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by

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Normalized Double Sine Functions

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Abstract

We express normalized double sine functions of integer periods (N_1, N_2) via the standard double sine function of period $(1, 1)$. As an application we give an Euler product expression using the di-logarithm for the double zeta function $\zeta(s, \mathbf{F}_p^{N_1}) \otimes \zeta(s, \mathbf{F}_p^{N_2})$ for a prime number p and integers N_1, N_2 .

1 Definitions and Results

Normalized multiple sine functions are generalizations of the usual sine function. We studied their basic properties in previous papers [KuKo] [KoKu1] [KoKu2] with some applications.

For $\omega_1, \dots, \omega_r > 0$ and $x > 0$, the multiple Hurwitz zeta function is defined by Barnes [B] as

$$\zeta_r(s, x, (\omega_1, \dots, \omega_r)) = \sum_{n_1, \dots, n_r=0}^{\infty} (n_1\omega_1 + \dots + n_r\omega_r + x)^{-s}$$

in $\text{Re}(s) > r$. This has the analytic continuation to all $s \in \mathbb{C}$ as a meromorphic function, and it is holomorphic at $s = 0$. Then the normalized multiple gamma function is defined as

$$\Gamma_r(x, (\omega_1, \dots, \omega_r)) = \exp \left(\left. \frac{\partial}{\partial s} \zeta_r(s, x, (\omega_1, \dots, \omega_r)) \right|_{s=0} \right).$$

This is a constant multiple of the multiple gamma function $\Gamma_r^B(x, (\omega_1, \dots, \omega_r))$ of Barnes [B]:

$$\Gamma_r(x, (\omega_1, \dots, \omega_r)) = \Gamma_r^B(x, (\omega_1, \dots, \omega_r)) / \rho_r(\omega_1, \dots, \omega_r).$$

Now, the normalized multiple sine function is

$$S_r(x, (\omega_1, \dots, \omega_r)) = \Gamma_r(x, (\omega_1, \dots, \omega_r))^{-1} \Gamma_r(\omega_1 + \dots + \omega_r - x, (\omega_1, \dots, \omega_r))^{(-1)^r}.$$

For example

$$\begin{aligned} S_1(x, \omega) &= \Gamma_1(x, \omega)^{-1} \Gamma_1(\omega - x, \omega)^{-1} \\ &= 2 \sin(\pi x / \omega), \end{aligned}$$

since we have $\Gamma_1(x, \omega) = (2\pi)^{-1/2} \Gamma(x/\omega) \omega^{\frac{x}{\omega} - \frac{1}{2}}$ from $\zeta_1(s, x, \omega) = \omega^{-s} \zeta(s, x/\omega)$.

To simplify the notation we put $S_r(x) = S_r(x, (1, \dots, 1))$, $\Gamma_r(x) = \Gamma_r(x, (1, \dots, 1))$ and $\zeta_r(s, x) = \zeta_r(s, x, (1, \dots, 1))$. Hence

$$S_r(x) = \Gamma_r(x)^{-1} \Gamma_r(r - x)^{(-1)^r}$$

and

$$\Gamma_r(x) = \exp \left(\left. \frac{\partial}{\partial s} \zeta_r(s, x) \right|_{s=0} \right).$$

Here we investigate normalized double sine functions, especially in the rational periods case: $S_2(x, (\omega_1, \omega_w))$ with $\omega_2/\omega_1 \in \mathbb{Q}$. The following theorem expresses them in terms of $S_2(x)$:

Theorem 1.1 *Let N_1, N_2 be positive integers with the greatest common divisor N_0 . Then we have*

$$S_2(x, (N_1, N_2)) = \prod_{k_1=0}^{(N_2/N_0)-1} \prod_{k_2=0}^{(N_1/N_0)-1} S_2 \left(\frac{x + N_1 k_1 + N_2 k_2}{N_1 N_2 / N_0} \right). \quad (1.1)$$

As application of Theorem 1.1, we compute the absolute tensor product of the Hasse zeta functions of finite fields with p^{N_1} and p^{N_2} elements:

Theorem 1.2 *Let N_1, N_2 be positive integers with the greatest common divisor N_0 . The absolute tensor product of the Hasse zeta functions for finite fields $\mathbf{F}_{p^{N_1}}$ and $\mathbf{F}_{p^{N_2}}$ is given as follows:*

$$\begin{aligned} &\zeta(s, \mathbf{F}_{p^{N_1}}) \otimes \zeta(s, \mathbf{F}_{p^{N_2}}) \\ &= \exp \left(-\frac{1}{2\pi i} \frac{N_0^2}{N_1 N_2} \sum_{n=1}^{\infty} \frac{p^{-snN_1N_2/N_0}}{n^2} + \left(\frac{isN_0 \log p}{2\pi} - 1 \right) \sum_{n=1}^{\infty} \frac{p^{-snN_1N_2/N_0}}{n} \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \frac{p^{-snN_1}}{n} f_1(n) + \sum_{n=1}^{\infty} \frac{p^{-snN_2}}{n} f_2(n) + Q_p(s) \right), \end{aligned}$$

where

$$f_1(n) = \begin{cases} (e^{2\pi i n N_1 / N_2} - 1)^{-1} & \left(\frac{N_2}{N_0} \mid n \right) \\ \frac{N_2 - N_0}{2N_0} & \left(\frac{N_2}{N_0} \nmid n \right) \end{cases}, \quad f_2(n) = \begin{cases} (e^{2\pi i n N_2 / N_1} - 1)^{-1} & \left(\frac{N_1}{N_0} \mid n \right) \\ \frac{N_1 - N_0}{2N_0} & \left(\frac{N_1}{N_0} \nmid n \right) \end{cases}$$

and $Q_p(s)$ is a quadratic polynomial in s .

2 Proof of Theorem 1.1

It suffices to show when $N_0 = 1$, since the homogeneity [KuKo, Theorem 2.1(e)] of the multiple sine functions gives

$$S_2(x, (N_1, N_2)) = S_2\left(\frac{x}{N_0}, \left(\frac{N_1}{N_0}, \frac{N_2}{N_0}\right)\right).$$

In case $N_0 = 1$, the right hand side of (1.1) is calculated as follows:

$$\begin{aligned} & \prod_{k_1=0}^{N_2-1} \prod_{k_2=0}^{N_1-1} S_2\left(\frac{x + N_1 k_1 + N_2 k_2}{N_1 N_2}\right) \\ = & \prod_{k_1=0}^{N_2-1} \prod_{k_2=0}^{N_1-1} \Gamma_2\left(\frac{x + N_1 k_1 + N_2 k_2}{N_1 N_2}\right)^{-1} \Gamma_2\left(2 - \frac{x + N_1 k_1 + N_2 k_2}{N_1 N_2}\right) \\ = & \prod_{k_1=0}^{N_2-1} \prod_{k_2=0}^{N_1-1} \exp\left(\left.\frac{\partial}{\partial s}\right|_{s=0} \left(-\zeta_2\left(s, \frac{x + N_1 k_1 + N_2 k_2}{N_1 N_2}\right) + \zeta_2\left(s, 2 - \frac{x + N_1 k_1 + N_2 k_2}{N_1 N_2}\right)\right)\right) \\ = & \exp\left(\left.\frac{\partial}{\partial s}\right|_{s=0} \sum_{k_1=0}^{N_2-1} \sum_{k_2=0}^{N_1-1} \left(-\zeta_2\left(s, \frac{x + N_1 k_1 + N_2 k_2}{N_1 N_2}\right) + \zeta_2\left(s, 2 - \frac{x + N_1 k_1 + N_2 k_2}{N_1 N_2}\right)\right)\right) \end{aligned} \quad (2.1)$$

The double sum is computed as follows:

$$\begin{aligned} & \sum_{k_1=0}^{N_2-1} \sum_{k_2=0}^{N_1-1} \left(-\sum_{m_1, m_2 \geq 0} \left(m_1 + m_2 + \frac{x + N_1 k_1 + N_2 k_2}{N_1 N_2}\right)^{-s} \right. \\ & \quad \left. + \sum_{m_1, m_2 \geq 0} \left(m_1 + m_2 + 2 - \frac{x + N_1 k_1 + N_2 k_2}{N_1 N_2}\right)^{-s}\right) \\ = & (N_1 N_2)^s \sum_{k_1=0}^{N_2-1} \sum_{k_2=0}^{N_1-1} \left(-\sum_{m_1, m_2 \geq 0} \left((m_1 N_2 + k_1)N_1 + (m_2 N_1 + k_2)N_2 + x\right)^{-s} \right. \\ & \quad \left. + \sum_{m_1, m_2 \geq 0} \left((m_1 N_2 + N_2 - k_1 - 1)N_1 + (m_2 N_1 + N_1 - k_2 - 1)N_2 + N_1 + N_2 - x\right)^{-s}\right) \\ = & (N_1 N_2)^s \left(-\zeta_2(s, x, (N_1, N_2)) + \zeta_2(s, N_1 + N_2 - x, (N_1, N_2))\right). \end{aligned}$$

We previously obtained in the proof of [KuKo, Theorem 2.1(b)] that the function

$$-\zeta_2(s, x, (N_1, N_2)) + \zeta_2(s, N_1 + N_2 - x, (N_1, N_2))$$

has zeros at even nonnegative integers s . In particular it vanishes at $s = 0$, thus (2.1) equals

$$\begin{aligned} & \exp\left(\left.\frac{\partial}{\partial s}\right|_{s=0}\left(-\zeta_2(s, x, (N_1, N_2)) + \zeta_2(s, N_1 + N_2 - x, (N_1, N_2))\right)\right) \\ &= \Gamma_2(x, (N_1, N_2))^{-1} \Gamma_2(N_1 + N_2 - x, (N_1, N_2)) \\ &= S_2(x, (N_1, N_2)). \end{aligned}$$

This completes the proof of Theorem 1.1. ■

3 Application

In this section we compute the absolute tensor product of the Hasse zeta functions of finite fields $\mathbf{F}_{p^{N_1}}$ and $\mathbf{F}_{p^{N_2}}$. We first recall the definition of the absolute tensor product of meromorphic functions. Let Z_j ($j = 1, 2$) be meromorphic functions of order μ_j . We put the Hadamard product as

$$Z_j(s) = s^{k_j} e^{Q_j(s)} \prod'_{\rho \in \mathbb{C}} P_{\mu_j} \left(\frac{s}{\rho} \right)^{m_j(\rho)}, \quad (3.1)$$

where $P_r(u) := (1 - u) \exp(u + \frac{u^2}{2} + \cdots + \frac{u^r}{r})$, m_j denotes the multiplicity function with $k_j := m_j(0)$, and Q_j is a polynomial with $\deg Q_j \leq \mu_j$. Here the product over $\rho \in \mathbb{C}$ means $\lim_{R \rightarrow \infty} \prod_{0 < |\rho| < R} P_{\mu_j} \left(\frac{s}{\rho} \right)^{m_j(\rho)}$. The absolute tensor product is defined by

$$(Z_1 \otimes Z_2)(s) := s^{k_1 k_2} e^{Q(s)} \prod'_{\rho_1, \rho_2 \in \mathbb{C}} P_{\mu_1 + \mu_2} \left(\frac{s}{\rho_1 + \rho_2} \right)^{m(\rho_1, \rho_2)}, \quad (3.2)$$

where $Q(s)$ is a polynomial with $\deg Q \leq \mu_1 + \mu_2$ and

$$m(\rho_1, \rho_2) := m_1(\rho_1) m_2(\rho_2) \times \begin{cases} 1 & \text{if } \operatorname{Im}(\rho_1), \operatorname{Im}(\rho_2) \geq 0, \\ -1 & \text{if } \operatorname{Im}(\rho_1), \operatorname{Im}(\rho_2) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here we do not give the precise definition of the polynomial $Q(s)$, since it is not necessary for our purpose.

In this section we will compute this absolute tensor product for the Hasse zeta functions for finite fields:

$$\begin{aligned} Z_1(s) &= \zeta(s, \mathbf{F}_{p^{N_1}}) = (1 - p^{-N_1 s})^{-1}, \\ Z_2(s) &= \zeta(s, \mathbf{F}_{p^{N_2}}) = (1 - p^{-N_2 s})^{-1}, \end{aligned}$$

with p a prime number and N_1, N_2 positive integers.

The following theorem extends our previous results on $\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q)$ for a prime p in [KoKu2].

Proposition 3.1 *The absolute tensor product of the Hasse zeta functions for finite fields $\mathbf{F}_{p^{N_1}}$ and $\mathbf{F}_{p^{N_2}}$ is given as follows:*

$$\begin{aligned} \zeta(s, \mathbf{F}_{p^{N_1}}) \otimes \zeta(s, \mathbf{F}_{p^{N_2}}) &= e^{Q(s)} S_2 \left(\frac{isN_1N_2 \log p}{2\pi}, (N_1, N_2) \right) \\ &= e^{Q(s)} \prod_{k_1=0}^{(N_2/N_0)-1} \prod_{k_2=0}^{(N_1/N_0)-1} S_2 \left(N_0 \left(\frac{is \log p}{2\pi} + \frac{k_1}{N_2} + \frac{k_2}{N_1} \right) \right), \end{aligned}$$

where $N_0 = (N_1, N_2)$ and $Q(s)$ is a polynomial of degree at most two, which depends on p .

Proof. The second equality is seen from Theorem 1.1. In what follows we prove the first one. The Hadamard product (3.1) for the Hasse zeta function is given for $j = 1, 2$ by

$$\zeta(s, \mathbf{F}_{p^{N_j}}) = s^{-1} e^{\tilde{Q}_{N_j}(s)} \prod'_{n=-\infty}^{\infty} P_1 \left(\frac{s}{\frac{2\pi i}{N_j \log p} n} \right)^{-1}$$

with $\tilde{Q}_{p,j}(s)$ a linear polynomial depending on N_j . Thus by the definition (3.2) of the absolute tensor product,

$$\zeta(s, \mathbf{F}_{p^{N_1}}) \otimes \zeta(s, \mathbf{F}_{p^{N_2}}) = s e^{\tilde{Q}_{N_1, N_2}(s)} \prod'_{k, n \in \mathbf{Z}} P_2 \left(\frac{s}{\frac{2\pi i}{N_1 \log p} k + \frac{2\pi i}{N_2 \log p} n} \right)^{m_{k, n}},$$

where $\tilde{Q}_{N_1, N_2}(s)$ is a polynomial of degree at most two and

$$m_{k, n} := m \left(\frac{2\pi i}{N_1 \log p} k, \frac{2\pi i}{N_2 \log p} n \right) = \begin{cases} 1 & \text{if } k, n \geq 0 \\ -1 & \text{if } k, n < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\zeta(s, \mathbf{F}_{p^{N_1}}) \otimes \zeta(s, \mathbf{F}_{p^{N_2}}) = e^{\tilde{Q}_{N_1, N_2}(s)} s \frac{\prod'_{k, n=0}^{\infty} P_2 \left(\frac{s}{\frac{2\pi i}{N_1 \log p} k + \frac{2\pi i}{N_2 \log p} n} \right)}{\prod'_{k, n=1}^{\infty} P_2 \left(-\frac{s}{\frac{2\pi i}{N_1 \log p} k + \frac{2\pi i}{N_2 \log p} n} \right)}.$$

We appeal to the $r = 2$ case of the formula [KuKo, Proposition 2.4]:

$$S_2(z, (\omega_1, \omega_2)) = e^{Q_{\underline{\omega}}(z)} z \frac{\prod'_{k,n=0}^{\infty} P_2\left(-\frac{z}{\omega_1 k + \omega_2 n}\right)}{\prod_{k,n=1}^{\infty} P_2\left(\frac{z}{\omega_1 k + \omega_2 n}\right)}$$

where $Q_{\underline{\omega}}(z)$ a polynomial with $\deg Q_{\underline{\omega}} \leq 2$. The proof is complete. ■

Taking the following exponential expression of the normalized double sine functions into account, we see that the absolute tensor product in Theorem 1.2 has an “Euler product” expression for $\operatorname{Re}(s) > 0$.

Proposition 3.2 *The following expression holds for $\operatorname{Im}z > 0$.*

$$S_2(z) = \exp\left(-\frac{1}{2\pi i} \operatorname{Li}_2(e^{2\pi iz}) + (1-z) \log(1 - e^{2\pi iz}) + Q(z)\right),$$

where $Q(z) = \frac{\pi i}{2} z^2 - \pi iz + \frac{5\pi i}{12}$.

Proof. We recall the formulas of the double sine functions:

$$S_2(z) = \mathcal{S}_2(z)^{-1} \mathcal{S}_1(z), \quad ([\text{KuKo, Example 3.6}]) \quad (3.3)$$

where $\mathcal{S}_r(z)$ ($r = 1, 2$) are the primitive multiple sine functions [KuKo]. We have by definition

$$\mathcal{S}_1(z) = 2 \sin \pi z = \exp\left(-\pi iz + \frac{\pi i}{2} + \log(1 - e^{2\pi iz})\right)$$

and the expression [KuKo, Theorem 2.8 (2.12)]:

$$\mathcal{S}_2(z) = \exp\left(\frac{1}{2\pi i} \operatorname{Li}_2(e^{2\pi iz}) + z \log(1 - e^{2\pi iz}) - \frac{\pi i}{2} z^2 - \frac{\zeta(2)}{2\pi i}\right)$$

for $\operatorname{Im}(z) > 0$. Thus (3.3) equals

$$\exp\left(-\frac{1}{2\pi i} \operatorname{Li}_2(e^{2\pi iz}) + (1-z) \log(1 - e^{2\pi iz}) + \frac{\pi i}{2} z^2 - \pi iz + \frac{5\pi i}{12}\right). \blacksquare$$

Lemma 3.3 *Assume $r \in \mathbb{C}$ satisfies that $r^N = 1$. Then we have*

$$\frac{1}{N} \sum_{n=0}^{N-1} nr^n = \begin{cases} (r-1)^{-1} & (r \neq 1) \\ \frac{N-1}{2} & (r = 1). \end{cases}$$

Proof. The $r = 1$ case is well-known. Differentiating the formula $\sum_{n=0}^{N-1} r^n = (1-r^N)/(1-r)$ in case $r \neq 1$ leads to the result. ■

Proof of Theorem 1.2: By the above propositions all we should compute is the following product:

$$\prod_{k_1=0}^{(N_2/N_0)-1} \prod_{k_2=0}^{(N_1/N_0)-1} S_2 \left(N_0 \left(\frac{is \log p}{2\pi} + \frac{k_1}{N_2} + \frac{k_2}{N_1} \right) \right)$$

$$= \exp \left(\sum_{k_1=0}^{(N_2/N_0)-1} \sum_{k_2=0}^{(N_1/N_0)-1} \left(-\frac{1}{2\pi i} \text{Li}_2 \left(e^{-N_0 s \log p + 2\pi i \left(\frac{N_0 k_1}{N_2} + \frac{N_0 k_2}{N_1} \right)} \right) \right) \right) \quad (3.4)$$

$$+ \left(1 - \frac{is N_0 \log p}{2\pi} - \frac{N_0 k_1}{N_2} - \frac{N_0 k_2}{N_1} \right) \log \left(1 - e^{-s N_0 \log p + 2\pi i \left(\frac{N_0 k_1}{N_2} + \frac{N_0 k_2}{N_1} \right)} \right) \quad (3.5)$$

$$+ Q \left(\frac{is N_0 \log p}{2\pi} + \frac{N_0 k_1}{N_2} + \frac{N_0 k_2}{N_1} \right). \quad (3.6)$$

Put $N_1 = N_0 N'_1$ and $N_2 = N_0 N'_2$. First the double sum in (3.4) is computed as follows:

$$- \sum_{k_1=0}^{N'_2-1} \sum_{k_2=0}^{N'_1-1} \frac{1}{2\pi i} \text{Li}_2 \left(e^{-N_0 s \log p + 2\pi i \left(\frac{k_1}{N'_2} + \frac{k_2}{N'_1} \right)} \right)$$

$$= -\frac{1}{2\pi i} \sum_{k_1=0}^{N'_2-1} \sum_{k_2=0}^{N'_1-1} \sum_{n=1}^{\infty} \frac{\left(p^{-s N_0} e^{2\pi i \left(\frac{k_1}{N'_2} + \frac{k_2}{N'_1} \right)} \right)^n}{n^2}$$

$$= -\frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{p^{-s N_0 n}}{n^2} \left(\sum_{k_1=0}^{N'_2-1} e^{2\pi i \frac{n k_1}{N'_2}} \right) \left(\sum_{k_2=0}^{N'_1-1} e^{2\pi i \frac{n k_2}{N'_1}} \right)$$

$$= \frac{1}{2\pi i} - \sum_{N'_1 N'_2 | n} \frac{p^{-s N_0 n}}{n^2} N'_1 N'_2$$

$$= \frac{1}{2\pi i} - \frac{1}{N'_1 N'_2} \sum_{n=1}^{\infty} \frac{p^{-s N_0 N'_1 N'_2 n}}{n^2}$$

$$= \frac{1}{2\pi i} - \frac{N_0^2}{N_1 N_2} \sum_{n=1}^{\infty} \frac{p^{-s n N_1 N_2 / N_0}}{n^2}.$$

Next we compute the double sum of (3.5) over k_1, k_2 :

$$\begin{aligned}
& \sum_{k_1=0}^{N'_2-1} \sum_{k_2=0}^{N'_1-1} \left(1 - \frac{isN_0 \log p}{2\pi} - \frac{k_1}{N'_2} - \frac{k_2}{N'_1} \right) \log \left(1 - e^{-sN_0 \log p + 2\pi i \left(\frac{k_1}{N'_2} + \frac{k_2}{N'_1} \right)} \right) \\
&= - \left(1 - \frac{isN_0 \log p}{2\pi} \right) \sum_{n=1}^{\infty} \frac{p^{-sN_0 n}}{n} \sum_{k_1=0}^{N'_2-1} e^{2\pi i \frac{nk_1}{N'_2}} \sum_{k_2=0}^{N'_1-1} e^{2\pi i \frac{nk_2}{N'_1}} \\
&\quad + \sum_{n=1}^{\infty} \frac{p^{-sN_0 n}}{n} \left(\sum_{k_1=0}^{N'_2-1} \sum_{k_2=0}^{N'_1-1} \left(\frac{k_1}{N'_2} + \frac{k_2}{N'_1} \right) e^{2\pi i \left(\frac{k_1}{N'_2} + \frac{k_2}{N'_1} \right) n} \right) \\
&= \left(\frac{isN_0 \log p}{2\pi} - 1 \right) \sum_{n=1}^{\infty} \frac{p^{-sN_0 n N'_1 N'_2}}{n} \\
&\quad + \sum_{n=1}^{\infty} \frac{p^{-sN_0 n}}{n} \left(\sum_{k_1=0}^{N'_2-1} \frac{k_1}{N'_2} e^{2\pi i \frac{k_1 n}{N'_2}} \sum_{k_2=0}^{N'_1-1} e^{2\pi i \frac{k_2 n}{N'_1}} + \sum_{k_2=0}^{N'_1-1} \frac{k_2}{N'_1} e^{2\pi i \frac{k_2 n}{N'_1}} \sum_{k_1=0}^{N'_2-1} e^{2\pi i \frac{k_1 n}{N'_2}} \right) \\
&= \left(\frac{isN_0 \log p}{2\pi} - 1 \right) \sum_{n=1}^{\infty} \frac{p^{-sN_0 n N'_1 N'_2}}{n} \\
&\quad + \sum_{n=1}^{\infty} \frac{p^{-snN_0 N'_1}}{n N'_1} \sum_{k_1=0}^{N'_2-1} \frac{k_1}{N'_2} e^{2\pi i \frac{k_1 n N'_1}{N'_2}} N'_1 + \sum_{n=1}^{\infty} \frac{p^{-snN_0 N'_2}}{n N'_2} \sum_{k_2=0}^{N'_1-1} \frac{k_2}{N'_1} e^{2\pi i \frac{k_2 n N'_2}{N'_1}} N'_2 \\
&= \left(\frac{isN_0 \log p}{2\pi} - 1 \right) \sum_{n=1}^{\infty} \frac{p^{-sN_0 n N'_1 N'_2}}{n} \\
&\quad + \sum_{n=1}^{\infty} \frac{p^{-snN_0 N'_1}}{n N'_2} \sum_{k_1=0}^{N'_2-1} k_1 e^{2\pi i \frac{k_1 n N'_1}{N'_2}} + \sum_{n=1}^{\infty} \frac{p^{-snN_0 N'_2}}{n N'_1} \sum_{k_2=0}^{N'_1-1} k_2 e^{2\pi i \frac{k_2 n N'_2}{N'_1}} \\
&= \left(\frac{isN_0 \log p}{2\pi} - 1 \right) \sum_{n=1}^{\infty} \frac{p^{-snN_1 N_2 / N_0}}{n} + \sum_{n=1}^{\infty} \frac{p^{-snN_1}}{n} f_1(n) + \sum_{n=1}^{\infty} \frac{p^{-snN_2}}{n} f_2(n)
\end{aligned}$$

by Lemma 3.3. ■

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