. m. over $\mathbb{R}$. Let $Q \in \mathbf{M}^+_d$.*

(i) *Suppose that $\{\Lambda(B)\}$ has finite log-moment. Then Langevin equation* (1.3) *based on Λ and Q has a unique semi-stationary solution $\{Z_s\}$ with period p. This solution has expression* (4.5) *and $\mathcal{L}(Z_s) \in L_0(e^{-p}, Q)$ for all s.*

(ii) *Suppose that*

$$E[\log^+|\Lambda((0,p])|] = \infty. \tag{4.8}$$

Then, Langevin equation (1.3) *based on Λ and Q has no semi-stationary solution with period p. Moreover, it has no solution mild at $-\infty$.*

Corollary 4.5. *Let $\{\Lambda(B)\colon B \in \mathcal{B}^0_{\mathbb{R}}\}$ be an $\mathbb{R}^d$-valued homogeneous i. s. r. m. over $\mathbb{R}$. Let $Q \in \mathbf{M}^+_d$.*

(i) *Suppose that $\{\Lambda(B)\}$ has finite log-moment. Then Langevin equation* (1.3) *based on Λ and Q has a unique stationary solution $\{Z_s\}$. This solution has expression* (4.5) *and $\mathcal{L}(Z_s)$ is Q-selfdecomposable.*

(ii) *Suppose that $\{\Lambda(B)\}$ does not have finite log-moment. Then, Langevin equation* (1.3) *based on Λ and Q does not have a stationary solution.*

Actually the result in Corollary 4.5 was given in [20]. Our Theorem 4.4 is an extension of it.

In order to prove Theorem 4.4, we prepare two lemmas.

Lemma 4.6. *Let $\{\Lambda(B)\colon B \in \mathcal{B}^0_{\mathbb{R}}\}$ be an $\mathbb{R}^d$-valued i. s. r. m. over $\mathbb{R}$ and let $Q \in \mathbf{M}^+_d$. Let $\{Z_s\colon s \in \mathbb{R}\}$ be an OU type process generated by Λ and Q. If $\{Z_s\}$ is stationary or, more generally, semi-stationary, then $\{Z_s\}$ is mild at $-\infty$.*

Proof. Suppose that $\{Z_s\}$ is semi-stationary with period p. Let $\eta_s = \mathcal{L}(Z_s)$. It follows from (1.3) or (4.1) that $\{Z_s\}$ is continuous in probability. Hence η_s is continuous in s. Thus we see that $\{\eta_s\colon s \in [0,p]\}$ is a compact set in the topology of the weak convergence. This set equals $\{\eta_s\colon s \in \mathbb{R}\}$ by semi-stationarity. Hence $\{\eta_s\colon s \in \mathbb{R}\}$ is tight. We have an estimate $\|e^{sQ}\| \leqslant c_2 e^{c_1 s}$ for $s \leqslant 0$ with $c_1, c_2 > 0$ (see [18]). Using this we see that, for any $\varepsilon > 0$,

$$P[\,|e^{sQ}Z_s| > \varepsilon\,] \leqslant P[\,c_2 e^{c_1 s}|Z_s| > \varepsilon\,] \leqslant \sup_u \eta_u(\{|x| > \varepsilon c_2^{-1} e^{c_1|s|}\}) \to 0$$

as $s \to -\infty$. That is, $e^{sQ}Z_s \to 0$ in probability. □

Lemma 4.7. *Let $\{\Lambda(B)\colon B \in \mathcal{B}^0_{\mathbb{R}}\}$ be an $\mathbb{R}^d$-valued p-periodic i. s. r. m. over $\mathbb{R}$. Fix $t_0 \in \mathbb{R}$ and define*

$$\widetilde{\Lambda}(B) = \Lambda(t_0 - B) \quad \text{for } B \in \mathcal{B}^0_{\mathbb{R}}\,. \tag{4.9}$$

Then $\{\widetilde{\Lambda}(B)\colon B \in \mathcal{B}^0_{\mathbb{R}}\}$ is a p-periodic i. s. r. m. and $\widetilde{\Lambda}((0,p]) \stackrel{\mathrm{d}}{=} \Lambda((0,p])$. Let $F(s)$ be an $\mathbf{M}_{l\times d}$-valued function on $\mathbb{R}$. Then $F(s)$ is $\widetilde{\Lambda}$-integrable if and only if $F(t_0 - s)$ is Λ-integrable. In this case,

$$\int_B F(s)\widetilde{\Lambda}(ds) = \int_{t_0 - B} F(t_0 - s)\Lambda(ds) \quad \text{a. s. for } B \in \mathcal{B}^0_{\mathbb{R}}\,. \tag{4.10}$$

Proof. It is easy to see that $\widetilde{\Lambda}(B)$ is a p-periodic i. s. r. m. over $\mathbb{R}$. To see that $\widetilde{\Lambda}((0,p]) \stackrel{\mathrm{d}}{=} \Lambda((0,p])$, note that $\widetilde{\Lambda}((0,p]) = \Lambda([t_0 - p, t_0)) = \Lambda((t_0 - p, t_0])$ a. s. and that, choosing $n \in \mathbb{Z}$ such that $t_0 - p < np \leqslant t_0$, $\Lambda((np, t_0]) \stackrel{\mathrm{d}}{=} \Lambda((0, t_0 - np])$ and $\Lambda((t_0 - p, np]) \stackrel{\mathrm{d}}{=} \Lambda((t_0 - np, p])$. If $F(s)$ is a simple function (3.6), then $F(t_0 - u) = \sum_{j=1}^n 1_{B_j}(t_0 - u)R_j = \sum_{j=1}^n 1_{t_0 - B_j}(u)R_j$ and hence $\int_B F(s)\widetilde{\Lambda}(ds) = \sum_{j=1}^n R_j \widetilde{\Lambda}(B \cap B_j) = \sum_{j=1}^n R_j \Lambda((t_0 - B) \cap (t_0 - B_j)) = \int_{t_0 - B} F(t_0 - u)\Lambda(du)$, which is (4.10). The rest of proof is straightforward. □

Proof of Theorem 4.4. (i) We assume that $\{\Lambda(B)\}$ has finite log-moment. Given $t_0 \in \mathbb{R}$, define $\{\widetilde{\Lambda}(B)\}$ by (4.9). Then, by Lemma 4.7, $\{\widetilde{\Lambda}(B)\}$ is a p-periodic i. s. r. m. with finite log-moment. By Theorem 5.2 of [18], $\int_0^\infty e^{-sQ}\widetilde{\Lambda}(ds)$ is definable. Hence, by (4.10), $\int_{-\infty}^{t_0} e^{-(t_0 - s)Q}\Lambda(ds)$ is definable and so is $\int_{-\infty}^0 e^{sQ}\Lambda(ds)$. Thus, by Theorem 4.3, there is a unique OU type process $\{Z_s\}$ mild at $-\infty$ generated by Λ and Q. It is expressed by (4.5). Since

$$e^{-t_0 Q}\int_{-\infty}^{t_0} e^{sQ}\Lambda(ds) = \int_0^\infty e^{-sQ}\widetilde{\Lambda}(ds) \quad \text{a. s.},$$

$\mathcal{L}(Z_{t_0}) \in L_0(e^{-p}, Q)$ for any t_0 by virtue of Theorem 5.2 of [18].

We have

$$Z_{s+p} = e^{-(s+p)Q} \int_{-\infty}^{s+p} e^{uQ} \Lambda(du) = e^{-sQ} \int_{-\infty}^{s} e^{vQ} \Lambda^\sharp(dv),$$

where we define $\Lambda^\sharp(B) = \Lambda(B+p)$. Since $\{\Lambda^\sharp(B)\} \stackrel{\mathrm{d}}{=} \{\Lambda(B)\}$, we get $Z_{s+p} \stackrel{\mathrm{d}}{=} e^{-sQ}\int_{-\infty}^{s} e^{vQ}\Lambda(dv) = Z_s$. Similarly, for any $s_1 < s_2 < \cdots < s_n$, $(Z_{s_j+p})_{j=1,\dots,n} \stackrel{\mathrm{d}}{=} (Z_{s_j})_{j=1,\dots,n}$, which is semi-stationarity of $\{Z_s\}$ with period p. By Lemma 4.6 semi-stationarity implies mildness at $-\infty$. Hence, by Theorem 4.3, a semi-stationary solution is unique.

(ii) We assume (4.8). Then, using Theorem 5.4 of [18] and Lemma 4.7, we see that $\int_{-\infty}^{t} e^{sQ}\Lambda(ds)$ is not definable. Hence, by Theorem 4.3, there is no solution mild at $-\infty$ of Langevin equation. Lemma 4.6 tells us that, a fortiori, there is no semi-stationary solution. □

Remark. Let $\{\Lambda(B)\}$ be an $\mathbb{R}^d$-valued i. s. r. m. over $\mathbb{R}$ and $Q \in \mathbf{M}_d$. If there is a semi-stationary solution $\{Z_s\}$ with period p of Langevin equation (1.3), then $\{\Lambda(B)\}$ is p-periodic. Indeed, it follows from (1.3) and $\{Z_{s+p}\} \stackrel{\mathrm{d}}{=} \{Z_s\}$ that

$$Z_{s_2+p} - Z_{s_1+p} + Q\int_{s_1}^{s_2} Z_{u+p}du \stackrel{\mathrm{d}}{=} Z_{s_2} - Z_{s_1} + Q\int_{s_1}^{s_2} Z_u du \ ,$$

that is, $\Lambda((s_1+p, s_2+p]) \stackrel{\mathrm{d}}{=} \Lambda((s_1, s_2])$. Similarly, if there is a stationary solution, then $\{\Lambda(B)\}$ is homogeneous.

Remark. In [19] solutions mild at $-\infty$ of Langevin equations based on a class of non-periodic i. s. r. m. will be given. Namely, it will be shown that, if $\{\Lambda(B)\}$ is an $\mathbb{R}^d$-valued i. s. r. m. over $\mathbb{R}$ such that the process $\{X_t \colon t \geqslant 0\}$ defined by $X_t = \Lambda((-t, 0])$ is a Q-semi-selfsimilar additive process in law for some $Q \in \mathbf{M}_d^+$, then, for any $R \in \mathbf{M}_d^+$, Langevin equation based on Λ and R has a unique solution mild at $-\infty$. In this case $E\log^+|\Lambda((-t,0])|$ may possibly be infinite.

Theorem 4.4 shows that, when we restrict our attention to p-periodic i. s. r. m., the integrals $\int_{-\infty}^{0} e^{sQ}\Lambda(ds)$ (if definable) with $Q \in \mathbf{M}_d^+$ have distributions in a restricted class. But, in the case of general i. s. r. m., the integrals can have arbitrary distributions. In fact, we can show the following.

Proposition 4.8. *Let $F(s)$ be an $\mathbf{M}_d$-valued continuous function on $(-\infty, 0]$ such that, for every s, $F(s)$ is an invertible matrix. Then, for any $\mu \in ID(\mathbb{R}^d)$, there is an $\mathbb{R}^d$-valued i. s. r. m. $\{\Lambda(B) \colon B \in \mathcal{B}^0_{(-\infty,0]}\}$ over $(-\infty, 0]$ such that $\int_{-\infty}^{0} F(s)\Lambda(ds)$ is definable and has distribution μ.*

Proof. Let $\{Y_s^\sharp\colon s \geqslant 0\}$ be a Lévy process with $\mathcal{L}(Y_1^\sharp) = \mu$. Define

$$\Lambda(B) = \int_{\exp B} (F(\log u))^{-1} dY_u^\sharp \quad \text{for } B \in \mathcal{B}^0_{(-\infty,0]} .$$

Then $\{\Lambda(B)\}$ is an $\mathbb{R}^d$-valued i. s. r. m. over $(-\infty, 0]$. We have, by Proposition 3.13,

$$Ee^{i\langle z,\Lambda(B)\rangle} = \exp \int_{\exp B} \log \widehat{\mu}((F(\log u)')^{-1} z) du = \exp \int_B e^v \log \widehat{\mu}((F(v)')^{-1} z) dv .$$

Thus we can choose a differential representation $(\{\rho_s\}, \sigma)$ of $\{\Lambda(B)\}$ such that $\widehat{\rho_s}(z) = \widehat{\mu}((F(s)')^{-1}z)^{e^s}$ and $\sigma =$ Lebesgue. Hence, by Proposition 3.13,

$$E \exp \left[i \left\langle z, \int_{s_1}^{s_2} F(s)\Lambda(ds) \right\rangle \right] = \exp \int_{s_1}^{s_2} \log \widehat{\rho_s}(F(s)'z) ds = \exp \left[\int_{s_1}^{s_2} e^s ds \log \widehat{\mu}(z) \right],$$

which tends to 1 as $s_1, s_2 \to -\infty$. It follows that $\int_{-\infty}^0 F(s)\Lambda(ds)$ is definable and that

$$E \exp \left[i \left\langle z, \int_{-\infty}^{0} F(s)\Lambda(ds) \right\rangle \right] = \exp \left[\int_{-\infty}^{0} e^s ds \log \widehat{\mu}(z) \right] = \widehat{\mu}(z),$$

that is, $\mathcal{L}\left(\int_{-\infty}^0 F(s)\Lambda(ds)\right) = \mu$. □

5. Proofs of main results

Let us prove the three theorems formulated in Section 1.

Proof of Theorem 1.1. Let $\{M(B)\colon B \in \mathcal{B}^0_{[0,\infty)}\}$ be the $\mathbb{R}^d$-valued i. s. r. m. over $[0, \infty)$ induced by $\{X_t\colon t \geqslant 0\}$ (Proposition 3.3). Let $M_0(B) = M(B)$ for $B \in \mathcal{B}^0_{(0,\infty)}$. Then $\{M_0(B)\colon B \in \mathcal{B}^0_{(0,\infty)}\}$ is an i. s. r. m. over $(0, \infty)$, which is the restriction of $\{M(B)\colon B \in \mathcal{B}^0_{[0,\infty)}\}$ to $\mathcal{B}^0_{(0,\infty)}$. The function t^{-Q} is M_0-integrable by Proposition 3.11. If $B \in \mathcal{B}^0_{\mathbb{R}}$, then $\exp B \in \mathcal{B}^0_{(0,\infty)}$ and hence we can define $\int_{\exp B} t^{-Q} M_0(dt)$. We denote this integral by $\int_{\exp B} t^{-Q} dX_t$. The right-hand side of (1.5) means this integral. By Proposition 3.13, $\{\Lambda(B)\}$ thus defined by (1.5) is an $\mathbb{R}^d$-valued i. s. r. m. over $\mathbb{R}$. Using Proposition 3.11 again, we can prove that, if $\varepsilon > 0$, then, for all $B \in \mathcal{B}^0_{\mathbb{R}}$ with $\exp B \subset [\varepsilon, \infty)$, $\int_{\exp B} t^{-Q} M_0(dt) = \int_{\exp B} F(t) dX_t$ a. s., where $F(t)$ is a continuous function on $[0, \infty)$ satisfying $F(t) = t^{-Q}$ on $[\varepsilon/2, \infty)$. Let $X_t^\sharp = X_{at}$. Using Theorem 4.10 of [18] and recalling $\{X_t^\sharp\} \stackrel{\mathrm{d}}{=} \{a^Q X_t\}$, we get

$$\Lambda(B + p) = \int_{a \exp B} t^{-Q} dX_t = \int_{\exp B} (at)^{-Q} dX_t^\sharp \stackrel{\mathrm{d}}{=} \int_{\exp B} t^{-Q} dX_t = \Lambda(B) .$$

Hence $\{\Lambda(B)\}$ is p-periodic. Define $\Lambda^\sharp(B) = \Lambda(\log B)$ for $B \in \mathcal{B}^0_{(0,\infty)}$. Then $\{\Lambda^\sharp(B)\}$ is an i. s. r. m. over $(0, \infty)$ and $\Lambda^\sharp(B) = \int_B t^{-Q} dX_t$ under similar interpretation of the integral. Use of analogues of Theorems 4.6 and 4.10 of [18] gives, for $0 < t_1 < t_2$,

$$\int_{\log t_1}^{\log t_2} e^{uQ}\Lambda(du) = \int_{t_1}^{t_2} u^Q \Lambda^\sharp(du) = \int_{t_1}^{t_2} u^Q u^{-Q} dX_u = X_{t_2} - X_{t_1}.$$

As $t_1 \downarrow 0$, $X_{t_1} \to 0$ a.s. Hence $\int_{-\infty}^{\log t_2} e^{uQ}\Lambda(du)$ is definable. It follows from Theorem 4.4 (ii) that $\{\Lambda(B)\}$ has finite log-moment. We get also the expression (1.6).

By (1.4) and (1.6), we get (1.7). Hence, by Theorem 4.4, $\{Z_s\}$ is the semi-stationary OU type process with period p generated by Λ and Q. □

Proof of Theorem 1.2. Existence and uniqueness of $\{Z_s \colon s \in \mathbb{R}\}$, the semi-stationary OU type process with period p generated by Λ and Q, are shown in Theorem 4.4. It is expressed by (4.5). Hence X_t has the expression (1.6) for $t > 0$. As $t \downarrow 0$, $X_t = \int_{-\infty}^{\log t} e^{sQ}\Lambda(ds) \to 0$ in probability. It follows from (1.6) that $\{X_t\}$ has independent increments. Since $\{X_t\}$ is continuous in probability for $t \geqslant 0$, it is an additive process in law and thus has a cadlag modification ([17] Theorem 11.5). On the other hand, $\{X_t\}$ is itself cadlag for $t > 0$ a.s. since $\{Z_s\}$ is cadlag. It follows that $\{X_t\}$ is cadlag for $t \geqslant 0$ a.s. Let $\Lambda^\sharp(B) = \Lambda(B+p)$. We have

$$X_{at} = \int_{-\infty}^{p+\log t} e^{sQ}\Lambda(ds) = \int_{-\infty}^{\log t} e^{(s+p)Q}\Lambda^\sharp(ds) \stackrel{\mathrm{d}}{=} a^Q \int_{-\infty}^{\log t} e^{sQ}\Lambda(ds) = a^Q X_t,$$

and similarly for joint distributions. Thus $\{X_{at}\} \stackrel{\mathrm{d}}{=} \{a^Q X_t\}$. Hence $\{X_t\}$ is a Q-semi-selfsimilar additive process with epoch a. Define $X_t^\sharp = X_{1+t} - X_1$ and $X_t^{\sharp\sharp} = X^\sharp_{e^t-1}$. Since $X_t^\sharp = \int_0^{\log(1+t)} e^{sQ}\Lambda(ds)$, $\{X_t^\sharp \colon t \geqslant 0\}$ is a natural additive process, by Propositions 3.3 and 3.13. Then $X_t^{\sharp\sharp} = X_{e^t} - X_1 = \int_0^t e^{sQ}\Lambda(ds)$. If $s \geqslant 0$, then

$$\int_1^{e^s} t^{-Q} dX_t = \int_0^{e^s-1} (1+t)^{-Q} dX_t^\sharp = \int_0^s (e^t)^{-Q} dX_t^{\sharp\sharp} = \Lambda((0,s]),$$

where the second equality is by Theorem 4.10 of [18] and the third is by Theorem 4.6 of [18]. If $s < 0$, then

$$\int_1^{e^s} t^{-Q} dX_t = -\int_{e^s}^1 t^{-Q} dX_t = -\Lambda((s,0])$$

similarly. Hence we obtain (1.5). That is, $\{\Lambda(B)\}$ is recovered from $\{X_t\}$ as in Theorem 1.1. The expression (1.4) of $\{Z_s\}$ by $\{X_t\}$ follows from (1.8). □

The following lemma is an extension of Theorem 10 of [10].

Lemma 5.1. *Let $Q \in \mathbf{M}_d^+$ and $b \in (0,1)$. A distribution μ is in $L_0(b,Q)$ if and only if there exists a natural Q-semi-selfsimilar additive process $\{X_t \colon t \geqslant 0\}$ with epoch b^{-1} such that $\mathcal{L}(X_1) = \mu$.*

Proof. We write $a = b^{-1}$.

The 'if' part. We have $\{X_{at}\} \stackrel{\mathrm{d}}{=} \{a^Q X_t\}$. Hence $b^Q X_1 \stackrel{\mathrm{d}}{=} X_b$. It follows that

$$Ee^{i\langle z,X_1\rangle} = Ee^{i\langle z,X_b\rangle} Ee^{i\langle z,X_1-X_b\rangle} = Ee^{i\langle b^{Q'}z,X_1\rangle} Ee^{i\langle z,X_1-X_b\rangle}.$$

Since $\mathcal{L}(X_1 - X_b)$ is infinitely divisible, this means $\mathcal{L}(X_1) \in L_0(b, Q)$. (Here we do not use naturalness. Similarly we can prove $\mathcal{L}(X_t) \in L_0(b, Q)$ for all t.)

The 'only if' part. If we construct from $\mu \in L_0(b, Q)$ a system of distributions $\{\mu_t\colon 1 \leqslant t \leqslant a\}$ on $\mathbb{R}^d$ such that (1) $\mu_1 = \mu$, (2) $\widehat{\mu}_a(z) = \widehat{\mu}(a^{Q'}z)$, (3) there is a distribution $\mu_{s,t}$ for $1 \leqslant s \leqslant t \leqslant a$ such that $\mu_t = \mu_s * \mu_{s,t}$, and (4) $\widehat{\mu}_t(z)$ is continuous in $t \in [1, a]$, then there is, uniquely in law, a Q-semi-selfsimilar additive process $\{X_t\colon t \geqslant 0\}$ with epoch a such that $\mathcal{L}(X_t) = \mu_t$ for $t \in [1, a]$. This is verified in the same way as the proof of Theorem 7 of [10]. A construction of such a system $\{\mu_t\}$ is as follows. Recall that $\mu \in ID$ (see [11]). Define μ_t for $1 \leqslant t \leqslant a$ by

$$\widehat{\mu}_t(z) = \widehat{\mu}(z)^{1-h(t)}\widehat{\mu}(a^{Q'}z)^{h(t)} \tag{5.1}$$

with a continuous increasing function $h(t)$ satisfying $h(1) = 0$ and $h(a) = 1$. Then $\{\mu_t\}$ satisfies conditions (1)–(4) above. Indeed, (1), (2), and (4) are obvious. To see (3), let $1 \leqslant s \leqslant t \leqslant a$. Notice that

$$\widehat{\mu}_t(z) = \widehat{\mu}(z)^{1-h(t)}\widehat{\mu}(a^{Q'}z)^{h(s)}\widehat{\mu}(a^{Q'}z)^{h(t)-h(s)},$$
$$\widehat{\mu}_s(z) = \widehat{\mu}(z)^{1-h(t)}\widehat{\mu}(z)^{h(t)-h(s)}\widehat{\mu}(a^{Q'}z)^{h(s)}.$$

Since $\mu \in L_0(a^{-1}, Q)$, there is $\rho \in ID$ such that $\widehat{\mu}(z) = \widehat{\mu}(a^{-Q'}z)\widehat{\rho}(z)$, that is, $\widehat{\mu}(a^{Q'}z) = \widehat{\mu}(z)\widehat{\rho}(a^{Q'}z)$. Hence $\widehat{\mu}_t(z) = \widehat{\mu}_s(z)\widehat{\rho}(a^{Q'}z)^{h(t)-h(s)}$, which shows that condition (3) is satisfied. It follows from (5.1) that the location parameter in the triplet (A_t, ν_t, γ_t) of μ_t satisfies $\gamma_t = (1 - h(t))\gamma_1 + h(t)\gamma_a$, which is of bounded variation in $t \in [1, a]$. Hence the process $\{X_t\}$ constructed is natural by Theorem 2.13 of [18]. □

Proof of Theorem 1.3. *The 'only if' part.* Let $\mu = \mathcal{L}(X_1) = \mathcal{L}(Z_0)$, where $\{X_t\}$ and $\{Z_s\}$ are the processes in Theorem 1.1 or 1.2. Then $\mu \in L_0(a^{-1}, Q)$ by Lemma 5.1.

The 'if' part. Given $\mu \in L_0(a^{-1}, Q)$, use the process $\{X_t\}$ in Lemma 5.1 as the process in Theorem 1.1. □

Remark. The 'only if' part of Theorem 1.3 can be strengthened as follows: the distributions $\mathcal{L}(X_t)$ for all $t \geqslant 0$ and $\mathcal{L}(Z_s)$ for all $s \in \mathbb{R}$ are (a^{-1}, Q)-decomposable.

Remark. In the proof of the 'only if' part of Lemma 5.1, the construction of $\{X_t\}$ has freedom of choice of the function $h(t)$ on $[1, a]$. Freedom of choice of systems

$\{\mu_t\colon 1 \leqslant t \leqslant a\}$ is even larger, since there exist systems not of the form (5.1). See examples in [10] in the case $Q = cI$ with $c > 0$. This corresponds to the variety of processes $\{X_t\}$ and $\{Z_s\}$ that express the same μ in Theorem 1.3. See also a remark below Proposition 7.4. This is in contrast to the situation in the non-"semi" case, which we will formulate in Section 6.

Corollary 5.2. *Let $Q \in \mathbf{M}_d^+$ and $a > 1$. A distribution μ on $\mathbb{R}^d$ is (a^{-1}, Q)-decomposable if and only if μ is expressible as*

$$\mu = \mathcal{L}\left(\int_0^\infty e^{-tQ} d\widetilde{Y}_t\right) \tag{5.2}$$

by a natural semi-Lévy process $\{\widetilde{Y}_t\colon t \geqslant 0\}$ with period $\log a$ *with finite log-moment. In particular, a distribution μ on $\mathbb{R}^d$ is semi-selfdecomposable if and only if μ is expressible as $\mu = \mathcal{L}\left(\int_0^\infty e^{-tI} d\widetilde{Y}_t\right)$ by a natural semi-Lévy process $\{\widetilde{Y}_t\colon t \geqslant 0\}$ with finite log-moment.*

Proof. Use Theorem 1.3 and Lemma 4.7. □

6. Selfsimilar additive processes, stationary OU type processes, and homogeneous independently scattered random measures

Relations of the three objects in the title of this section are formulated below. These are consequences of Theorems 1.1–1.3 except the uniqueness assertions in Theorem 6.3 and Corollary 6.4. When the basic matrix Q equals the identity matrix I, these are new formulations of essentially known results.

Theorem 6.1. *Let $Q \in \mathbf{M}_d^+$. Let $\{X_t\colon t \geqslant 0\}$ be an arbitrary Q-selfsimilar additive process on $\mathbb{R}^d$. Define $\{Z_s\colon s \in \mathbb{R}\}$ and $\{\Lambda(B)\colon B \in \mathcal{B}_{\mathbb{R}}^0\}$ by* (1.4) *and* (1.5)*, respectively. Then $\{\Lambda(B)\}$ is an $\mathbb{R}^d$-valued homogeneous i. s. r. m. over $\mathbb{R}$ with finite log-moment. The process $\{X_t\colon t \geqslant 0\}$ is expressed by Λ in the form of* (1.6)*. The process $\{Z_s\colon s \in \mathbb{R}\}$ is the unique stationary OU type process generated by Λ and Q; it is expressible in the form of* (1.7)*.*

Theorem 6.2. *Let $Q \in \mathbf{M}_d^+$ and let $\{\Lambda(B)\colon B \in \mathcal{B}_{\mathbb{R}}^0\}$ be an arbitrary $\mathbb{R}^d$-valued homogeneous i. s. r. m. over $\mathbb{R}$ with finite log-moment. Let $\{Z_s\colon s \in \mathbb{R}\}$ be the stationary OU type process generated by Λ and Q. Define $\{X_t\colon t \geqslant 0\}$ by* (1.8)*. Then $\{X_t\}$ is a Q-selfsimilar additive process on $\mathbb{R}^d$; $\{Z_s\}$ and $\{\Lambda(B)\}$ are recovered from $\{X_t\}$ in the form of* (1.4) *and* (1.5)*.*

Theorem 6.3. *Fix $Q \in \mathbf{M}_d^+$. A distribution μ on $\mathbb{R}^d$ given by $\mu = \mathcal{L}(X_1) = \mathcal{L}(Z_0)$ in Theorem 6.1 or 6.2 is Q-selfdecomposable. Conversely, for any Q-selfdecomposable distribution μ on $\mathbb{R}^d$, there is, uniquely in law, an $\mathbb{R}^d$-valued homogeneous i. s. r. m. over $\mathbb{R}$ with finite log-moment in Theorem 6.2 such that $\mu = \mathcal{L}(X_1) = \mathcal{L}(Z_0)$.*

Concerning the relation of $\{Z_t\}$ and μ, Theorem 6.3 was proved by [20] and [21]. Concerning the relation of $\{X_t\}$ and μ, it was proved by [16].

Corollary 6.4. *Fix $Q \in \mathbf{M}_d^+$. A distribution μ on $\mathbb{R}^d$ is Q-selfdecomposable if and only if*

$$\mu = \mathcal{L}\left(\int_0^\infty e^{-tQ} dY_t\right) \tag{6.1}$$

with $\{Y_t \colon t \geqslant 0\}$ being a Lévy process on $\mathbb{R}^d$ with finite log-moment. In this case, $\{Y_t \colon t \geqslant 0\}$ is determined by μ uniquely in law.

This result was directly proved by Wolfe [27] and Jurek [5].

For completeness, we give a proof of the uniqueness assertion in Theorem 6.3. Let $\{\Lambda(B) \colon B \in \mathcal{B}_{\mathbb{R}}^0\}$ be an $\mathbb{R}^d$-valued homogeneous i. s. r. m. over $\mathbb{R}$ and let us define $\{Z_s \colon s \in \mathbb{R}\}$ and $\{X_t \colon t \geqslant 0\}$ as in Theorem 6.2. Let $\mu = \mathcal{L}(X_1) = \mathcal{L}(Z_0)$. Since $\{X_t\}$ is a Q-selfsimilar additive process, its distribution as a stochastic process is determined by μ. Hence, by (1.5), the distribution of $\{\Lambda(B)\}$ is determined by μ.

7. Results and examples related to semi-stability

In this section, let $Q \in \mathbf{M}_d^+$ and $b \in (0,1)$. For $\alpha > 0$, a distribution μ on $\mathbb{R}^d$ is called semi-stable with index α and span b^{-1} if $\mu \in ID$ and

$$\widehat{\mu}(z)^{b^\alpha} = \widehat{\mu}(bz) e^{i\langle \gamma, z\rangle} \quad \text{for } z \in \mathbb{R}^d \tag{7.1}$$

for some $\gamma \in \mathbb{R}^d$. In order that such a nontrivial (that is, not concentrated at a point) distribution μ exists, we must have $\alpha \leqslant 2$. We extend this notion. Considering the definition of the class $OSS(b,Q)$ of operator semi-stable distributions in [12], we call a distribution μ on $\mathbb{R}^d$ *(b,Q)-semi-stable* if $\mu \in ID$ and, for some $a \in (0,1)$ and $\gamma \in \mathbb{R}^d$,

$$\widehat{\mu}(z)^a = \widehat{\mu}(b^{Q'} z) e^{i\langle \gamma, z\rangle} \quad \text{for } z \in \mathbb{R}^d. \tag{7.2}$$

Expressing a explicitly, we say that μ is *(b,Q,a)-semi-stable* if $\mu \in ID$ and (7.2) holds with some γ. In this terminology, semi-stability with index α and span b^{-1} is (b, I, b^α)-semi-stability. If γ can be chosen to be 0 in (7.2), we say that μ is *strictly (b,Q,a)-semi-stable*. An additive or Lévy process $\{X_t \colon t \geqslant 0\}$ is said to be (b,Q,a)-semi-stable (resp. strictly (b,Q,a)-semi-stable) if $\mathcal{L}(X_t)$ is (b,Q,a)-semi-stable (resp. strictly (b,Q,a)-semi-stable) for all t. In this section we give some remarks on representations of (b,Q,a)-semi-stable distributions in application of our main theorems. We also give examples of Q-semi-selfsimilar processes connected with processes in the study of diffusion processes in random environments.

We give two basic lemmas.

Lemma 7.1. *If μ is (b,Q,a)-semi-stable on $\mathbb{R}^d$, then it is (b,Q)-decomposable, that is, $\mu \in L_0(b,Q)$.*

Proof. It follows from (7.2) that

$$\widehat{\mu}(z) = \widehat{\mu}(b^{Q'}z)^{a^{-1}} e^{i\langle a^{-1}\gamma, z\rangle} = \widehat{\mu}(b^{Q'}z)\widehat{\mu}(b^{Q'}z)^{a^{-1}-1} e^{i\langle a^{-1}\gamma, z\rangle} .$$

Since $\widehat{\mu}(b^{Q'}z)^{a^{-1}-1} e^{i\langle a^{-1}\gamma, z\rangle}$ is infinitely divisible, we have (1.1) with $\rho_b \in ID$. $\square$

Lemma 7.2. *If μ is (b,Q,a)-semi-stable on $\mathbb{R}^d$, then there is $c \in (0,\infty)$ such that $\int_{\mathbb{R}^d} |x|^c \mu(dx) < \infty$.*

Proof. See Łuczak [8]. The special case of $Q = I$ is treated in [17]. $\square$

It follows from this lemma that any (b,Q,a)-semi-stable distribution has finite log-moment.

Proposition 7.3. *Let $\{U_t : t \geqslant 0\}$ be a Lévy process on $\mathbb{R}^d$ with finite log-moment. Then $\mathcal{L}\left(\int_0^\infty e^{-tQ} dU_t\right)$ is (b,Q,a)-semi-stable if and only if $\{U_t\}$ is a (b,Q,a)-semi-stable Lévy process. The statement with the word 'strictly' added in both conditions is also true.*

Proof. Let $\mu = \mathcal{L}(U_1)$ and $\rho = \mathcal{L}\left(\int_0^\infty e^{-tQ} dU_t\right)$. Since $\{U_t\}$ is a Lévy process, it is (b,Q,a)-semi-stable if μ is (b,Q,a)-semi-stable. We have, by Theorem 5.2 of [18],

$$\int_0^\infty \sup_{|z|\leqslant a} |\log \widehat{\mu}(e^{-tQ'}z)| dt < \infty \quad \text{for } a \in (0,\infty)$$

and $\log \widehat{\rho}(z) = \int_0^\infty \log \widehat{\mu}(e^{-tQ'}z) dt$.

If μ is (b,Q,a)-semi-stable, then, with some γ,

$$\begin{aligned} \log \widehat{\rho}(b^{Q'}z) &= \int_0^\infty \log \widehat{\mu}(e^{-tQ'}b^{Q'}z) dt = \int_0^\infty (a \log \widehat{\mu}(e^{-tQ'}z) - i\langle \gamma, e^{-tQ'}z\rangle) dt \\ &= a \log \widehat{\rho}(z) - i\langle Q^{-1}\gamma, z\rangle, \end{aligned}$$

that is, ρ is (b,Q,a)-semi-stable.

Conversely, assume that ρ is (b, Q, a)-semi-stable. Then, with some γ,

$$\int_0^\infty \log \widehat{\mu}(e^{-tQ'} b^{Q'} z)dt = a \int_0^\infty \log \widehat{\mu}(e^{-tQ'} z)dt - i\langle \gamma, z \rangle \ .$$

Since z is arbitrary, we have

$$\int_0^\infty \log \widehat{\mu}(b^{Q'} e^{-(t+u)Q'} z)dt = a \int_0^\infty \log \widehat{\mu}(e^{-(t+u)Q'} z)dt - i\langle \gamma, e^{-uQ'} z \rangle$$

for $u \in \mathbb{R}$. That is,

$$\int_u^\infty \log \widehat{\mu}(b^{Q'} e^{-tQ'} z)dt = a \int_u^\infty \log \widehat{\mu}(e^{-tQ'} z)dt - i\langle \gamma, e^{-uQ'} z \rangle \ .$$

Differentiating in u and letting $u = 0$, we obtain

$$\log \widehat{\mu}(b^{Q'} z) = a \log \widehat{\mu}(z) - i\langle Q\gamma, z \rangle \ ,$$

which shows that μ is (b, Q, a)-semi-stable. The assertion for strict (b, Q, a)-semi-stability is proved with $\gamma = 0$. □

Let us say that μ is Q-semi-stable if μ is (b, Q, a)-semi-stable with some b and a. It is known that the class of Q-semi-stable distributions on $\mathbb{R}^d$ neither includes, nor is included by, the class of Q-semi-selfdecomposable. In the case $Q = I$, this fact is seen from the description of their triplets in [17]. Concerning the intersection of the two classes we have the following assertion.

Proposition 7.4. *A distribution μ on $\mathbb{R}^d$ is Q-selfdecomposable and (b, Q, a)-semi-stable if and only if*

$$\mu = \mathcal{L}\left(\int_0^\infty e^{-tQ} dU_t\right) \tag{7.3}$$

with some (b, Q, a)-semi-stable Lévy process $\{U_t\}$ on $\mathbb{R}^d$. The statement with the word 'strictly' added in both conditions is also true.

Proof. The 'if' part. By Lemma 7.2, the integral in (7.3) is definable. It follows from (7.3) that μ is Q-selfdecomposable by Corollary 6.4 and that μ is (b, Q, a)-semi-stable by Proposition 7.3.

The 'only if' part. By Q-selfdecomposability, μ is represented in the form of (7.3) with a unique (in law) Lévy process $\{U_t\}$ with finite log-moment by Corollary 6.4. Then, using Proposition 7.3, we see that $\{U_t\}$ is (b, Q, a)-semi-stable. The case of strict (b, Q, a)-semi-stability is similar. □

Remark. Let μ be Q-selfdecomposable and (b, Q, a)-semi-stable on $\mathbb{R}^d$. Then μ is (c, Q)-decomposable for any $c \in (0, 1)$. Thus μ has a unique representation

(7.3) with a Lévy process $\{U_t\}$ on one hand and representations (5.2) with natural semi-Lévy processes $\{\widetilde{Y}_t\}$ on the other. Of course (7.3) is a special case of (5.2). That is, $\{U_t\}$ in (7.3) is one of many choices of $\{\widetilde{Y}_t\}$ in (5.2). There is a unique (in law) Q-selfsimilar additive process $\{X_t\}$ with $\mathcal{L}(X_1) = \mu$ and the process $\{U_t\}$ is connected with this $\{X_t\}$ (Theorems 6.1–6.3 and Lemma 4.7). We can construct Q-semi-selfsimilar natural additive processes $\{X_t^\sharp\}$ with epoch c^{-1} with $\mathcal{L}(X_1^\sharp) = \mu$ (see Lemma 5.1) and the processes $\{\widetilde{Y}_t\}$ are connected with these $\{X_t^\sharp\}$ (Corollary 5.2). Thus $\{X_t\}$ is a special case of the processes $\{X_t^\sharp\}$. But, in general, no choice of the function $h(t)$ on $[1, c^{-1}]$ in the proof of Lemma 5.1 gives the Q-selfsimilar process $\{X_t\}$. In fact, with $c = b$, it follows from (7.2) that the distribution μ_t in (5.1) satisfies

$$\widehat{\mu}_t(z) = \widehat{\mu}(z)^{1-h(t)}\widehat{\mu}(b^{-Q'}z)^{h(t)} = \widehat{\mu}(z)^{1-h(t)+a^{-1}h(t)}e^{i\langle a^{-1}h(t)b^{-Q}\gamma,z\rangle}$$

for $1 \leqslant t \leqslant b^{-1}$, since $\widehat{\mu}(b^{-Q'}z) = \widehat{\mu}(z)^{a^{-1}}e^{i\langle a^{-1}b^{-Q}\gamma,z\rangle}$. If this system $\{\mu_t\}$ satisfies $\mu_t = \mathcal{L}(X_t)$ for a Q-selfsimilar additive process $\{X_t\}$, then $\widehat{\mu}_{rt}(z) = \widehat{\mu}_t(r^{Q'}z)$ for $t > 0$ and $r > 0$ and hence $\widehat{\mu}_t(z) = \widehat{\mu}(t^{Q'}z)$. Thus, in this case, the Lévy measure ν of μ satisfies, for $1 \leqslant t \leqslant b^{-1}$,

$$\nu(t^{-Q}B) = \nu_t(B) = (1 + (a^{-1} - 1)h(t))\,\nu(B), \qquad B \in \mathcal{B}(\mathbb{R}^d),$$

where ν_t is the Lévy measure of μ_t. In general, no choice of the function $h(t)$ validates this relation. For example, consider a (b, I, b^α)-semi-stable distribution μ with Lévy measure $\nu = \sum_{n=-\infty}^{\infty} b^{n\alpha}\delta_{b^{-n}c}$ with $0 < \alpha < 2$ and $1 < |c| \leqslant b^{-1}$ in Remark 14.4 of [17]. Then

$$\nu(t^{-I}B) = \sum_{n=-\infty}^{\infty} b^{n\alpha}\delta_{tb^{-n}c}(B)$$

while

$$(1 + (b^{-\alpha} - 1)h(t))\,\nu(B) = (1 + (b^{-\alpha} - 1)h(t))\sum_{n=-\infty}^{\infty} b^{n\alpha}\delta_{b^{-n}c}(B).$$

In the rest of this section we consider some examples appearing in the study of diffusion processes in random environments. It consists of two parts.

1. Let $\{X_t \colon t \geqslant\}$ be a c-semi-selfsimilar process on $\mathbb{R}$ with epoch a, that is,

$$\{X_{at}\} \stackrel{\mathrm{d}}{=} \{a^c X_t\}, \tag{7.4}$$

where $c > 0$ and $a > 1$. Assume that $\{X_t\}$ has cadlag paths and that

$$\limsup_{t\to\infty}\left(X_t - \inf_{s\leqslant t} X_s\right) = \infty \quad \text{a.s.} \tag{7.5}$$

Define, for $t \geqslant 0$,

$$M_t = \inf\{u \geqslant 0\colon X_u - \inf_{s \leqslant u} X_s \geqslant t\} ,$$
$$V_t = -\inf\{X_s\colon s \leqslant M_t\} ,$$
$$N_t = \inf\{u \in [0, M_t]\colon X_u \wedge X_{u-} = -V_t\} ,$$

where we understand $X_{0-} = X_0$. Let us show that $\{(M_t, V_t, N_t)'\colon t \geqslant 0\}$ is a Q-semi-selfsimilar process on $\mathbb{R}^3$ with some Q. Denote by $\mathrm{diag}(a_1, \ldots, a_d)$ a $d \times d$ diagonal matrix with (j, j)-entry equal to a_j.

Proposition 7.5. *Let $Q = \mathrm{diag}(c^{-1}, 1, c^{-1})$. Under the assumptions above,*

$$\{(M_{a^c t}, V_{a^c t}, N_{a^c t})'\colon t \geqslant 0\} \stackrel{\mathrm{d}}{=} \{a^{cQ}(M_t, V_t, N_t)'\colon t \geqslant 0\} , \tag{7.6}$$

that is, the process $\{(M_t, V_t, N_t)'\}$ is Q-semi-selfsimilar with epoch a^c.

Proof. Rewriting (7.4) to $\{a^{-c}X_t\} \stackrel{\mathrm{d}}{=} \{X_{a^{-1}t}\}$, we get $M_{a^c t} \stackrel{\mathrm{d}}{=} aM_t$, since

$$M_{a^c t} = \inf\left\{u \geqslant 0\colon a^{-c}\left(X_u - \inf_{s \leqslant u} X_s\right) \geqslant t\right\} \stackrel{\mathrm{d}}{=} \inf\{u \geqslant 0\colon X_{a^{-1}u} - \inf_{s \leqslant u} X_{a^{-1}s} \geqslant t\}$$
$$= \inf\{u \geqslant 0\colon X_{a^{-1}u} - \inf_{s \leqslant a^{-1}u} X_s \geqslant t\} = a \inf\{u \geqslant 0\colon X_u - \inf_{s \leqslant u} X_s \geqslant t\} = aM_t .$$

Similarly we have $V_{a^c t} \stackrel{\mathrm{d}}{=} a^c V_t$ and $N_{a^c t} \stackrel{\mathrm{d}}{=} aN_t$ in the following way:

$$V_{a^c t} = -\inf\{X_s\colon s \leqslant M_{a^c t}\} = -a^c \inf\{a^{-c}X_s\colon s \leqslant M_{a^c t}\}$$
$$\stackrel{\mathrm{d}}{=} -a^c\{X_{a^{-1}s}\colon s \leqslant aM_t\} = -a^c V_t ,$$
$$N_{a^c t} = \inf\{u \in [0, M_{a^c t}]\colon X_u \wedge X_{u-} = -V_{a^c t}\}$$
$$= \inf\{u \in [0, M_{a^c t}]\colon (a^{-c}X_u) \wedge (a^{-c}X_{u-}) = -a^{-c}V_{a^c t}\}$$
$$\stackrel{\mathrm{d}}{=} \inf\{u \in [0, aM_t]\colon X_{a^{-1}u} \wedge X_{a^{-1}u-} = -V_t\} = aN_t .$$

In checking the identities in law above, the only transformation involved is that of $\{a^{-c}X_t\} \stackrel{\mathrm{d}}{=} \{X_{a^{-1}t}\}$. Hence the same proof applies to the joint distributions of the three processes $\{M_t\}$, $\{X_t\}$, and $\{N_t\}$. Thus we get $\{(M_{a^c t}, V_{a^c t}, N_{a^c t})'\} \stackrel{\mathrm{d}}{=} \{(aM_t, a^c V_t, aN_t)'\}$. Since $a^{cQ} = \mathrm{diag}(a, a^c, a)$, this means (7.6). □

2. Suppose that $\{X_t\colon t \geqslant 0\}$ is a strictly (b, I, b^α)-semi-stable Lévy process on $\mathbb{R}$ with $0 < b < 1$ and $0 < \alpha \leqslant 2$. That is, $\{X_t\}$ is a Lévy process satisfying $\{X_{b^\alpha t}\} \stackrel{\mathrm{d}}{=} \{bX_t\}$. Hence, $\{X_t\}$ is a c-semi-selfsimilar Lévy process on $\mathbb{R}$ with epoch a, where $c = \alpha^{-1}$ and $a = b^{-\alpha}$. Approaching a generalization of Tanaka's paper [24] on diffusion processes in selfsimilar environments, Takahashi [23] obtains the following results for this process.

Proposition 7.6. *Assume that* $\{X_t\}$ *satisfies* (7.5). *Then* $\{N_t\}$ *is an additive process* (*hence, it is an* α*-semi-selfsimilar additive process with epoch* b^{-1}).

Proposition 7.7. *Assume, in addition to* (7.5), *that* $\{X_t\}$ *does not have positive jumps. Then* $\{M_t\}$ *and* $\{V_t\}$ *are also additive processes* (*hence,* $\{M_t\}$ *and* $\{V_t\}$ *are, respectively,* α*-semi-selfsimilar and* 1*-semi-selfsimilar additive processes with epoch* b^{-1}).

A process $\{X_t\}$ on $\mathbb{R}$ satisfies the assumptions in Proposition 7.7 if and only if $\{X_t\}$ is either a nonzero constant multiple of Brownian motion or a nonzero strictly (b, I, b^{α})-semi-stable Lévy process with $0 < b < 1$ and $1 < \alpha < 2$ having Lévy measure concentrated on the negative axis.

K. Kawazu finds that, in the case of Brownian motion on $\mathbb{R}$, the process $\{(V_t, N_t)\}$ is an additive process on $\mathbb{R}^2$ but the process $\{(M_t, V_t, N_t)\}$ is not an additive process on $\mathbb{R}^3$ (see Example 3.3 of [16]). We do not know to what extent this fact can be generalized.

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Makoto Maejima
Department of Mathematics
Keio University
Hiyoshi
Yokohama 223-8522
Japan
maejima@math.keio.ac.jp

Ken-iti Sato

Hachiman-yama 1101-5-103

Tenpaku-ku

Nagoya 468-0074

Japan

ken-iti.sato@nifty.ne.jp