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Arithmetic forms of Selberg zeta functions with applications to prime geodesic theorem

by

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Running title. Selberg zeta functions and applications

Abstract. We obtain an arithmetic expression of the Selberg zeta function for cocompact subgroup defined via an indefinite division quaternion algebra over $\mathbb{Q}$. The proof makes use of the theorems due to Eichler [E]. As the application for that expression to the prime geodesic theorem, we give the uniformity of the distribution which is called the Brun-Tichmarsh type.

Key words. Indefinite division quaternion algebra; Selberg zeta function; explicit form; Brun-Tichmarsh type prime geodesic theorem.

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1. Introduction

Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$ containing -1_2 with finite covolume $v(\Gamma\backslash\mathfrak{H})$, $\mathfrak{H}$ denoting the upper half plane. The Selberg zeta function attached to Γ is defined by

$$Z_\Gamma(s) := \prod_{\{P\}_\Gamma} \prod_{m=0}^{\infty} (1 - N(P)^{-s-m}), \qquad (\mathrm{Re}(s) > 1)$$

where $\{P\}_\Gamma$ runs through all primitive hyperbolic conjugacy classes of Γ with $\mathrm{tr}(P) > 2$, and $N(P) := |\rho|^2$ with ρ the eigenvalue of $P \in \Gamma$ such that $|\rho| > 1$. When $\Gamma = SL(2,\mathbb{Z})$, Sarnak[S] obtains an arithmetic form of $Z_\Gamma(s)$:

$$(1.1) \qquad Z_\Gamma(s) = \prod_{D>0} \prod_{n=0}^{\infty} \left(1 - \varepsilon_D^{-2(s+n)}\right)^{h(D)},$$

where D runs through the discriminants of real quadratic fields with ε_D the fundamental unit, and $h(D)$ is the class number (see also [H2, p.518]). Such an arithmetic expression is proved for some congruence subgroups as well [S], but is not known for arithmetic cocompact groups.

The chief concern of this paper is to obtain such an arithmetic expression of the Selberg zeta function for cocompact Γ defined via an indefinite division quaternion algebra over $\mathbb{Q}$.

Let $B = \left(\frac{a,b}{\mathbb{Q}}\right)$ be an indefinite division quaternion algebra over $\mathbb{Q}$ with a and b positive integers which are relatively prime and squarefree. We write a typical element of B in the form

$$q = q_0 + q_1\alpha + q_2\beta + q_3\alpha\beta,$$

where $\alpha^2 = a$, $\beta^2 = b$, $\alpha\beta = -\beta\alpha$, and $q_i \in \mathbb{Q}$ ($i = 0,1,2,3$). We denote by $q \longmapsto \bar{q}$ the canonical involution of B and put $n(q) = q\bar{q}$, $\mathrm{tr}(q) = q + \bar{q}$. We choose and fix a maximal order $\mathcal{O}$ of B. Let B^1 (resp. $\mathcal{O}^1$) be the group consisting of all elements q of B (resp, $\mathcal{O}$) with $n(q) = 1$. Since the $\mathbb{R}$ algebra $B \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to $M_2(\mathbb{R})$, B^1 is injectively embedded into $SL_2(\mathbb{R})$ via this isomorphism. The unit group $\mathcal{O}^1$ can be identified with a cocompact discrete subgroup $\Gamma_{\mathcal{O}}$ of $SL_2(\mathbb{R})$ which is the image of the following injection:

(1.2)
$$\begin{array}{ccc} \mathcal{O}^1 & \hookrightarrow & SL_2(\mathbb{R}) \\ q & \longmapsto & \begin{pmatrix} q_0 + q_1\sqrt{a} & q_2\sqrt{b} + q_3\sqrt{a}\sqrt{b} \\ q_2\sqrt{b} - q_3\sqrt{a}\sqrt{b} & q_0 - q_1\sqrt{a} \end{pmatrix} \end{array}$$

We write $Z_{\mathcal{O}^1}(s) := Z_{\Gamma_{\mathcal{O}}}(s)$ with this identification. Since B is indefinite over $\mathbb{Q}$, there is a unique maximal order $\mathcal{O}$ of B up to $B^\times$-conjugation. Therefore, $Z_{\mathcal{O}^1}(s)$ depends only on B and not on the choice of $\mathcal{O}$. We simply write $Z_B(s)$ for the Selberg zeta function $Z_{\mathcal{O}^1}(s)$.

For any basis $\{u_i\}$ of $\mathcal{O}$ over $\mathbb{Z}$, set

$$d(B) = |\det(\mathrm{tr}(u_i u_j))|^{\frac{1}{2}}.$$

This number is independent of the choice of $\mathcal{O}$ and $\{u_i\}$, and denotes the product of prime integers p which ramify at $B/\mathbb{Q}$.

Put

$$\mathcal{D} := \{D \in \mathbb{Z}_{>0} \mid D \equiv 0,1 \pmod 4, \text{ not a square}\}.$$

Let $\mathfrak{o}$ be an order of $K = \mathbb{Q}(\sqrt{D})$ and $h(\mathfrak{o}) = h(D)$ be the number of classes of proper $\mathfrak{o}$-ideals in the narrow sense. We moreover set

$$\lambda(K) = \prod_{p|d(B)} \left(1 - \left(\frac{K}{p}\right)\right),$$

where $\left(\frac{K}{p}\right)$ denotes the Artin symbol for $K = \mathbb{Q}(\sqrt{D})$.

Let $\varepsilon_D = \dfrac{\alpha + \beta\sqrt{D}}{2}$ with $(\alpha,\ \beta)$ being the minimal solution of the Pell equation: $x^2 - Dy^2 = 4$. The main theorem of this paper is as follows:

Theorem 1.1. *Let B be a division indefinite quaternion algebra over $\mathbb{Q}$. Then*

$$Z_B(s) = \prod_{D>0}^{*} \prod_{n=0}^{\infty} \left(1 - \varepsilon_D^{-2(s+n)}\right)^{h(D)\lambda(D)},$$

and

$$\frac{Z_B'}{Z_B}(s) = \sum_{D>0}^{*} \sum_{m=1}^{\infty} h(D)\lambda(D) \log \varepsilon_D^2 \cdot \frac{\varepsilon_D^{-2ms}}{1-\varepsilon_D^{-2m}},$$

where $\lambda(D) = \lambda(\mathbb{Q}(\sqrt{D}))$ and the symbol $*$ indicates that D runs through all elements in $\mathcal{D}$ satisfying the following conditions.

(Pr-i) $\left(\frac{K}{p}\right) \neq 1$ *for any prime integers $p \mid d(B)$.*

(Pr-ii) $(f(D), d(B)) = 1$, *where the positive integer $f(D)$ is given by $D = f(D)^2 D_K$, D_K being the discriminant of K.*

Remark 1.2. *Though for the proof of Theorem 1.1 we have used the theory of optimal embeddings due to Eichler, the theorem would also be deduced from the result of* [BJ] *and* [S1]([S2]).

Theorem 1.1 has an application for improving the prime geodesic theorem:

$$\pi_\Gamma(x) \sim \mathrm{li}(x) \sim \frac{x}{\log x}, \tag{1.3}$$

where $\pi_\Gamma(x)$ is the number of primitive hyperbolic conjugacy classes P of Γ whose norm $N(P)$ satisfies that $N(P) \le x$, and the relation "$\sim$" means that the quotient of both sides goes to 1 as $x \to \infty$. The formula (1.3) is an average estimate in the sense that it just counts the number of elements in the whole interval $(1, x]$. When we are interested in more refined version, we need to estimate for smaller interval such as $(x, x+y]$ $(0 < y < x)$ for sufficiently large x. If we were able to prove

$$\pi_\Gamma(x+y) - \pi_\Gamma(x) \sim \mathrm{li}(x+y) - \mathrm{li}(x) \sim \frac{y}{\log x},$$

then it would mean the uniformity of the distribution. We call such an estimate the Brun-Titchmarsh type prime geodesic theorem. When $\Gamma = SL(2, \mathbb{Z})$, Iwaniec [I, Lemma 4] proved that

$$\pi_\Gamma(x+y) - \pi_\Gamma(x) \ll y$$

for $x^{\frac{1}{2}}(\log x)^2 < y < x$. He uses the arithmetic form (1.1) of $Z_\Gamma(s)$, and the method is applicable to our case using Theorem 1.1. We prove:

Theorem 1.3. *Let B be a division indefinite quaternion algebra over $\mathbb{Q}$. Put $\pi_B(x) = \pi_{\mathcal{O}^1}(x)$. Then for $x^{\frac{1}{2}}(\log x)^2 < y < x$, we have*

$$\pi_B(x+y) - \pi_B(x) \ll y. \tag{1.4}$$

The implied constant depends only on B.

Remark 1.4. (a) *Theorem 1.3 gives the best possible range of y in view of the multiplicities of the length spectrum in the following sense: It is known that $N(P)$ is a function of $|\mathrm{tr}(P)|$ and grows like $|\mathrm{tr}(P)|^2$. When $x \in \mathbb{Z}^2 = \{n^2 \mid n \in \mathbb{Z}\}$, there exist at least $\sqrt{x}$ different P's which satisfy $|\mathrm{tr}(P)|^2 = x$. It means $\pi_\Gamma(x)$ jumps by as much as $\sqrt{x}$ at that moment. Therefore* (1.4) *is not true for $y < \sqrt{x}$. Hence the exponent 1/2 in the lower bound of y in Theorem 1.3 is the best possible.*

(b) *Theorem 1.3 gives the best possible exponents of x and y according to the conjectural form (??).*

(c) *The current best error term of* (1.3) *for arithmetic cocompact groups is obtained by Koyama*[K]*:*

$$\pi_B(x) = \mathrm{li}(x) + O(x^{\frac{7}{10}+\varepsilon}). \tag{1.5}$$

By using this error term one easily computes that Theorem 1.3 is valid for $x^{\frac{7}{10}+\varepsilon} < y < x$. Hence Theorem 1.3 is nontrivial for $x^{\frac{1}{2}}(\log x)^2 < y \le x^{\frac{7}{10}}$.

(d) *This estimate (1.5) together with Theorem 1.1 implies the following estimates for class numbers:*

$$\sum_{0<\varepsilon_D \le x}^{*} h(D)\lambda(D) = \mathrm{li}(x^2) + O(x^{\frac{7}{5}+\varepsilon})$$

$$\sum_{0<\varepsilon_D \le x}^{*} h(D)\lambda(D)\log\varepsilon_D = \frac{x^2}{2} + O(x^{\frac{7}{5}+\varepsilon}),$$

which should be compared with [S, Theorem 4.11] *and* [H2, p.519, Proposition 2.9].

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2. Explicit Form

For obtaining the explicit form of $Z_B(s)$, we introduce the following two theorems due to Eichler.

Theorem 2.1 (Eichler [E]). *Let K be a quadratic field over $\mathbb{Q}$ and $\mathfrak{o}_K$ the maximal order of K. Each order $\mathfrak{o}$ of K has an expression: $\mathfrak{o} = \mathbb{Z} + f\mathfrak{o}_K$ for some positive integer $f = f(\mathfrak{o})$. The discriminant of $\mathfrak{o}$ is given by $D(\mathfrak{o}) := f^2 D_K$, D_K being the discriminant of K. Then,* (i) *There exists a $\mathbb{Q}$-isomorphism*

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φ of K into B, if and only if $\left(\frac{K}{p}\right) \neq 1$ for all prime integers $p \mid d(B)$. (ii) *Let K satisfy the condition of* (i) *and $\mathfrak{o}$ an order of K. Then there exists a $\mathbb{Q}$-isomorphism φ of K into B such that $\varphi(\mathfrak{o}) = \varphi(K) \cap \mathcal{O}$, if and only if $(f(\mathfrak{o}), d(B)) = 1$.*

Let K and $\mathfrak{o}$ be the same as in the theorem. Denote by $I(K, \mathfrak{o})$ the set of all $\mathbb{Q}$-isomorphisms φ of K into B such that $\varphi(\mathfrak{o}) = \varphi(K) \cap \mathcal{O}$. We say that, for φ, $\varphi' \in I(K, \mathfrak{o})$, φ' is $\mathcal{O}^1$-equivalent to φ, if there exists some $\varepsilon \in \mathcal{O}^1$ such that $\varphi'(z) = \varepsilon\varphi(z)\varepsilon^{-1}$ for any $z \in K$. Denote by $I(K, \mathfrak{o})/\mathcal{O}^1$ the set of all the $\mathcal{O}^1$-equivalence classes in $I(K, \mathfrak{o})$.

We denote the cardinarity of a finite set S by $\sharp(S)$.

Theorem 2.2 (Eichler [E]). *We have*

$$\sharp\Big(I(K, \mathfrak{o})/\mathcal{O}^1\Big) = h(\mathfrak{o})\lambda(K).$$

For a proof we refer to Shimizu [Sh] (see also [A]).

Now we need the relation between the quadratic field over $\mathbb{Q}$ and the quaternion algebra.

Set

$$\widetilde{L} := \{x \in \mathbb{Z} + 2\mathcal{O} \mid \mathrm{tr}(x) = 0\}.$$

Any non zero element $x \in \widetilde{L}$ is called primitive, if it cannot be expressed as $x = my$ with $m \in \mathbb{Z}$, $m \neq \pm 1$, $y \in \widetilde{L}$. Denote by $\widetilde{L}_{pr}$ the subset of $\widetilde{L}$ consisting of primitive elements of $\widetilde{L}$. For each positive discriminant D let

$$\mathcal{C}^{pr}(D) := \{\xi \in \widetilde{L}_{pr} \mid n(\xi) = -D\}.$$

In view of Theorem 2.1 we see the following relation;

Lemma 2.3. *We have $\mathcal{C}^{pr}(D) \neq \phi$, if and only if D satisfies the conditions (Pr-i) and (Pr-ii).*

Proof. For each $x \in \mathcal{C}^{pr}(D)$ we form an isomorphism $\varphi_x : K \longrightarrow B$ by $\varphi_x(\sqrt{D}) = x$.

Let $\mathfrak{o}$ be an order of K with discriminant D. We put $x = p + 2\xi$ for $p \in \mathbb{Z}$ and $\xi \in \mathcal{O}$. Because $\mathrm{tr}(x) = 0$, we have

$$n(x) + p^2 = 4n(\xi).$$

From $n(x) = -D$ and $n(\xi) \in \mathbb{Z}$, we have $p^2 \equiv D \pmod 4$.

When $D \equiv 1 \pmod 4$, we have $1 + p \in 2\mathbb{Z}$ and

$$1 + x = 1 + p + 2\xi \in 2\mathbb{Z} + 2\mathcal{O} \subset 2\mathcal{O}.$$

In the case of $D \equiv 0 \pmod 4$, we have $p \in 2\mathbb{Z}$ and

$$x = p + 2\xi \in 2\mathbb{Z} + 2\mathcal{O} \subset 2\mathcal{O}.$$

By the isomorphism φ_x, we have

$$\varphi_x(\mathfrak{o}) = \begin{cases} \mathbb{Z} + \dfrac{1+x}{2}\mathbb{Z} & \text{if } D \equiv 1 \pmod 4, \\ \mathbb{Z} + \dfrac{x}{2}\mathbb{Z} & \text{if } D \equiv 0 \pmod 4. \end{cases}$$

Then we have $\varphi_x(\mathfrak{o}) \subset \mathcal{O}$. From the primitivity of x, there doesn't exist any $n \geq 2$ satisfying $\varphi_x(\mathfrak{o}) \subset n\mathcal{O}$. Applying Theorem 2.1 leads that (Pr-i) and (Pr-ii).

Conversely, we assume (Pr-i) and (Pr-ii). Let $\mathfrak{o}$ be the order of $K = \mathbb{Q}(\sqrt{D})$ with discriminant D. From Theorem 2.1, there exists $\mathbb{Q}$-isomorphism $\varphi : K \to B$ with $\varphi(\mathfrak{o}) = \varphi(K) \cap \mathcal{O}$. We form $x := \varphi(\sqrt{D})$. Since $\sqrt{D} \in \mathbb{Z} + 2\mathfrak{o}$, we have $x \in \mathbb{Z} + 2\mathcal{O}$. Since $\mathfrak{o}_K$ is the maximal order of K, $\mathfrak{o} = \mathbb{Z} + f(\mathfrak{o})\mathfrak{o}_K$ is as follows;

(2.1)

$$\mathfrak{o} = \begin{cases} \mathbb{Z} + \dfrac{f(\mathfrak{o}) + \sqrt{D}}{2}\mathbb{Z}, & D_K \equiv 1 \pmod 4, \\ \mathbb{Z} + \dfrac{\sqrt{D}}{2}\mathbb{Z}, & D_K \equiv 0 \pmod 4. \end{cases}$$

Then since there doesn't exist $n \geq 2$ such that $\frac{\sqrt{D}}{n} \in \mathbb{Z} + 2\mathfrak{o}$, x is primitive. It follows that $x \in \mathcal{C}^{pr}(D)$. ∎

Set

$$\mathcal{C}^{pr} := \bigcup_{D>0}^{*} \mathcal{C}^{pr}(D),$$

where D runs over all positive discriminants satisfying the conditions (Pr-i) and (Pr-ii).

Denote by $Prm^+(\mathcal{O}^1)$ the set of primitive elements γ of $\mathcal{O}^1$ with $\mathrm{tr}(\gamma) > 2$. For $\varepsilon \in Prm^+(\mathcal{O}^1)$, we put $\mathbb{Q}(\varepsilon) := \mathbb{Q} + \mathbb{Q}\varepsilon$. Since B is a division quaternion algebra, $\mathbb{Q}(\varepsilon)$ is a quadratic extension over $\mathbb{Q}$ and is isomorphic to $K = \mathbb{Q}(\sqrt{d^2-4})$ over $\mathbb{Q}$ with $d = \mathrm{tr}(\varepsilon)$. We denote this isomorphism by $\varphi : K \longrightarrow \mathbb{Q}(\varepsilon)$ given by $\varphi((d + \sqrt{d^2-4})/2) = \varepsilon$. We put $\mathbf{o} := \mathbb{Q}(\varepsilon) \cap \mathcal{O}$ which is an order of $\mathbb{Q}(\varepsilon)$, then $\mathfrak{o} := \varphi^{-1}(\mathbf{o})$ is an order of K. One can write $\mathfrak{o} = \mathbb{Z} + f(\mathfrak{o})\mathfrak{o}_K$ with $f(\mathfrak{o}) \in \mathbb{Z}_{>0}$, $\mathfrak{o}_K$ being the maximal order of K. If we set $D = f(\mathfrak{o})^2 D_K$, then D is the discriminant of $\mathfrak{o}$. Since $\varphi(\mathfrak{o}) = \mathbf{o} = \mathbb{Q}(\varepsilon) \cap \mathcal{O}$, Theorem 2.1 implies $(f(\mathfrak{o}), d(B)) = 1$. We see that $\mathcal{C}^{pr}(D) \neq \phi$ for D determined by the order of $\mathbb{Q}(\varepsilon)$.

Lemma 2.4. *It holds that*

$$\varphi^{-1}(\varepsilon) = \varepsilon_D,$$

where D is the discriminant of $\mathfrak{o}$, and $\varepsilon_D = \dfrac{\alpha + \beta\sqrt{D}}{2}$ with (α, β) ($\alpha, \beta \in \mathbb{Z}_{>0}$) being the minimal solution of the Pell equation $x^2 - Dy^2 = 4$.

Proof. We have

$$\varphi^{-1}(\varepsilon) = \frac{d + \sqrt{d^2-4}}{2}.$$

We put $\alpha := d$ and $\beta^2 D := d^2 - 4$, where $D = f(\mathfrak{o})^2 D_K$. Then (α, β) is the minimal. We will prove it. By the reduction to absurdity, we assume (α_0, β_0) is the minimal solution, which is not (α, β). Then there exists $n \in \mathbb{Z}(\neq 1)$ such that

$$\frac{\alpha + \beta\sqrt{D}}{2} = \left(\frac{\alpha_0 + \beta_0\sqrt{D}}{2}\right)^n.$$

By $\mathbb{Q}$-isomorphism φ, we have

$$\varepsilon = \varphi\left(\frac{\alpha + \beta\sqrt{D}}{2}\right) = \varphi\left(\left(\frac{\alpha_0 + \beta_0\sqrt{D}}{2}\right)^n\right) = \varphi\left(\frac{\alpha_0 + \beta_0\sqrt{D}}{2}\right)^n.$$

This contradicts ε is primitive. ∎

Now we have $K = \mathbb{Q}(\sqrt{d^2-4}) = \mathbb{Q}(\sqrt{D})$. By using the correspondence in Lemma 2.4, we have the following lemma.

Lemma 2.5. *Let the notation be the same as in Lemma 2.4. The map* $Prm^+(\mathcal{O}^1) \in \varepsilon \longmapsto \xi \in C^{pr}$, *where* ξ *is given by* $\xi = \dfrac{2\varepsilon - \alpha}{\beta}$, *is a bijection.*

Proof. Let $\varepsilon \in Prm^+(\mathcal{O}^1)$ be given. We put α, β and D be the same as in the proof of Lemma 2.4. Set $\xi = \dfrac{2\varepsilon - \alpha}{\beta}$, then we have $\mathrm{tr}(\xi) = 0$. From $n(\varepsilon) = n\left(\dfrac{\alpha + \beta\xi}{2}\right) = 1$, ξ satisfies

$$\alpha^2 + \beta^2 n(\xi) = 4.$$

Since (α, β) is the solution of the Pell equation $x^2 - Dy^2 = 4$, we have $n(\xi) = -D$. By using

$$\varphi^{-1}(\varepsilon) = \varepsilon_D = \frac{\alpha + \beta\sqrt{D}}{2}$$

as Lemma 2.4, we have

$$\varphi^{-1}(\xi) = \sqrt{D}.$$

The definition of D gives $\sqrt{D} = f(\mathfrak{o})\sqrt{D_K}$. Because of $\sqrt{D_K} \in \mathfrak{o}_K$ and $\mathfrak{o} = \mathbb{Z} + f(\mathfrak{o})\mathfrak{o}_K$, we have $\sqrt{D} \in \mathfrak{o}$.

From (2.1) we get $\sqrt{D} \in \mathbb{Z} + 2\mathfrak{o}$. Since ε is a primitive element, (α, β) is the minimal solution. It shows that there doesn't exist $n \geq 2$ such that $\frac{\sqrt{D}}{n} \in \mathbb{Z} + 2\mathfrak{o}$. From $\varphi(\mathfrak{o}) = \mathbf{o} = \mathbb{Q}(\varepsilon) \cap \mathcal{O}$ and $\varphi(\sqrt{D}) = \xi$, we have $\xi \in \mathbb{Z} + 2(\mathbb{Q}(\varepsilon) \cap \mathcal{O}) \subset \mathbb{Z} + 2\mathcal{O}$ and also we have that ξ is a primitive element in $\widetilde{L}$. Therefore $\xi \in C^{pr}(D)$.

Conversely, we choose and fix an element ξ in C^{pr} and put $D := -n(\xi)$. Let $(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}$ be the minimal solution of the Pell equation $x^2 - Dy^2 = 4$, and set $\varepsilon := \dfrac{\alpha + \beta\xi}{2}$. Then we have

$$n(\varepsilon) = \frac{\alpha^2 + \beta^2 n(\xi)}{4} = 1,$$

and by $\xi \in \mathbb{Z} + 2\mathcal{O}$, we also have $\alpha + \beta\xi \in 2\mathcal{O}$. Thus we have $\varepsilon \in \mathcal{O}^1$. If ε is written in the form $\varepsilon = \varepsilon_1^m$ with an integer $m > 0$ and $\varepsilon_1 \in \mathcal{O}^1$, then ε_1 commutes with any element of $\mathbb{Q}(\varepsilon)$ and hence ε_1 lies in the order $\mathbf{o} = \mathbb{Q}(\varepsilon) \cap \mathcal{O}$. Therefore, ε_1 is a unit of the order $\mathbf{o}$ with norm one. Since we see from the definition of ε that ε is a fundamental unit of $\mathbf{o}$ with norm one, we have $m = 1$. This means ε is a primitive hyperbolic element in $\mathcal{O}^1$, which completes the proof. ∎

We denote by $C^{pr}/\mathcal{O}^1$ (resp. $C^{pr}(D)/\mathcal{O}^1$) the set of $\mathcal{O}^1$- conjugacy classes of C^{pr} (resp. $C^{pr}(D)$).

Lemma 2.6. *The correspondence in Lemma 2.4 induces a bijection of* $Prm^+(\mathcal{O}^1)/\mathcal{O}^1$ *onto* $C^{pr}/\mathcal{O}^1$.

Proof. Let $\varepsilon, \varepsilon' \in Prm^+(\mathcal{O}^1)$. When ε is $\mathcal{O}^1$-conjugate to ε', there exists $\gamma \in \mathcal{O}^1$ such that $\varepsilon' = \gamma\varepsilon\gamma^{-1}$. Since $\mathbb{Q}(\varepsilon') \cap \mathcal{O} = \gamma(\mathbb{Q}(\varepsilon) \cap \mathcal{O})\gamma^{-1}$ and both of ε, ε' are primitive, the corresponding minimal solutions of the Pell equations are the same. Therefore we may write

$$\varepsilon = \frac{\alpha + \beta\xi}{2} \quad \text{and} \quad \varepsilon' = \frac{\alpha + \beta\xi'}{2}$$

with α, $\beta \in \mathbb{Z}_{>0}$. Thus $\xi' = \gamma\xi\gamma^{-1}$. ∎

Let $D \in \mathbb{Z}_{>0}$ be a discriminant satisfying the conditions (Pr-i) and (Pr-ii). From Lemma 2.3, we see easily that there exists a bijection from $C^{pr}(D)$ to $I(K, \mathfrak{o})$, where $K = \mathbb{Q}(\sqrt{D})$ and $\mathfrak{o}$ is the order of K with discriminant D. This induces the following lemma.

Lemma 2.7. *It holds a bijection from* $C^{pr}(D)/\mathcal{O}^1$ *to* $I(K, \mathfrak{o})/\mathcal{O}^1$.

Proof. For $x, x' \in C^{pr}(D)$, take φ_x and $\varphi_{x'} \in I(K, \mathfrak{o})$ such that $\varphi_x(\sqrt{D}) = x$ and $\varphi_{x'}(\sqrt{D}) = x'$. Then for $z = p + q\sqrt{D} \in K$, where $K = \mathbb{Q}(\sqrt{D})$ and $p, q \in \mathbb{Q}$, we have

$$\text{(2.2)} \qquad \varphi_x(z) = p + qx \quad \text{and} \quad \varphi_{x'}(z) = p + qx'.$$

When x' is $\mathcal{O}^1$-equivalent to x, there exists $\gamma \in \mathcal{O}^1$ such that $x' = \gamma x \gamma^{-1}$. Then we have

$$\gamma\varphi_x(z)\gamma^{-1} = \gamma(p + qx)\gamma^{-1} = p + qx' = \varphi_{x'}(z).$$

Conversely, assume φ_x is $\mathcal{O}^1$-equivalent to $\varphi_{x'}$. Then there exists $\gamma \in \mathcal{O}^1$ such that $\gamma\varphi_x(z)\gamma^{-1} = \varphi_{x'}(z)$. Taking (2.2) into account, since $\gamma\varphi_x(z)\gamma^{-1} = p + q\gamma x\gamma^{-1}$, we get $\gamma x \gamma^{-1} = x'$. ∎

In view of the theorem of Eichler (Theorem 2.2), by applying Lemma 2.7 we have

Proposition 2.8. *Let $D \in \mathbb{Z}_{>0}$ be a discriminant satisfying the conditions* (Pr-i), (Pr-ii). *Then*

$$\sharp\Big(C^{pr}(D)/\mathcal{O}^1\Big) = h(D)\lambda(D).$$

The eigenvalues λ of the element of $\Gamma_{\mathcal{O}} \subset SL_2(\mathbb{R})$ associated to $\varepsilon \in \mathcal{O}^1$ by injection (1.2) are given by

$$\lambda = \frac{d \pm \sqrt{d^2-4}}{2},$$

where $d = \mathrm{tr}(\varepsilon)$. Now we write $N_B(\varepsilon)$ for the norm of the element associated to ε. From the correspondence in Lemma 2.5, we have

$$N_B(\varepsilon) = \Big(\varphi^{-1}(\varepsilon)\Big)^2.$$

The Selberg zeta function attached to $\mathcal{O}^1$ is denoted by

$$Z_B(s) = \prod_{\varepsilon \in Prm^+(\mathcal{O}^1)/\mathcal{O}^1} \prod_{m=0}^{\infty} (1 - N_B(\varepsilon)^{-s-m}).$$

From Lemmas 2.4, 2.6 and Proposition 2.8, we have Theorem 1.1.

3. Brun-Tichmarsh type prime geodesic theorem

We introduce the following two theorems.

Theorem 3.1 (Landau [L], p.196). *Let D be a positive discriminant. Then we have*

$$h(D) = \frac{\sqrt{D}}{\log \varepsilon_D} \sum_{n=1}^{\infty} \frac{\chi_D(n)}{n},$$

where $\chi_D(n) = \left(\frac{D}{n}\right)$ is a Kronecker's symbol.

Theorem 3.2. *For $0 < Y < t$, put $S(Y,t)$ to be the character sum*

$$S(Y,t) := \sum_{Y \le n \le t} \chi_D(n).$$

Then it gives that

$$\Big|S(Y,t)\Big| \ll |D|^{\frac{1}{2}} \log |D|.$$

For a proof we refer to Davenport [D, p.135].

These estimates lead to the following proposition.

Proposition 3.3. *Let $D \in \mathbb{Z}_{>0}$ be a positive discriminant. Then*

$$h(D) \ll D^{\frac{1}{2}}$$

as $D \to \infty$.

Proof. We estimate $\sum_{n=1}^{\infty} \frac{\chi_D(n)}{n}$ by breaking up the sum into $n < Y$ and $n \ge Y$, Y to be determined.

For the first sum, we use a trivial bound;

$$\Big|\sum_{n<Y} \frac{\chi_D(n)}{n}\Big| \le \sum_{n<Y} \frac{1}{n} \ll \log Y.$$

On the second sum, since the summation by parts gives $\sum_{n \ge Y} \frac{\chi_D(n)}{n} = \int_Y^{\infty} \frac{S(Y,t)}{t^2} dt$, Theorem 3.2 leads to

$$\sum_{n \ge Y} \frac{\chi_D(n)}{n} \ll \int_Y^{\infty} \frac{D^{\frac{1}{2}} \log D}{t^2} dt = \frac{D^{\frac{1}{2}} \log D}{Y}.$$

These give

$$\Big|\sum_{n=1}^{\infty} \frac{\chi_D(n)}{n}\Big| \ll \log Y + \frac{D^{\frac{1}{2}} \log D}{Y}.$$

On taking $Y = D^{\frac{1}{2}}$, we get

$$\Big|\sum_{n=1}^{\infty} \frac{\chi_D(n)}{n}\Big| \ll \log D.$$

Since $\log \varepsilon_D \gg \log D$ by definition of ε_D, we have the proposition from Theorem 3.1. ∎

By using Proposition 3.3 and the following estimates for the divisor function $\tau(u)$ of a natural number u, Theorem 1.3 will be proved.

Lemma 3.4. *For any $\alpha > 1$ and $x \ge 2$,*

$$\sum_{u < \sqrt{x}} \frac{\tau(u)}{u^{\alpha}} \ll_{\alpha} 1 \quad \textit{and} \quad \sum_{u<\sqrt{x}} \frac{\tau(u)}{u} \ll (\log x)^2.$$

where for the first inequality the implied constant depends only on α.

Lemma 3.5. *We have*

$$\sharp\{n \mid n^2 \equiv 4 \pmod{u^2},\ n < u^2\} \ll \tau(u),$$

where u and n are integers.

Proof of Theorem 1.3. Let B, $\mathcal{O}$, and $\mathcal{O}^1$ be the same as before. Set $\Gamma = \mathcal{O}^1 \subset SL_2(\mathbb{R})$. By the definition of $\pi_B(x)$,

$$\pi_B(x+y) - \pi_B(x) = \sum_{\substack{\varepsilon \\ x < N_B(\varepsilon) \le x+y}} 1,$$

where the sum is taken over $\varepsilon \in Prm^+(\mathcal{O}^1)/\mathcal{O}^1$ satisfying $x < N_B(\varepsilon) \le x+y$. We write this sum in terms of positive discriminants D satisfying the conditions (Pr-i) and (Pr-ii) in Section 1:

$$\pi_B(x+y) - \pi_B(x) = \sideset{}{^*}\sum_{\substack{D>0 \\ \sqrt{x} < \varepsilon_D \le \sqrt{x+y}}} h(D)\lambda(D).$$

Let $t(B)$ denote the number of distinct primes dividing $d(B)$. Then obviously,

$$\lambda(D) \leq 2^{t(B)}.$$

We have

$$\pi_B(x+y) - \pi_B(x) \leq 2^{t(B)} \sum_{\substack{D \\ \sqrt{x}<\varepsilon_D\leq\sqrt{x+y}}}^{*} D^{\frac{1}{2}} \ll \sum_{\substack{D \\ \sqrt{x}<\varepsilon_D\leq\sqrt{x+y}}} D^{\frac{1}{2}}.$$

The estimate of the right hand side is proved by Iwaniec [I]. We give here a more detailed presentation of that proof. Put $\varepsilon_D = \frac{\alpha+\beta\sqrt{D}}{2}$ with $\alpha,\ \beta \in \mathbb{Z}_{>0}$. From the condition on ε_D, there follows that

$$2\sqrt{x} < \alpha + \beta\sqrt{D} \leq 2\sqrt{x+y}, \tag{3.1}$$

and the inverse of each term gives

$$\frac{2}{\sqrt{x+y}} \leq \alpha - \beta\sqrt{D} < \frac{2}{\sqrt{x}}, \tag{3.2}$$

since (α,β) is a solution of the Pell equation.

From (3.2) we have $\alpha = \beta\sqrt{D} + T$ with

$$\frac{2}{\sqrt{x+y}} \leq T \leq \frac{2}{\sqrt{x}}.$$

By combining this with (3.1), we have

$$\sqrt{x} + \frac{T}{2} < \alpha \leq \sqrt{x+y} + \frac{T}{2}. \tag{3.3}$$

By expanding

$$\sqrt{x+y} = \sqrt{x} + \frac{y}{2\sqrt{x}} + E$$

with E the error term satisfying $E = O(x^{-\frac{3}{2}}y^2)$ as $y < x$, (3.3) can be written by

$$\sqrt{x} + \frac{T}{2} < \alpha \leq \sqrt{x} + \frac{T}{2} + \frac{y}{2\sqrt{x}} + E. \tag{3.4}$$

We denote the region of α expressed in (3.4) by $\mathcal{T}$. Then we have

$$\pi_B(x+y) - \pi_B(x) \ll \sum_{\alpha\in\mathcal{T}} \sum_{\substack{\beta \\ \alpha^2-D\beta^2=4}} D^{\frac{1}{2}}.$$

By the Pell equation, we have $D \ll \left(\frac{\alpha}{\beta}\right)^2$. Hence

$$\pi_B(x+y) - \pi_B(x) \ll \sqrt{x} \sum_{\beta<2\sqrt{x}} \frac{1}{\beta} \sum_{\substack{\alpha\in\mathcal{T} \\ \alpha^2\equiv 4 \pmod{\beta^2}}} 1.$$

The last sum over α is estimated by

$$\tau(\beta)\Big(\frac{1}{\beta^2}\Big(\frac{y}{\sqrt{x}} + E\Big) + 1\Big)$$

from Lemma 3.5. The estimates in Lemma 3.4 now give

$$\pi_B(x+y) - \pi_B(x) \ll y + \frac{y^2}{x} + \sqrt{x}(\log x)^2.$$

It is estimated by y as long as $x^{\frac{1}{2}}(\log x)^2 < y < x$. ∎

Remark 3.6. *Zeev Rudnick pointed out that for $1 < y < x$ we can prove*

$$\pi_B(x+y) - \pi_B(x) \ll y \log x$$

by omitting the congruence condition at the cost of increasing the number of solutions in the proof of Theorem 1.3.

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