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Remarks on the Conformally Invariant Quantization by means of a Finsler Function

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Abstract

Let (M, F) be a Finsler manifold. We construct a 1-cocycle on $\mathrm{Diff}(M)$ with values in the space of differential operators acting on tensor fields, by means of the Finsler function F. This is a first step toward the existence of Schwarzian derivatives for Finsler structure. We, furthermore, discuss some properties of the conformally invariant quantization map by means of a (Sazaki type) metric on the slit bundle $TM\backslash 0$ induced by F.

1 Introduction

The notion of equivariant quantization was first introduced by Duval-Lecomte-Ovsienko in [9, 10, 12]. It is defined as an identification between the space of symbols and the the space of differential operators, equivariant with respect to a (finite dimensional) group G acting locally on a manifold M–see also [3, 6, 8, 11, 14] for related works. The computation was carried out for the projective group $\mathrm{SL}(n+1, \mathbb{R})$ and the conformal group $\mathrm{O}(p+1, q+1)$ in [9, 10, 12]. It turns out that these maps make sense on any manifold, not necessarily flat (see [6, 10]). For example, the conformally equivariant map has the property that it does not depend on the rescaling of the (not necessarily conformally flat) pseudo-Riemannian metric. The existence of such maps induces naturally cohomology classes on the group $\mathrm{Diff}(M)$ with values in the space of differential operators acting on tensor fields. These classes were given explicitly in [4, 5, 7], and interpreted as *projective and conformal* multi-dimensional analogous of the celebrate Schwarzian derivative (see [4, 5, 7] for more details).

A Riemannian metric is a particular case of more general functions called *Finsler functions.* Recall that a Finsler function F is a function on the tangent bundle TM satisfying some extra conditions (see section 2)–see [1] who provide some examples of Finsler functions appearing in Physics. In this paper, we generalize one of the two 1-cocycles introduced in [4] as conformal Schwarzian derivatives, to the more general framework of Finsler structure. In particular, this new 1-cocycle coincide with the conformal 1-cocycle when the Finsler function is Riemannian. However, the generalization of the second 1-cocycle is still unknown.

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The Finsler function gives rise to a Riemannian metric, say $\mathbf{m}$, on the slit bundle $TM\backslash 0$. we shall apply the Duval-Ovsienko quantization procedure through the metric $\mathbf{m}$. That means that we associate with functions on the cotangent bundle of the manifold $TM\backslash 0$, differential operators acting on the space of λ-densities of $TM\backslash 0$. It turns out that this map cannot descend as an operator acting on the space of densities on M unless F is Riemannian. We give its explicit expression when F is Riemannian.

2 Introduction to a Finsler structure

We will follow verbatim the notation of [2]. Let M be a manifold of dimension n. A local system of coordinates $(x^i), i = 1, \dots, n$ on M gives rise to a local system of coordinates (x^i, y^i) on the tangent bundle TM through

$$y = y^i \frac{\partial}{\partial x^i}, \qquad i = 1, \dots, n.$$

A Finsler structure on M is a function $F : TM \to [0, \infty)$ satisfying the following conditions:

(i) the function F is differentiable away from the origin;

(ii) the function F is homogeneous of degree one in y: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$;

(iii) the $n \times n$ matrix

$$\mathrm{g}_{ij} := \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j}(F^2)$$

is positive-definite at every point of $TM\backslash 0$.

Example 2.1 (i) Let (M, a) be a Riemannian manifold. The function $F := \sqrt{a_{ij}y^iy^j}$[1] satisfies the conditions (i), (ii) and (iii). In this case, the Finsler function is called *Riemannian.*

(ii) Let (M, a) be a Riemannian manifold and α a closed 1-form on M. We put $F = \sqrt{a_{ij}y^iy^j} + \alpha_i y^i$. One can prove that F satisfies (i), (ii) and (iii) if and only if $\|\alpha \cdot \alpha\|_a < 1$ (see e.g. [2]). In this case, F is Riemannian if and only if the 1-form α is identically zero.

Denote by π the natural projection $TM\backslash 0 \to M$. We define the pull-back bundle of T^*M by the commutative diagram

$$\begin{array}{ccc} \pi^*(T^*M) & \longrightarrow & T^*M \\ \downarrow & & \downarrow \\ TM\backslash 0 & \xrightarrow{\pi} & M \end{array} \tag{2.1}$$

The components g_{ij} in (iii) of the definition above define a section of the pulled-back bundle $\pi^*(T^*M) \otimes \pi^*(T^*M)$.

The geometric object g in (iii) is called *fundamental* tensor; it depends on x and on y as well.

The (symmetric) tensor

$$A := A_{ijk}\, dx^i \otimes dx^j \otimes dx^k, \tag{2.2}$$

[1] Here and bellow summation is understood over repeated indices.

where $A_{ijk} := F/2 \cdot \partial g_{ij}/\partial y^k$, is called *Cartan* tensor. It defines a section of the pulled-back bundle $(\pi^*(T^*M))^{\otimes 3}$. The Cartan tensor is identically zero if and only if the Finsler function F is Riemannian.

The tensor

$$\omega := \omega_i \, dx^i, \tag{2.3}$$

where $\omega_i := \partial F/\partial y^i$, is called *Hilbert* form. It defines a section of the pulled-back bundle $\pi^*(T^*M)$. Denote by ω^i its dual with respect to the fundamental tensor g.

To simplify the computation, we shall introduce adapted basis for the bundle $T^*(TM\backslash 0)$ and for the bundle $T(TM\backslash 0)$. The following 1-forms and vector fields are dual to each other:

$$\left\{ \begin{array}{lcl} dx^i, & & \\ \dfrac{\delta y^i}{F} & := & \dfrac{1}{F}\left(dy^i + N^i_m \, dx^m\right), \end{array} \right. \qquad \left\{ \begin{array}{lcl} \dfrac{\delta}{\delta x^i} & := & \dfrac{\partial}{\partial x^i} - N^m_i \dfrac{\partial}{\partial y^m}, \\ F\dfrac{\partial}{\partial y^i}, & & \end{array} \right. \tag{2.4}$$

where $N^i_m = 1/4 \; \partial/\partial y^m \; \left(g^{is}\left(\partial g_{sk}/\partial x^j + \partial g_{sj}/\partial x^k - \partial g_{kj}/\partial x^s\right) y^j y^k\right)$. A straightforward computation shows that the above objects behave under coordinate changes as follows: for local changes on M, say (x^i) and their inverses $(\tilde{x}^i)$, one has

$$\left\{ \begin{array}{lcl} \dfrac{\delta}{\delta \tilde{x}^i} & = & \dfrac{\partial x^p}{\partial \tilde{x}^i}\dfrac{\delta}{\delta x^p}, \\ F\dfrac{\partial}{\partial \tilde{y}^i} & = & \dfrac{\partial x^p}{\partial \tilde{x}^i} F \dfrac{\partial}{\partial y^p}, \end{array} \right. \qquad \left\{ \begin{array}{lcl} d\tilde{x}^i & = & \dfrac{\partial \tilde{x}^i}{\partial x^p} dx^p, \\ \dfrac{\delta y^i}{F} & = & \dfrac{\partial \tilde{x}^i}{\partial x^p}\dfrac{\delta y^p}{F}, \end{array} \right. \tag{2.5}$$

(cf. [2]).

Through this paper we will use the following notation: on the manifold $TM\backslash 0$, the index i runs with respect to the basis dx^i or $\delta/\delta x^i$, and the index $\bar{i}$ runs with respect to the basis $\delta y^i/F$ or $F\partial/\partial y^i$.

3 The space of densities and the space of linear differential operators

Let E be a vector bundle of rank p over an (oriented) manifold. We define the space of λ-densities of E as the space of sections of the line bundle $(\wedge^p E)^{\otimes\lambda}$. Denote by $\mathcal{F}_\lambda(M)$ the space of λ-densities of the bundle $T^*M \to M$ and denote by $\mathcal{F}_\lambda(\pi^*(T^*M))$ the space of λ-densities of the bundle $\pi^*(T^*M) \to TM\backslash 0$ (see (2.1)). Both $\mathcal{F}_\lambda(M)$ and $\mathcal{F}_\lambda(\pi^*(T^*M))$ are modules over the group of diffeomorphisms Diff(M) : for $f \in$ Diff(M), $\phi \in \mathcal{F}_\lambda(M)$ and $\tilde{\phi} \in \mathcal{F}_\lambda(\pi^*(T^*M))$, the action is given in local coordinates (x, y) by

$$f^*\phi = \phi \circ f^{-1} \cdot (J_{f^{-1}})^\lambda, \tag{3.1}$$

$$f^*\tilde{\phi} = \tilde{\phi}(f^{-1}, y) \cdot (J_{f^{-1}})^\lambda, \tag{3.2}$$

where $J_f = |Df/Dx|$ is the Jacobian of f.

By differentiating this action, one can obtain the action of the Lie algebra of vector fields Vect(M).

Consider now $\mathcal{D}(\mathcal{F}_\lambda(M), \mathcal{F}_\mu(M))$, the space of linear differential operators

$$T : \mathcal{F}_\lambda(M) \to \mathcal{F}_\mu(M). \tag{3.3}$$

The action of Diff(M) on $\mathcal{D}(\mathcal{F}_\lambda(M), \mathcal{F}_\mu(M))$ depends on the two parameters λ and μ. This action is given by the equation

$$f_{\lambda,\mu}(T) = f^* \circ T \circ f^{*-1}, \tag{3.4}$$

where f^* is the action (3.1) of Diff(M) on $\mathcal{F}_\lambda(M)$.
In the same spirit, we define $\mathcal{D}(\mathcal{F}_\lambda(\pi^*(T^*M)), \mathcal{F}_\mu(\pi^*(T^*M))$, the space of linear differential operator

$$\tilde{T} : \mathcal{F}_\lambda(\pi^*(T^*M)) \to \mathcal{F}_\mu(\pi^*(T^*M)), \tag{3.5}$$

with the action

$$f_{\lambda,\mu}(\tilde{T}) = f^* \circ \tilde{T} \circ f^{*-1}, \tag{3.6}$$

where f^* is the action (3.2) of Diff(M) on $\mathcal{F}_\lambda(\pi^*(T^*M))$.

By differentiating the actions (3.4), (3.6), one can obtain the action of the Lie algebra Vect(M).

The formulæ (3.1), (3.2), (3.4) and (3.6) do not depend on the choice of the system of coordinates.

Denote by $\mathcal{D}^2(\mathcal{F}_\lambda(M), \mathcal{F}_\mu(M))$ the space of second-order linear differential operators with the Diff(M)-module structure given by (3.4). The space $\mathcal{D}^2(\mathcal{F}_\lambda(M), \mathcal{F}_\mu(M))$ is in fact a Diff(M)-submodule of $\mathcal{D}(\mathcal{F}_\lambda(M), \mathcal{F}_\mu(M))$.

Example 3.1 The space of Sturm-Liouville operators $\frac{d^2}{dx^2} + u(x) : \mathcal{F}_{-1/2} \to \mathcal{F}_{3/2}$ on S^1, where $u(x) \in \mathcal{F}_2$ is the potential, is a submodule of $\mathcal{D}^2_{-\frac{1}{2},\frac{3}{2}}(S^1)$ (see [15]).

4 Symbols space

Let N be an oriented manifold. The space of symbols Pol(T^*N) is defined as the space of functions on the cotangent bundle T^*N that are polynomials on the fibers. This space is naturally isomorphic to the space $\oplus_{p\geq 0} S\Gamma(TN^{\otimes p})$ of symmetric contravariant tensor fields on N. In local coordinates (z^i, ξ_i), one can write $P \in$ Pol(T^*N) in the form

$$P = \sum_{l\geq 0} P^{i_1,\dots,i_l} \xi_{i_1} \cdots \xi_{i_l},$$

with $P^{i_1,\dots,i_l}(z) \in C^\infty(N)$.

We define a one parameter family of Diff(N)-module on the space of symbols by

$$\mathrm{Pol}_\delta(T^*N) := \mathrm{Pol}(T^*N) \otimes \mathcal{F}_\delta(N).$$

For $f \in$ Diff(N) and $P \in$ Pol$_\delta(T^*N)$, in local coordinates (z^i), the action is defined by

$$f_\delta(P) \;=\; f^*P \cdot (J_{f^{-1}})^\delta, \tag{4.1}$$

where $J_f = |Df/Dz|$ is the Jacobian of f, and f^* is the natural action of Diff(N) on Pol(T^*N).

We then have a graduation of Diff(N)-modules given by

$$\mathrm{Pol}_\delta(T^*N) = \bigoplus_{k=0}^{\infty} \mathrm{Pol}_\delta^k(T^*N),$$

where $\mathrm{Pol}_\delta^k(T^*N)$ is the space of polynomials of degree k endowed with the Diff(N)-module structure (4.1).

5 Schwarzian derivative for Finsler structures

Let (M, F) be a Finsler manifold. There exists a symmetric connection $D : \Gamma(\pi^*(T^*M)) \to \Gamma(\pi^*(T^*M) \otimes T^*(TM\backslash 0))$ whose Christoffel symbols are given by

$$\gamma_{ij}^k = \frac{1}{2}\mathrm{g}^{ks}\left(\frac{\delta \mathrm{g}_{si}}{\delta x^j} + \frac{\delta \mathrm{g}_{sj}}{\delta x^i} - \frac{\delta \mathrm{g}_{ij}}{\delta x^s}\right),$$

where the operator $\delta/\delta x^i$ is defined in (2.4). This connection is called *Chern connection* and has the following properties:

(i) the connection 1-form has no dy dependence;

(ii) the connection D is almost g-compatible in the sense that $D_s(\mathrm{g}_{ij}) = 0$ and $D_{\bar{s}}(\mathrm{g}_{ij}) = \frac{1}{2}A_{ijs}$, where A is the Cartan tensor (2.2);

(iii) in general, the Chern connection is not a connection on M; however, the Chern connection can descend to a connection on M when F is Riemannian. In this case, it coincides with the Levi-Civita connection associated with the metric g.

It is well known that the difference between two connections is a well-defined tensor field. It follows therefore that the difference

$$\ell(f) := f^*\gamma - \gamma, \tag{5.1}$$

where $f \in \mathrm{Diff}(M)$, is a well-defined tensor on $S\Gamma(\pi^*(T^*M)^{\otimes 2}) \otimes (\pi^*(TM))$.

It is easy to see that the map

$$f \mapsto \ell(f^{-1})$$

defines a non-trivial 1-cocycle on Diff(M) with values in $S\Gamma(\pi^*(T^*M)^{\otimes 2}) \otimes (\pi^*(TM))$.

Our main definition is the linear differential operator $\mathcal{A}(f)$ acting from $S\Gamma(\pi^*(TM)^{\otimes 2}) \otimes \mathcal{F}_\delta(\pi^*(T^*M))$ to $\Gamma(\pi^*(TM)) \otimes \mathcal{F}_\delta(\pi^*(T^*M))$ defined by

$$\begin{aligned} \mathcal{A}(f)_{ij}^k \; := \; & f^{*-1}\left(\mathrm{g}^{sk}\,\mathrm{g}_{ij}\,D_s\right) - \mathrm{g}^{sk}\,\mathrm{g}_{ij}\,D_s + (2-\delta n)\left(\ell(f)_{ij}^k - \tfrac{1}{n}\,\mathrm{Sym}_{i,j}\,\delta_i^k\,\ell(f)_{tj}^t\right) \\ & + \mathrm{g}^{kl}\left(\mathrm{Sym}_{i,j}\,\mathrm{g}_{sj}B_{li}^s - \delta B_{lt}^t\right) - f^{-1*}\left(\mathrm{g}^{kl}\left(\mathrm{Sym}_{i,j}\,\mathrm{g}_{sj}B_{li}^s - \delta B_{lt}^t\right)\right) \\ & + (2-\delta n)\left(f^{-1*}B_{ij}^k - B_{ij}^k - \tfrac{1}{n}\mathrm{Sym}_{i,j}\,\delta_i^k(f^{-1*}B_{jt}^t - B_{jt}^t)\right), \end{aligned} \tag{5.2}$$

where we have put

$$B_{ij}^k := \left(-A_{ij}^k\,\omega^r - A_{ij}^r\,\omega^k + A_i^{kr}\,\omega_j + A_j^{kr}\,\omega_i - A_{is}^k\,A_j^{sr} - A_{js}^k\,A_i^{sr} + A_u^{rk}\,A_{ij}^u\right) D_r(\log F),$$

to avoid clutter; D is the Chern connection, A is the Cartan tensor (2.2), ω is the Hilbert form (2.3) and $\ell(f)_{ij}^k$ are the components of the tensor (5.1)

Theorem 5.1 (i) *For all $\delta \neq 2/n$, the map $f \mapsto \mathcal{A}(f^{-1})$ defines a non-trivial 1-cocycle on* $\mathrm{Diff}(M)$ *with values in* $\mathcal{D}(S\Gamma(\pi^*(TM)^{\otimes 2}) \otimes \mathcal{F}_\delta(\pi^*(T^*M)), \Gamma(\pi^*(TM)) \otimes \mathcal{F}_\delta(\pi^*(T^*M)))$;
(ii) *the operator (5.2) does not depend on the rescaling of the Finsler function F by any nonzero positive function on M;*
(iii) *if $M := \mathbb{R}^n$ and F is Riemannian such that the metric* g *is the flat metric, this operator vanishes on the conformal group* $\mathrm{O}(n+1,1)$.

Proof. To prove (i) we have to verify the 1-cocycle condition

$$\mathcal{A}(f \circ h) = h^{*-1}\mathcal{A}(f) + \mathcal{A}(h), \quad \text{for all } f, h \in \mathrm{Diff}(M),$$

where h^* is the natural action on $\mathcal{D}(S\Gamma(\pi^*(TM)^{\otimes 2})\otimes\mathcal{F}_\delta(\pi^*(T^*M)), \Gamma(\pi^*(TM))\otimes\mathcal{F}_\delta(\pi^*(T^*M)))$. This condition holds because, in the expression of the operator (5.2), ℓ is a 1-cocycle and the rest is a coboundary.

Let us proof that this 1-cocycle is not trivial for $\delta \neq 2/n$. Suppose that there is a first-order differential operator $A_{ij}^k = u_{ij}^{sk} D_s + v_{ij}^k$ such that

$$\mathcal{A}(f) = f^{*-1}A - A. \tag{5.3}$$

From (5.3), it is easy to see that

$$\begin{aligned} f^{*-1}v_{ij}^k - v_{ij}^k &= (2-\delta n)\left(\ell(f)_{ij}^k - \frac{1}{n}\,\mathrm{Sym}_{i,j}\,\delta_i^k\,\ell(f)_{tj}^t\right) \\ &\quad + \mathrm{g}^{kl}\left(\mathrm{Sym}_{i,j}\,\mathrm{g}_{sj}B_{li}^s - \delta B_{lt}^t\right) - f^{-1^*}\left(\mathrm{g}^{kl}\left(\mathrm{Sym}_{i,j}\,\mathrm{g}_{sj}B_{li}^s - \delta B_{lt}^t\right)\right) \\ &\quad + (2-\delta n)\left(f^{-1^*}B_{ij}^k - B_{ij}^k - \frac{1}{n}\mathrm{Sym}_{i,j}\,\delta_i^k(f^{-1^*}B_{jt}^t - B_{jt}^t)\right). \end{aligned}$$

The right-hand side of this equation depends on the second jet of the diffeomorphism f, while the left-hand side depends on the first jet of f, which is absurd.

For $\delta = 2/n$, one can easily see that the 1-cocycle (5.2) is a coboundary.

Let us prove (ii). Consider a Finsler function $\tilde{F} = \sqrt{\psi}\cdot F$, where ψ is a non-zero positive function on M. Denote by $\tilde{\mathcal{A}}(f)$ the operator (5.2) written with the function $\tilde{F}$. We have to prove that $\tilde{\mathcal{A}}(f) = \mathcal{A}(f)$. The Chern connections associated with the functions F and $\tilde{F}$ are related by

$$\begin{aligned} \tilde{\gamma}_{ij}^k &= \gamma_{ij}^k + \frac{1}{2\psi}\left(\psi_i\,\delta_j^k + \psi_j\,\delta_i^k - \psi_t\,\mathrm{g}^{tk}\mathrm{g}_{ij}\right) \\ &\quad \frac{1}{2\psi}\left(-A_{ij}^k\,\omega^r - A_{ij}^r\,\omega^k + A_i^{kr}\,\omega_j + A_j^{kr}\,\omega_i - A_{is}^k\,A_j^{sr} - A_{js}^k\,A_i^{sr} + A_u^{rk}\,A_{ij}^u\right)\psi_r, \end{aligned} \tag{5.4}$$

where $\psi_i = \partial_i\psi$, A is the Cartan tensor (2.2) and ω is the Hilbert form (2.3).
We need some formulæ: denote by $\ell(f)$ the tensor (5.1) written with the function $\tilde{F}$, then we have

$$\begin{aligned} \tilde{D}_k P^{ij} &= D_k P^{ij} + \frac{1}{2\psi}\left(\mathrm{Sym}_{i,j}P^{mi}\left(\psi_m\delta_k^j - \psi_t\,\mathrm{g}^{tj}\mathrm{g}_{km}\right) + (2-n\delta)\,P^{ij}\psi_k\right), \\ &\quad + \mathrm{Sym}_{i,j}P^{sj}C_{ks}^i - \delta P^{ij}C_{tk}^t \\ \tilde{\ell}(f)_{ij}^k &= \ell(f)_{ij}^k + f^{-1^*}\left(\frac{1}{2\psi}\left(\mathrm{Sym}_{i,j}\psi_i\delta_j^k - \psi_t\,\mathrm{g}^{tk}\mathrm{g}_{ij}\right)\right) - \frac{1}{2\psi}\left(\mathrm{Sym}_{i,j}\,\psi_i\,\delta_j^k - \psi_t\,\mathrm{g}^{tk}\mathrm{g}_{ij}\right) \\ &\quad + f^{-1^*}C_{ij}^k - C_{ij}^k, \end{aligned} \tag{5.5}$$

where we have put

$$C_{ij}^{k} := \frac{1}{2\psi}\left(-A_{ij}^{k}\,\omega^{r} - A_{ij}^{r}\,\omega^{k} + A_{i}^{kr}\,\omega_{j} + A_{j}^{kr}\,\omega_{i} - A_{is}^{k}\,A_{j}^{sr} - A_{js}^{k}\,A_{i}^{sr} + A_{u}^{rk}\,A_{ij}^{u}\right)\psi_{r},$$

to avoid clutter; for all $P^{ij} \in S\Gamma(\pi^{*}(TM)^{\otimes 2}) \otimes \mathcal{F}_{\delta}(\pi^{*}(T^{*}M))$.
By substituting the formulæ (5.5) into (5.2) we get $\mathcal{A}(f) = \tilde{\mathcal{A}}(f)$.

Let us prove (iii). Suppose that F is Riemannian (i.e. $F = \sqrt{\mathrm{g}_{ij}y^{i}y^{j}}$). In this case, the Cartan tensor A is identically zero; the Chern connection D can descend to a connection on M and coincide with Levi-Civita connection associated with g. It follows that the operator (5.2) takes the form

$$\mathcal{A}(f)_{ij}^{k} \;:=\; f^{*-1}\left(\mathrm{g}^{sk}\,\mathrm{g}_{ij}\,D_{s}\right) - \mathrm{g}^{sk}\,\mathrm{g}_{ij}\,D_{s} + (2-\delta n)\left(\ell(f)_{ij}^{k} - \tfrac{1}{n}\,\mathrm{Sym}_{i,j}\,\delta_{i}^{k}\,\ell(f)_{tj}^{t}\right) \tag{5.6}$$

The operator above is nothing but one of the conformally invariant operators introduced in [6]. If, furthermore, M is $\mathbb{R}^{n}$ and g is the flat metric then the restriction of the operator (5.6) on $\Gamma(TM^{\otimes 2}) \otimes \mathcal{F}_{\delta}(M)$ vanishes on the conformal group $\mathrm{O}(n+1,1)$ (see [6]). ∎

6 Conformally invariant quantization by means of Sazaki type metric

6.1 Conformally invariant quantization

Let (N, a) be a Riemannian manifold of dimension n. Denote by ∇ the Levi-Civita connection associated with the metric a. We recall the following theorem.

Theorem 6.1 ([10]) *For $n > 2$ and for all $\delta \neq \frac{2}{n}, \frac{n+2}{2n}, \frac{n+1}{n}, \frac{n+2}{n}$, there exists an isomorphism*

$$\mathcal{Q}_{\lambda,\mu}^{a} : \mathrm{Pol}_{\delta}^{2}(T^{*}N) \to \mathcal{D}^{2}(\mathcal{F}_{\lambda}(N), \mathcal{F}_{\mu}(N))$$

given as follows: for all $P = P^{ij}\xi_{i}\xi_{j} \in \mathrm{Pol}_{\delta}^{2}(T^{*}N)$, one can associate a linear differential operator given by

$$\begin{aligned} \mathcal{Q}_{\lambda,\mu}^{a}(P) \;=\; & P^{ij}\nabla_{i}\nabla_{j} \\ & +(\beta_{1}(\lambda,\delta)\nabla_{i}P^{ij} + \beta_{2}(\lambda,\delta)\,a^{ij}\,\nabla_{i}(a_{kl}P^{kl}))\nabla_{j} \qquad (6.1) \\ & +\beta_{3}(\lambda,\delta)\nabla_{i}\nabla_{j}P^{ij} + \beta_{4}(\lambda,\delta)\,a^{st}\,\nabla_{s}\nabla_{t}(a_{ij}P^{ij}) + \beta_{5}(\lambda,\delta)R_{ij}P^{ij} + \beta_{6}(\lambda,\delta)R\,a_{ij}\,P^{ij}, \end{aligned}$$

where R_{ij} (resp. R) are the Ricci tensor components (resp. the scalar curvature) of the *metric a, constants $\beta_1, \ldots, \beta_6$ are given by*

$$
\begin{aligned}
\beta_1(\lambda, \delta) &= \frac{2(n\lambda + 1)}{2 + n(1 - \delta)}, \\
\beta_2(\lambda, \delta) &= \frac{n(2\lambda + \delta - 1)}{(2 + n(1 - \delta))(2 - n\delta)}, \\
\beta_3(\lambda, \delta) &= \frac{n\lambda(n\lambda + 1)}{(1 + n(1 - \delta))(2 + n(1 - \delta))}, \\
\beta_4(\lambda, \delta) &= \frac{n\lambda(n^2\mu(2 - 2\lambda - \delta) + 2(n\lambda + 1)^2 - n(n + 1))}{(1 + n(1 - \delta))(2 + n(1 - \delta))(2 + n(1 - 2\delta))(2 - n\delta)}, \\
\beta_5(\lambda, \delta) &= \frac{n^2\lambda(\lambda + \delta - 1)}{(n - 2)(1 + n(1 - \delta))}, \\
\beta_6(\lambda, \delta) &= \frac{(n\delta - 2)}{(n - 1)(2 + n(1 - 2\delta))}\beta_5(\lambda, \delta).
\end{aligned}
\tag{6.2}
$$

the operator $\mathcal{Q}^a_{\lambda,\mu}(P)$ has the following properties:
(i) it does not depend on the rescaling of the metric a;
(ii) if $N = \mathbb{R}^n$ is endowed with a flat conformal structure, it is unique, equivariant with respect to the action of the group $\mathrm{O}(p + 1, q + 1) \subset \mathrm{Diff}(\mathbb{R}^n)$.

6.2 A Sazaki type metric on $TM\backslash 0$

Let (M, F) be a Finsler manifold. The Finsler function F gives rise to a (Sazaki type) metric $\mathbf{m}$ on the manifold $TM\backslash 0$, given in the basis (2.4) by

$$
\mathbf{m} := \mathrm{g}_{ij} dx^i \otimes dx^j + \mathrm{g}_{ij} \frac{\delta y^i}{F} \otimes \frac{\delta y^j}{F}, \tag{6.3}
$$

where g_{ij} are the components of the fundamental tensor.

The Christoffel symbols of the Levi-Civita connection associated with the metric $\mathbf{m}$ (6.3) are given by

$$
\begin{aligned}
\Gamma^{k}_{ij}{}^{\mathbf{m}} &= \frac{1}{2}\mathrm{g}^{ks}\left(\frac{\delta \mathrm{g}_{si}}{\delta x^j} + \frac{\delta \mathrm{g}_{sj}}{\delta x^i} - \frac{\delta \mathrm{g}_{ij}}{\delta x^s}\right), & \Gamma^{k}_{i\bar{j}}{}^{\mathbf{m}} &= \mathrm{g}^{ks} A_{isj}, \\
\Gamma^{k}_{\bar{i}\bar{j}}{}^{\mathbf{m}} &= -\frac{1}{2}\mathrm{g}^{ks}\frac{\delta \mathrm{g}_{ij}}{\delta x^s}, & \Gamma^{\bar{k}}_{ij}{}^{\mathbf{m}} &= -\mathrm{g}^{ks} A_{sij}, \\
\Gamma^{\bar{k}}_{i\bar{j}}{}^{\mathbf{m}} &= \frac{1}{2}\mathrm{g}^{ks}\frac{\delta \mathrm{g}_{js}}{\delta x^i}, & \Gamma^{\bar{k}}_{\bar{i}\bar{j}}{}^{\mathbf{m}} &= \mathrm{g}^{ks} A_{sij},
\end{aligned}
\tag{6.4}
$$

where A_{ijk} are the components of the Cartan tensor.

Remark 6.2 Let us emphasize the difference between the geometric objects $\mathbf{m}$ and g. The metric $\mathbf{m}$ is a Riemannian metric on the bundle $TM\backslash 0$, whereas g defines a section of the bundle $\pi^*(T^*M) \otimes \pi^*(T^*M)$. When (and only when) F is Riemannian, g has no y dependence and can then descend to a metric on the manifold M.

Lemma 6.3 *Any tensor P^{ij} on $S\Gamma(TM^{\otimes 2})$ can be extended to a tensor on $S\Gamma(TM\backslash 0^{\otimes 2})$, given in the adapted basis (2.4) by*

$$
\tilde{P} = P^{ij} \frac{\delta}{\delta x^i} \frac{\delta}{\delta x^j} + P^{ij} F \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j}. \tag{6.5}
$$

Proof. Simple computation using the formulas (2.5).

Lemma 6.4 *The space* $\mathcal{F}_\lambda(M)$ *can be identified with the subspace of* $\mathcal{F}_{\frac{\lambda}{2}}(TM\backslash 0)$ *with elements of the form*

$$\phi(x)\left(dx^1 \wedge \cdots \wedge dx^n \wedge \frac{\delta y^1}{F} \wedge \cdots \wedge \frac{\delta y^n}{F}\right)^{\frac{\lambda}{2}}. \tag{6.6}$$

Proof. Simple computation using the formulas (2.5).

Theorem 6.5 *The quantization map* $\mathcal{Q}^{\mathbf{m}} : \mathrm{Pol}^2_\delta(T^*(TM\backslash 0)) \to \mathcal{D}^2(\mathcal{F}_\lambda(TM\backslash 0), \mathcal{F}_\mu(TM\backslash 0))$ *has the following properties:*

(i) if F *is not Riemannian then* $\mathcal{Q}^{\mathbf{m}}(\tilde{P})_{|\mathcal{F}_\lambda(M)} \notin \mathcal{F}_\mu(M)$, *for all* $P \in \mathrm{Pol}^2_\delta(T^*M)$;

(ii) if F *is Riemannian then* $\mathcal{Q}^{\mathbf{m}}(\tilde{P})_{|\mathcal{F}_\lambda(M)} \in \mathcal{F}_\mu(M)$ *and is given in terms of the metric* g *by*

$$\begin{aligned}
\mathcal{Q}^{\mathbf{m}}_{\lambda,\mu}(\tilde{P})_{|\mathcal{F}_\lambda(M)} \;=\;& P^{ij}\nabla^{\mathrm{g}}_i\nabla^{\mathrm{g}}_j \\
&+\left(\beta_1\left(\frac{\lambda}{2},\frac{\delta}{2}\right)\nabla^{\mathrm{g}}_i P^{ij} + 2\,\beta_2\left(\frac{\lambda}{2},\frac{\delta}{2}\right)\mathrm{g}^{ij}\,\nabla^{\mathrm{g}}_i(\mathrm{g}_{kl}P^{kl})\right)\nabla^{\mathrm{g}}_j \\
&+\beta_3\left(\frac{\lambda}{2},\frac{\delta}{2}\right)\nabla^{\mathrm{g}}_i\nabla^{\mathrm{g}}_j P^{ij} + 2\,\beta_4\left(\frac{\lambda}{2},\frac{\delta}{2}\right)\mathrm{g}^{st}\,\nabla^{\mathrm{g}}_s\nabla^{\mathrm{g}}_t(\mathrm{g}_{ij}P^{ij}) + \beta_5\left(\frac{\lambda}{2},\frac{\delta}{2}\right)R^{\mathrm{g}}_{ij}P^{ij} \\
&+2\,\beta_6\left(\frac{\lambda}{2},\frac{\delta}{2}\right)R^{\mathrm{g}}\,\mathrm{g}_{ij}\,P^{ij} + \beta_5\left(\frac{\lambda}{2},\frac{\delta}{2}\right)\Big(\partial_k\Gamma_j - \frac{1}{2}\Gamma^m_{ks}\Gamma^s_{jm} + \frac{1}{2}\mathrm{g}^{hm}\mathrm{g}_{ip}\Gamma^p_{jm}\Gamma^i_{kh} \\
&-\Gamma_h\Gamma^l_{sj}\mathrm{g}^{sh}\mathrm{g}_{lk} + \frac{1}{2}\Gamma^m_{hs}\Gamma^p_{ki}\mathrm{g}^{is}\mathrm{g}^{hq}\mathrm{g}_{mk}\mathrm{g}_{pj} - 2\,\Gamma^s_{im}\Gamma^l_{js}\mathrm{g}^{im}\mathrm{g}_{lk} + 2\,\mathrm{g}^{is}\mathrm{g}_{mk}\partial_i\Gamma^m_{sj}\Big)\,P^{jk} \\
&+2\,\beta_6\left(\frac{\lambda}{2},\frac{\delta}{2}\right)\Big(\partial_k\Gamma_j\mathrm{g}^{jk} - \frac{1}{2}\Gamma^m_{ks}\Gamma^s_{lm}\mathrm{g}^{lk} - \Gamma_h\Gamma_s\mathrm{g}^{sh} - 2\,\Gamma^s_{tm}\Gamma_s\mathrm{g}^{tm} \\
&+\Gamma^m_{sk}\Gamma^l_{tr}\mathrm{g}^{ts}\mathrm{g}^{rk}\mathrm{g}_{lm}\Big)\,\mathrm{g}_{ij}\,P^{ij},
\end{aligned}$$

where R^{g}_{ij} *(resp.* R^{g}*) are the Ricci tensor components (resp. scalar curvature) of the metric* g *and constants* $\beta_1, \ldots, \beta_6$ *are as in (6.2).*

Proof. (i) Suppose that F is not Riemannian. In this case, the metric $\mathbf{m}$ (6.3) depends, in any local coordinates (x^i, y^i), on x and on y as well. It is easy to see from the map (6.1) that $\mathcal{Q}^{\mathbf{m}}(\tilde{P})_{|\mathcal{F}_\lambda(M)}$ depends on y.

(ii) Suppose that F is Riemannian. The section g can then define a metric on the manifold M. Let us write the covariant derivative associated with the metric $\mathbf{m}$ in terms of the covariant derivative associated with the metric g. Namely,

$$\begin{aligned}
&\nabla^{\mathbf{m}}_{\bar{i}}(\phi) = 0, && \nabla^{\mathbf{m}}_i(\phi) = \nabla^{\mathrm{g}}_i(\phi), \\
&\nabla^{\mathbf{m}}_{\bar{i}}\nabla^{\mathbf{m}}_{\bar{j}}(\phi) = 0, && \nabla^{\mathbf{m}}_j\nabla^{\mathbf{m}}_i(\phi) = \nabla^{\mathrm{g}}_j\nabla^{\mathrm{g}}_i(\phi),
\end{aligned} \tag{6.7}$$

for all $\phi \in \mathcal{F}_\lambda(M)$, and

$$\begin{array}{ll} \nabla^{\mathbf{m}}_{i}(\mathbf{m}_{kl}\tilde{P}^{kl} + \mathbf{m}_{\bar{k}\bar{l}}\tilde{P}^{\bar{k}\bar{l}}) = 2\,\nabla^{\mathrm{g}}_{i}(\mathrm{g}_{kl}P^{kl}), & \nabla^{\mathbf{m}}_{\bar{j}}(\tilde{P}^{\bar{i}\bar{j}}) = 0, \\ \nabla^{\mathbf{m}}_{\bar{i}}\nabla^{\mathbf{m}}_{\bar{j}}(\tilde{P}^{\bar{i}\bar{j}}) = 0, & \nabla^{\mathbf{m}}_{j}\nabla^{\mathbf{m}}_{i}(\tilde{P}^{ij}) = \nabla^{\mathrm{g}}_{j}\nabla^{\mathrm{g}}_{i}(P^{ij}), \end{array} \tag{6.8}$$

for all $P^{ij} \in S\Gamma(TM^{\otimes 2}) \otimes \mathcal{F}_\delta(M)$.
A simple computation proves that

$$\begin{array}{rcl} R^{\mathbf{m}}_{jk}P^{jk} + R^{\mathbf{m}}_{\bar{j}\bar{k}}P^{jk} & = & \Big(R^{\mathrm{g}}_{jk} + \partial_k\Gamma_j - \frac{1}{2}\Gamma^m_{ks}\Gamma^s_{jm} + \frac{1}{2}\mathrm{g}^{hm}\mathrm{g}_{ip}\Gamma^p_{jm}\Gamma^i_{kh} - \Gamma_h\Gamma^l_{sj}\mathrm{g}^{sh}\mathrm{g}_{lk} \\ & & +\frac{1}{2}\Gamma^m_{hs}\Gamma^p_{ki}\mathrm{g}^{is}\mathrm{g}^{hq}\mathrm{g}_{mk}\mathrm{g}_{pj} - 2\,\Gamma^s_{im}\Gamma^l_{js}\mathrm{g}^{im}\mathrm{g}_{lk} + 2\,\mathrm{g}^{is}\mathrm{g}_{mk}\partial_i\Gamma^m_{sj}\Big)\,P^{jk}, \\ R^{\mathbf{m}} & = & R^{\mathrm{g}} + \partial_k\Gamma_j\mathrm{g}^{jk} - \frac{1}{2}\Gamma^m_{ks}\Gamma^s_{lm}\mathrm{g}^{lk} - \Gamma_h\Gamma_s\mathrm{g}^{sh} - 2\,\Gamma^s_{tm}\Gamma_s\mathrm{g}^{tm} \\ & & +\Gamma^m_{sk}\Gamma^l_{tr}\mathrm{g}^{ts}\mathrm{g}^{rk}\mathrm{g}_{lm}. \end{array} \tag{6.9}$$

Replace the formulas (6.7, 6.8, 6.9) into the formula (6.1) we prove that $\mathcal{Q}^{\mathbf{m}}(\tilde{P})_{|\mathcal{F}_\lambda(M)}$ is as above. ■

Remark 6.6 (i) When F is Riemannian, the operator $\mathcal{Q}^{\mathbf{m}}(\tilde{P})_{|\mathcal{F}_\lambda(M)}$ turns out to be defined only locally; it is defined globally only when the metric g is flat or $\lambda = 0$ or $\mu = 2$.

(ii) Theorem above shows that the quantity $\mathcal{Q}^{\mathbf{m}}(\tilde{P})_{|\mathcal{F}_\lambda(M)}$, for all $P \in \mathrm{Pol}^2_\delta(T^*M)$, does not coincide with the quantity $\mathcal{Q}^{\mathrm{g}}(P)$ even if F is Riemannian.

7 Open problems

1. Following [4], there exists two 1-cocycles on Diff(M), say c_1, c_2, with values in $\mathcal{D}(S\Gamma(TM^{\otimes 2})\otimes\mathcal{F}_\delta(M), \Gamma(TM)\otimes\mathcal{F}_\delta(M))$ and $\mathcal{D}(S\Gamma(TM^{\otimes 2})\otimes\mathcal{F}_\delta(M), \mathcal{F}_\delta(M))$ respectively, that are conformally invariants (i.e. they depend only on the conformal class of the Riemannian metric). These 1-cocycles were introduced in [4] as conformal multi-dimensional Schwarzian derivatives. In this paper, we have introduce the 1-cocycle $\mathcal{A}$ (see (5.2)) as the Finslerian analogous of the 1-cocycle c_1; however, the computation to generalize the 1-cocycle c_2 seems to be more intricate.
2. We ask the following question:
Is there a map

$$\mathcal{Q} : \mathrm{Pol}_\delta(T^*M) \to \mathcal{D}(\mathcal{F}_\lambda(\pi^*(T^*M)), \mathcal{F}_\mu(\pi^*(T^*M))),$$

having the following properties: (i) it does not depend on the rescaling of the Finsler function by a positive function on M, (ii) it coincides with the Duval-Ovsienko's conformally invariant map when F is Riemannian?
A positive answer to this question will automatically induce the existence of the 1-cocycle c_2.

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