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Prime geodesic theorem for hyperbolic 3-manifolds: General cofinite cases

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PRIME GEODESIC THEOREM FOR HYPERBOLIC 3-MANIFOLDS: GENERAL COFINITE CASES

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Running Title. Prime Geodesic Theorem for Hyperbolic 3-manifolds

Abstract. We obtain a lower bound for the error term of the prime geodesic theorem for hyperbolic 3-manifolds of finite volume. Under the assumption that the contribution of the discrete spectra is larger than that of the continuous spectra, our result is $\Omega_{\pm}\left(\frac{x(\log\log x)^{\frac{1}{3}}}{\log x}\right)$. Without the assumption we have $\Omega(x^{1-\varepsilon})$.

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1. Introduction

For a $(d+1)$-dimensional hyperbolic manifold with Γ being the fundamental group, the prime geodesic theorem is

$$\pi_\Gamma(x) = \mathrm{li}(x^d) + \sum_{n=1}^{M} \mathrm{li}(x^{s_n}) + (\text{error}), \tag{1.1}$$

where $\pi_\Gamma(x)$ is the number of prime geodesics P whose length $l(P)$ satisfies that $N(P) := e^{l(P)} \le x$, and $s_1, \ldots\ s_M$ are the zeros of the Selberg zeta function $Z_\Gamma(s)$ in the interval $(\frac{d}{2}, d)$. The chief concern of this paper is to give lower estimates of the error term in (1.1).

Hejhal [5][6] obtained a lower bound in two-dimensional cases i.e. $d = 1$, by using the explicit formula for $\Psi_1(x) := \int_1^x \Psi_\Gamma(t)dt$. Here we put $\Psi_\Gamma(x) = \sum_{\substack{\{P\}\\ N(P)\le x}} \Lambda_\Gamma(P)$, where $\Lambda_\Gamma(P)$ is defined by

$$\frac{Z_\Gamma{}'}{Z_\Gamma}(s) = \sum_{\substack{\{P\}\\ N(P)\le x}} \frac{\Lambda_\Gamma(P)}{N(P)^s}$$

and is an analogue of the von-Mangoldt function in the theory of the Riemann zeta function. His result is as follows:

Theorem 1.1. *When $\Gamma \subset PSL(2,\mathbf{R})$ is a cocompact subgroup or a cofinite subgroup satisfying that*

$$\sum_{\gamma_n>0} \frac{x^{\beta_n-\frac{1}{2}}}{{\gamma_n}^2} = O\left(\frac{1}{1+(\log x)^2}\right), \tag{1.2}$$

it holds that

$$\pi_\Gamma(x) = \mathrm{li}(x) + \sum_{n=1}^{M} \mathrm{li}(x^{s_n}) + \Omega_\pm\left(\frac{x^{\frac{1}{2}}(\log\log x)^{\frac{1}{2}}}{\log x}\right) \quad \text{as } x \to \infty,$$

where $\beta_n + i\gamma_n$ are poles of the scattering determinant.

As we announced in [10], we generalize it to three-dimensional cases:

Theorem 1.2. *When $\Gamma \subset PSL(2,\mathbf{C})$ is a cocompact subgroup or a cofinite subgroup satisfying that*

$$\sum_{\gamma_n>0} \frac{x^{\beta_n-1}}{{\gamma_n}^2} = O\left(\frac{1}{1+(\log x)^3}\right), \tag{1.3}$$

it holds that

$$\pi_\Gamma(x) = \mathrm{li}(x^2) + \sum_{n=1}^{M} \mathrm{li}(x^{s_n}) + \Omega_\pm\left(\frac{x(\log\log x)^{\frac{1}{3}}}{\log x}\right) \quad \text{as } x \to \infty,$$

where $\beta_n + i\gamma_n$ are poles of the scattering determinant.

Assumptions (1.2) and (1.3) mean we can ignore the contribution of the continuous spectra. Hence, the proofs are reduced to those of cocompact cases.

In this paper, we will loosen the assumption (1.3) by considering a generalization of Weyl's law:

Proposition 1.3. [2, p. 307 Theorem 5.4] *Let $\Gamma \subset PSL(2,\mathbf{C})$ be a cofinite group, λ_n be the eigenvalues of the Laplacian on $L^2(\Gamma\backslash\mathbf{H}^3)$ with and $\varphi(s)$ is the determinant of the scattering matrix. We put*

$$N_\Gamma(T) := \sharp\{\lambda_n | \lambda_n < 1 + T^2\}.$$

Then

$$N_\Gamma(T) - \frac{1}{4\pi}\int_{-T}^{T} \frac{\varphi'}{\varphi}(1+it)dt \sim \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}^3)}{6\pi^2}T^3 \tag{1.4}$$

as $T \to \infty$.

The following result is our main theorem:

Theorem 1.4. *For $\Gamma \subset PSL(2,\mathbf{C})$, which satisfies that*

$$N_\Gamma(T) \sim \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}^3)}{6\pi^2}T^3, \tag{1.5}$$

we have

$$\pi_\Gamma(x) = \mathrm{li}(x^2) + \sum_{n=1}^{M} \mathrm{li}(x^{s_n}) + \Omega_\pm\left(\frac{x(\log\log x)^{\frac{1}{3}}}{\log x}\right) \quad \text{as } x \to \infty.$$

Though we imitate the proof of Theorem 1.1 for those of Theorems 1.2 and 1.4, by the reason that the order of $Z_\Gamma(s)$ is three, abundance of the zeros of $Z_\Gamma(s)$ gives rise to a difficulty concerning the estimate of $\Psi_1(x)$. We overcame it by considering the explicit formula for $\Psi_2(x) := \int_1^x \Psi_1(t)dt$.

Remark. *The conjectural exponent of x in the error term in* (1.1) *is* $\frac{d}{2}$. *Theorems 1.1, 1.2 and 1.4 give sharp estimates in that sense.*

When the assumption (1.5) doesn't hold, a different approach gives a weaker Ω-result, $\Omega(x^{1-\varepsilon})$. Combinintg it with Theorem 1.4, we have the following theorem:

Theorem 1.5. *When* $\Gamma \subset PSL(2,\mathbf{C})$, *then we have*

$$\pi_\Gamma(x) = \mathrm{li}(x^2) + \sum_{n=1}^{M} \mathrm{li}(x^{s_n}) + \Omega(x^{1-\varepsilon}) \quad \text{as } x \to \infty,$$

where ε *is any positive constant.*

In Section 3, we deal with cocompact cases. Its generalization to cofinite cases when the contribution of the continuous spectra is small is given in Section 4, where we will also show the assumption (1.3) implies (1.5). As an example which satisfies the assumption (1.3), we will see any Bianchi group in Example 4.11. In Section 5, we will give the proof of Theorem 1.5.

2. Preliminaries

Throughout this paper we put G to be $PSL(2,\mathbf{C})$ and Γ to be a cofinite subgroup of G. Let j be an element in the quaternion field which satisfies $j^2 = -1$, $ij = -ji$, and let $\mathbf{H}^3$ be the 3-dimensional hyperbolic space:

$$\mathbf{H}^3 := \{v = z + yj \mid z = x_1 + x_2 i \in \mathbf{C}, y > 0\}$$

with the Riemannian metric

$$dv^2 = \frac{dx_1{}^2 + dx_2{}^2 + dy^2}{y^2}.$$

The hyperbolic distance $d(v, v')$ is given by

$$\cosh d(v, v') = \frac{|z - z'|^2 + y^2 + y'^2}{2yy'},$$

where $v = z + yj$ and $v' = z' + y'j$. The volume measure is given by

$$\frac{dx_1 dx_2 dy}{y^3}.$$

The group $PSL(2,\mathbf{C})$ acts on $\mathbf{H}^3$ transitively by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(v) := (av+b)(cv+d)^{-1} = \frac{(az+b)\overline{(cz+d)} + a\bar{c}y^2 + yj}{|cz+d|^2 + |c|^2y^2}.$$

The Laplacian for $\mathbf{H}^3$ is defined by

$$\Delta := -y^2\left(\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial}{\partial y}.$$

We denote the eigenvalues of Δ by $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_M \le 1 < \lambda_{M+1}\cdots$.

The classification of conjugacy classes is given as follows:

Definition 2.1. *An element $P \in \Gamma - \{1\}$ is called*

parabolic	*iff*	$\lvert \mathrm{tr(P)} \rvert = 2$	*and*	$\mathrm{tr(P)} \in \mathbf{R}$,
hyperbolic	*iff*	$\lvert \mathrm{tr(P)} \rvert > 2$	*and*	$\mathrm{tr(P)} \in \mathbf{R}$,
elliptic	*iff*	$\lvert \mathrm{tr(P)} \rvert < 2$	*and*	$\mathrm{tr(P)} \in \mathbf{R}$,

and loxodromic in all other cases. An element of $PSL(2,\mathbf{C})$ is called parabolic, elliptic, hyperbolic, loxodromic if its preimages in $SL(2,\mathbf{C})$ have this property. A conjugacy class $\{P\}$ in Γ is called hyperbolic, elliptic, parabolic if each P in the class has this property.

The norm of a hyperbolic or loxodromic element P is defined by $N(P) = |a(P)|^2$, if $a(P) \in \mathbf{C}$ is the eigenvalue of $P \in G$ such that $|a(P)| > 1$.

Definition 2.2. *An element $P \in \Gamma - \{1\}$ is called primitive if and only if it is not an essential power of any other element. A conjugacy class $\{P\}$ in Γ is called primitive if each P in the class has this property.*

For every hyperbolic matrix $P \in \Gamma$ there exist exactly one primitive hyperbolic element $P_0 \in \Gamma$ and exactly one $n \in \mathbf{N}$ such that $P = {P_0}^n$. We define that $\pi_\Gamma(x)$ is the number of P_0 which is primitive hyperbolic or loxodromic and satisfies $N(P_0) \le x$.

Definition 2.3. *For $\mathrm{Re}(s) > 2$, the Selberg zeta function for Γ is defined by*

$$Z_\Gamma(s) := \prod_{\{P_0\}}\prod_{(k,l)}(1 - a(P_0)^{-2k}\overline{a(P_0)}^{-2l}N(P_0)^{-s}),$$

where the product on $\{P_0\}$ is taken over all primitive hyperbolic or loxodromic conjugacy classes of Γ, and (k,l) runs through all the pairs of positive integers satisfying the following congruence relation: $k \equiv l \pmod{m(P_0)}$ with $m(P)$ the order of the torsion of the centralizer of P.

Elstrodt-Grunewald-Mennicke proves the following lemma:

Lemma 2.4. [2, p. 208, Lemma 4.2] *For $\mathrm{Re}(s) > 2$, we have*

$$\frac{Z'_\Gamma}{Z_\Gamma}(s) = \sum_{\{P\}} \frac{N(P)\log N(P_0)}{m(P)|a(P) - a(P)^{-1}|^2}N(P)^{-s},$$

where P_0 is a primitive element associated with P, and $\{P\}$ runs through the hyperbolic or loxodromic conjugacy classes of Γ.

Recall that the von-Mangoldt function $\Lambda(n)$ appears in the logarithmic derivative of the Riemann zeta function:

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}. \tag{2.1}$$

Comparing Lemma 2.4 and (2.1), the following definition is natural.

Definition 2.5. *For a hyperbolic or loxodromic element P of* Γ, *we define*

$$\Lambda_\Gamma(P) := \frac{N(P) \log N(P_0)}{m(P)|a(P) - a(P)^{-1}|^2},$$

and

$$\Psi_\Gamma(x) := \sum_{\substack{\{P\} \\ N(P) \le x}} \Lambda_\Gamma(P),$$

where P_0 is a primitive element associated with P, and $\{P\}$ runs through hyperbolic or loxodromic classes of Γ.

Then we have

$$\frac{Z'_\Gamma}{Z_\Gamma}(s) = \sum_{\{P\}} \Lambda_\Gamma(P) N(P)^{-s}. \tag{2.2}$$

In what follows that we simply write $Z(s)$ for $Z_\Gamma(s)$.

3. Ω-Result for cocompact groups

We will show some properties of $Z(s)$ for cocompact Γ.

A determinant expression of Selberg zeta functions was discovered by Sarnak[12] and Voros[16] for compact Riemann surfaces. Koyama generalized it to 3-dimensional Bianchi groups [8, Theorem 4.4]. He expressed $Z(s)$ multiplied with some gamma factors in terms of the determinant of the Laplacian.

In the case of cocompact Γ, we do not have any contribution from the parabolic classes and the continuous spectra in the formula in [8]. We introduce the spectral zeta function generalized by a variables s:

$$\zeta(w, s, \Delta) := \sum_{n=0}^{\infty} \frac{1}{(\lambda_n - s(2-s))^w} \quad \left(\mathrm{Re}(w) > \frac{3}{2}\right).$$

Theorem 3.1. *Let*

$$\hat{Z}(s) := Z_I(s) Z_E(s) Z(s)$$

with

$$Z_I(s) = \exp\left(-\frac{\mathrm{vol}(\Gamma \backslash \mathbf{H}^3)}{6}(s-1)^3\right),$$

$$Z_E(s) = \exp\left(\sum_{\{R\}} \frac{\log N(P_0)}{2m(R)} \sum_{m=0}^{\nu_R - 1} \left(1 - \cos\frac{2m\pi}{\nu_R}\right)^{-1} s\right),$$

where $\{R\}$ also runs through all the primitive elliptic conjugacy classes of Γ, and ν_R is the order of R. We denote by $m(R)$ the order of the maximal finite subgroup of the centralizer of R.

Then

$$\hat{Z}(s) = e^{c+c's(2-s)}\mathrm{det}_D(\Delta - s(2-s)),$$

where det_D is the determinant of the Laplacian composed of the discrete spectra :

$$\mathrm{det}_D(\Delta - s(2-s)) := \exp\left(-\left.\frac{\partial}{\partial\omega}\right|_{\omega=0} \zeta(\omega, s, \Delta)\right). \tag{3.1}$$

The equation (3.1) is the zeta-regularization of a divergent product $\prod_{n=0}^{\infty}(\lambda_n - s(2-s))$.

From Theorem 3.1, the zeros of $Z(s)$ from the dicrete spectra are expressed as $s_n = 1 + i\sqrt{\lambda_n - 1}$ and $\tilde{s}_n = 1 - i\sqrt{\lambda_n - 1}$. Let $t_n := \sqrt{\lambda_n - 1}$. Then it leads to the following proposition:

Proposition 3.2. *We have,*

$$\frac{Z'}{Z}(s) = \frac{1}{s-2} + \sum_{|s-s_n|<1} \frac{1}{s-s_n} + \sum_{|s-\tilde{s}_n|<1} \frac{1}{s-\tilde{s}_n} + O(|s|^2+1),$$

where $s_n = 1 + it_n$ and $\tilde{s}_n = 1 - it_n$ are the zeros of $Z(s)$ on $\mathrm{Re}(s) = 1$.

Proof. From Theorem 3.1, we get

$$\frac{Z'}{Z}(s) + \frac{Z_I'}{Z_I}(s) + \frac{Z_E'}{Z_E}(s) = c'(2s-2) + \frac{d}{ds}\left(\log(\mathrm{det}_D(\Delta - s(2-s)))\right), \tag{3.2}$$

together with

$$\frac{Z_I'}{Z_I}(s) = O(|s|^2+1), \tag{3.3}$$

and

$$\frac{Z_E'}{Z_E}(s) = O(1). \tag{3.4}$$

About the right hand side of (3.2), by Hadamard's theory, we have

$$\mathrm{det}_D\left(\Delta - s(2-s)\right) = e^{g(s)} \prod_{n=0}^{\infty} \left(1 - \frac{s}{s_n}\right)\left(1 - \frac{s}{\tilde{s}_n}\right) e^{\frac{s}{s_n} + \frac{s}{\tilde{s}_n}}, \tag{3.5}$$

where $g(s)$ is an integral function and the product is absolutely convergent for all s. Elstrodt-Grunewald-Mennicke [2, p. 215 Lemma 5.8] shows the order of (3.5) is three. Then we have

$$\frac{d}{ds}(\log(\det_D(\Delta - s(2-s))))$$
$$= O(|s|^2) + \frac{1}{s-2} + \sum_{n=0}^{\infty}\left(\frac{1}{s-s_n} + \frac{1}{s_n}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{s-\tilde{s}_n} + \frac{1}{\tilde{s}_n}\right). \quad (3.6)$$

Gathering together (3.2), (3.3), (3.4) and (3.6), we have Proposition 3.2. □

As the existence problem for eigenvalues of Δ, we have Weyl's law as follows:

Proposition 3.3. [2, p. 215, Theorem 5.6] *Let $\Gamma \subset PSL(2,\mathbf{C})$ be a cocompact subgroup, the counting function for eigenvalues of Δ in $L^2(\Gamma\backslash\mathbf{H}^3)$, which is denoted by $N_\Gamma(T) := \sharp\{\lambda_n \mid \lambda_n < 1 + T^2\}$, satisfies*

$$N_\Gamma(T) = \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}^3)}{6\pi^2}T^3 + O(T^2) \quad as \quad T \to \infty.$$

From Propositions 3.2 and 3.3, we obtain the following estimates:

Lemma 3.4. *For $\varepsilon > 0$, we get*

$$\frac{Z'}{Z}(1+\varepsilon+it) \ll \frac{|t|^2}{\varepsilon}, \quad (|t| \geq 2) \quad (3.7)$$

$$\frac{Z'}{Z}(2+\varepsilon+it) \ll \frac{1}{\varepsilon}, \quad (3.8)$$

$$\frac{Z'}{Z}(-\varepsilon+it) \ll |t|^2 + 1, \quad (3.9)$$

$$\left|\frac{Z'}{Z}(s)\right| \ll |t|^{2\max(0,2-\sigma)}\log|t| \;\; \left(s = \sigma + it,\ \sigma > 1 + \frac{1}{\log|t|}, |t| \geq 2\right). \quad (3.10)$$

Moreover, for any T there exists τ in $[T, T+1]$ such that

$$\int_0^2 \left|\frac{Z'}{Z}(\sigma + i\tau)\right| d\sigma \ll T^2 \log T. \quad (3.11)$$

Proof. By Propositions 3.2 and 3.3, we get

$$\frac{Z'}{Z}(1+\varepsilon+it) \ll \frac{1}{\varepsilon+it-1} + \frac{t^2}{\varepsilon} + O(|t|^2).$$

This implies (3.7). Here ε is any number with $0 < \varepsilon \leq \frac{1}{2}$.

Since Definition 2.3 converges for $\mathrm{Re}(s) > 2$, we have from Proposition 3.2 that

$$\frac{Z'}{Z}(s) = \frac{1}{s-2} + O(1) \quad \text{as} \quad s \to 2,$$

for $\mathrm{Re}(s) > 2$. It leads us to (3.8).

For proving (3.9), we again appeal to Proposition 3.2. Putting $s = -\varepsilon + it$ gives the conclusion.

The equation (3.10) is the consequence of (3.7) and (3.8) together with the Phragmen-Lindelöf principle.

To see (3.11), we integrate the left hand side of (3.11) over τ in

$$\mathcal{T} := \{\tau \mid T < \tau < T+1,\ |\tau - t_n| \geq T^{-3}\}. \tag{3.12}$$

By Propositions 3.2, 3.3 and (3.12), we estimate

$$\int_{\mathcal{T}} \int_0^2 \left| \frac{Z'}{Z}(\sigma + i\tau) \right| d\sigma d\tau \ll \int_0^2 \sum_{|s_n - T| \leq 2} \log T d\sigma$$
$$\ll T^2 \log T.$$

Since $|\mathcal{T}| = 1 + O(T^{-2}) > \frac{1}{2}$ for sufficiently large T, we have (3.11) for that T. For small T the estimate is trivial. □

Elstrodt-Grunewald-Mennicke also shows the functional equation:

Lemma 3.5. [2, p. 209, Corollary 4.4] *The Selberg zeta function* $Z(s)$ *satisfies*

$$Z(2-s) = \exp\left(-\frac{\mathrm{vol}(\Gamma \backslash \mathbf{H}^3)}{3\pi}(s-1)^3 + 2E(s-1) \right) Z(s),$$

where $E := \sum_{\{R\}} \frac{\log N(P_0)}{m(R)|(\mathrm{trR})^2 - 4|}$, *and the summation of* $\{R\}$ *is taken over all elliptic conjugacy classes of* Γ.

This lemma leads to

$$\frac{Z'}{Z}(s) + \frac{Z'}{Z}(2-s) = O(|s|^2). \tag{3.13}$$

We now consider $\Psi_\Gamma(x)$. Let $\Psi_1(x) := \int_1^x \Psi_\Gamma(t)dt$ and $\Psi_2(x) := \int_1^x \Psi_1(t)dt$.

By using the following theorem, we can express Ψ_1 and Ψ_2 in terms of $N(P)$ and $\Lambda_\Gamma(P)$.

Theorem 3.6. [7, Theorem A] *Let* $\lambda_1, \lambda_2, \cdots$, *be a real sequence which increases (in the wide sense) and has the limit infinity, and let*

$$C(x) = \sum_{\lambda_n \leq x} c_n,$$

where the c_n *may be real or complex, and the notation indicates a summation over the (finite) set of positive integers* n *for which* $\lambda_n \leq x$. *Then, if* $X \geq \lambda_1$ *and* $\phi(x)$ *has a continuous derivative, we have*

$$\sum_{\lambda_n \leq X} c_n \phi(\lambda_n) = -\int_{\lambda_1}^X C(x)\phi'(x)dx + C(X)\phi(X).$$

If, further, $C(X)\phi(X) \to 0$ as $X \to \infty$, then

$$\sum_{n=1}^{\infty} c_n \phi(\lambda_n) = -\int_{\lambda_1}^{\infty} C(x)\phi'(x)dx,$$

provided that either side is convergent.

We have the following lemma:

Lemma 3.7. *It holds that*

$$\Psi_1(x) = \sum_{N(P)\le x} (x - N(P))\Lambda_\Gamma(P),$$

$$2\Psi_2(x) = \sum_{N(P)\le x} (x - N(P))^2\Lambda_\Gamma(P).$$

Proof. The first assertion is obtained by substituting $n = P,\ \lambda_n = N(n),\ \phi(x) = x - N(n)$ and $c_n = \Lambda_\Gamma(n)$ in Theorem 3.6. Similarly the second one is obtained by putting $n = P,\ \lambda_n = N(n),\ \phi(x) = (x - N(n))^2$ and $c_n = \Lambda_\Gamma(n)$ in Theorem 3.6. □

Theorem 3.8. [7, p. 31, Theorem B] *If k is a positive integer, $c > 0$, $y > 0$, then*

$$\frac{1}{2\pi i}\int_{c-\infty si}^{c+\infty i} \frac{y^s ds}{s(s+1)\dots(s+k)} = \begin{cases} 0 & (y \le 1), \\ \frac{1}{k!}(1-\frac{1}{y})^k & (y \ge 1). \end{cases}$$

Substituting $k = 2$ and $y = \frac{x}{N(P)}$ in Theorem 3.8, by comparing to Lemma 3.7 the expression of $\Psi_2(x)$ is obtained as follows:

$$\Psi_2(x) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s+2}}{s(s+1)(s+2)} \sum_{N(P)\le x} \frac{\Lambda_\Gamma(P)}{N(P)^s} ds \quad \text{for} \quad c > 2.$$

From (2.2), we can express

$$\Psi_2(x) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s+2}}{s(s+1)(s+2)} \frac{Z'}{Z}(s)ds. \tag{3.14}$$

We now get the following theorem by estimating (3.14).

Theorem 3.9. *We have*

$$\Psi_1(x) = \alpha x + \beta x \log x + \alpha_1 + \sum_{n=0}^{M} \frac{x^{s_n+1}}{s_n(s_n+1)} + \sum_{n=0}^{M} \frac{x^{\tilde{s}_n+1}}{\tilde{s}_n(\tilde{s}_n+1)} + \sum_{t_n \ge 0} \frac{x^{s_n+1}}{s_n(s_n+1)} + \sum_{t_n > 0} \frac{x^{\tilde{s}_n+1}}{\tilde{s}_n(\tilde{s}_n+1)}$$

with some constants α, β and α_1, where $s_n = 1 + it_n$ and $\tilde{s}_n = 1 - it_n$ are the zeros of $Z(s)$.

Proof. Suppose $T \geq 1000$, and let $A := N + \frac{1}{2}$ where N is a positive integer. Let

$$R(A,T) := \{z \in \mathbf{C} \mid -A \leq \mathrm{Re}(z) \leq 3, -T \leq \mathrm{Im}(z) \leq T\}.$$

By applying the Cauchy's theorem on $R(A,T)$, we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_{3-iT}^{3+iT} \frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)ds \\ &\quad = \frac{1}{2\pi i}\left(\int_{-A-iT}^{-A+iT} + \int_{-A+iT}^{3+iT} - \int_{-A-iT}^{3-iT}\right)\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)ds \\ &\qquad + \sum_{z\in R(A,T)} \mathrm{Res}_{s=z}\left(\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)\right). \end{aligned} \tag{3.15}$$

We now estimate each integral in the right hand side of (3.15), which will be denoted by I_1, I_2 and I_3.

We first consider I_1. From (3.13), we have for $A \geq \frac{3}{2}$

$$\int_{-A+i}^{-A+iT} \frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)ds = \int_{-A+i}^{-A+iT} \frac{x^{s+2}}{s(s+1)(s+2)}\left(-\frac{Z'}{Z}(2-s) + O(|s|^2)\right)ds.$$

Since $-\frac{Z'}{Z}(2-s) = O(1)$, the right hand side of this equation is $O(x^{2-A})$. By taking it into account that

$$\left|\int_{-A}^{-A+i} \frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)ds\right| \leq 1 \times \left|\frac{x^{2-A}}{A^3} \times O(A^2)\right| = O(x^{2-A}),$$

we obtain

$$I_1 = O(x^{2-A}). \tag{3.16}$$

For the next integral I_2, we divide it into the following three parts,

$$\int_{-A+iT}^{3+iT} \frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)ds = \int_{-A+iT}^{-1+iT} + \int_{-1+iT}^{1+iT} + \int_{1+iT}^{3+iT}.$$

We put them to be J_1, J_2 and J_3, respectively.

By the functional equation (3.13), we have

$$J_1 + J_2 \leq \left(\int_{-A+iT}^{-1+iT} + \int_{-1+iT}^{1+iT}\right)\frac{x^{\sigma+2}}{T^3}\left\{\left|\frac{Z'}{Z}(2-s)\right| + O(|s|^2)\right\}|ds|.$$

Since $|\frac{Z'}{Z}(2-s)| \leq |\frac{Z'}{Z}(3)|$ and $\int_{-1+iT}^{1+iT}\left|\frac{Z'}{Z}(2-s)\right||ds| = -\int_{3-iT}^{1-iT}\left|\frac{Z'}{Z}(s)\right||ds|$, we have

$$J_1 + J_2 = O\left(\frac{x^2(1-x^{-A})}{T\log x}\right) + O\left(\int_{1-iT}^{3-iT}\frac{x^{\sigma+2}}{T^3}\left|\frac{Z'}{Z}(s)\right||ds|\right) + O\left(\frac{x^2(x-1)}{T\log x}\right), \tag{3.17}$$

where the first estimate in the right hand side is from J_1 and the second and third terms are from J_2.

For J_3, it holds obviously that

$$J_3 \leq \int_{1+iT}^{3+iT} \frac{x^{\sigma+2}}{T^3} \left| \frac{Z'}{Z}(s) \right| |ds|. \tag{3.18}$$

The equation (3.14) for $c = 3$ shows

$$\Psi_2(x) = \frac{1}{2\pi i} \int_{3-iT}^{3+iT} \frac{x^{s+2}}{s(s+1)(s+2)} \frac{Z'}{Z}(s) ds + O\left(\frac{x^5}{T^2}\right).$$

Therefore the equation (3.15) now becomes

$$\Psi_2(x) + O\left(\frac{x^5}{T^2}\right) = O(x^{2-A}) + O\left(\frac{x^3}{T\log x}\right) + O\left(\frac{1}{T^3}\int_{1+iT}^{3+iT} x^{\sigma+2}\left|\frac{Z'}{Z}(s)\right| |ds|\right) + \sum_{z\in R(A,T)} \mathrm{Res}_{s=z}\left(\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)\right) \tag{3.19}$$

by gathering together (3.16), (3.17) and (3.18).

We will estimate $\int_1^3 \frac{x^{\sigma+2}}{T^3}|\frac{Z'}{Z}(\sigma+iT)|d\sigma$. By using Lemma 3.4, we have

$$\int_1^3 \frac{x^{\sigma+2}}{T^3}\left|\frac{Z'}{Z}(\sigma+iT)\right| d\sigma \ll \int_1^{1+\varepsilon} \frac{x^{\sigma+2}}{T^3} T^2 d\sigma + \int_{1+\varepsilon}^2 \frac{x^{\sigma+2}}{T^3} T^{2(2-\sigma)} d\sigma + \int_2^3 \frac{x^{\sigma+2}}{T^3}\frac{1}{\varepsilon} d\sigma$$
$$\ll \frac{x^{3+\varepsilon}}{T\log x} + \frac{\frac{x^4}{T^3} - \frac{x^{3+\varepsilon}}{T^{1+2\varepsilon}}}{\log x - 2\log T} + \frac{x^4(x-1)}{T^3\log x} \ll \frac{x^5}{T(\log x - 2\log T)}.$$

Calculating the residues leads to

$$\begin{aligned}\Psi_2(x)+O\left(\frac{x^5}{T^2}\right) &= O(x^{2-A}) + O\left(\frac{x^3}{T\log x}\right) + O\left(\frac{x^5}{T(\log x - 2\log T)}\right)\\ &+ \sum_{n=0}^{M} \frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)} + \sum_{n=0}^{M} \frac{x^{\tilde{s}_n+2}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)}\\ &+ \sum_{t_n\geq 0} \frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)} + \sum_{t_n\geq 0} \frac{x^{\tilde{s}_n+2}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)}\\ &+ \alpha_0 x^2 + \beta_0 x^2\log x + \alpha_1 x + \alpha_2.\end{aligned} \tag{3.20}$$

Here we have used the following calculations for the residues:

For $s = s_n$ and $s = \tilde{s_n}$, it is easily seen that

$$\mathrm{Res}_{s=s_n}\left(\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)\right) = \frac{\mu(n)x^{s_n+2}}{s_n(s_n+1)(s_n+2)} \quad (n \geq 1),$$

$$\mathrm{Res}_{s=\tilde{s}_n}\left(\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)\right) = \frac{\mu(n)x^{\tilde{s}_n+2}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)} \quad (n \geq 0),$$

where $\mu(n)$ is the multiplicity of s_n.

Since we have by Theorem 3.1

$$\frac{Z'}{Z}(s) = \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}^3)}{2} - a + 2c' + \frac{1}{s} - \frac{1}{2} + \left(-\mathrm{vol}(\Gamma\backslash\mathbf{H}^3) - 2c' - \frac{1}{4}\right)s + \cdots$$

with

$$a := \sum_{\{R\}} \frac{\log N(P_0)}{2m(R)} \sum_{m=0}^{\nu_R-1} \left(1 - \cos\frac{2m\pi}{\nu_R}\right)^{-1},$$

we can express for $s = 0$

$$\mathrm{Res}_{s=0}\left(\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)\right) = \alpha_0 x^2 + \beta_0 x^2 \log x,$$

where $\alpha_0 := \frac{1}{2}\left(\frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}^3)}{2} - a + 2c' - 2\right)$ and $\beta_0 := \frac{1}{2}$.

For the case of $s = -1$ and $s = -2$, we have

$$\mathrm{Res}_{s=-1}\left(\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)\right) = \alpha_1 x,$$

$$\mathrm{Res}_{s=-2}\left(\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)\right) = \alpha_2,$$

with some constants α_1 and α_2.

As both A and T go to ∞ in (3.20), we obtain

$$\Psi_2(x) = \alpha_0 x^2 + \beta_0 x^2 \log x + \alpha_1 x + \alpha_2 + \sum_{n=0}^{M} \frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)} + \sum_{n=0}^{M} \frac{x^{\tilde{s}_n+2}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)}$$
$$+ \sum_{t_n \geq 0} \frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)} + \sum_{t_n \geq 0} \frac{x^{\tilde{s}_n+2}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)}. \quad (3.21)$$

Recall $\Psi_2(x) = \int_1^x \Psi_1(t)dt$, and we get the theorem. □

Our next goal is to show an Ω-result for

$$P(x) := \Psi_\Gamma(x) - \left(\alpha + \beta \log x + \beta + \sum_{n=0}^{M} \frac{x^{s_n}}{s_n} + \sum_{n=0}^{M} \frac{x^{\tilde{s}_n}}{\tilde{s}_n}\right). \quad (3.22)$$

Definition 3.10. *We define*

$$P_1(x) := \int_0^x P(t)dt, \quad (3.23)$$

$$P_2(x) := \int_0^x P_1(t)dt, \quad (3.24)$$

and further

$$\mathcal{P}(x) := P(x) - N(0)x, \quad (3.25)$$

$$\mathcal{P}_1(x) := P_1(x) - \frac{1}{2}N(0)x^2,$$

$$\mathcal{P}_2(x) := P_2(x) - \frac{1}{6}N(0)x^3.$$

Then we have

$$\mathcal{P}_1(x) = b_1 + \int_2^x \mathcal{P}(t)dt, \tag{3.26}$$

and

$$\mathcal{P}_2(x) = b_2 + \int_2^x \mathcal{P}_1(t)dt. \tag{3.27}$$

with constants b_1 and b_2.

The function $\mathcal{P}$ has the following property:

Lemma 3.11. *There exists $b_3 \in \mathbf{C}$ such that*

$$b_3 + \int_1^v \frac{\mathcal{P}(e^u)}{e^u}du = \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)}.$$

Proof. Put

$$F(v) := \beta_1 + \int_1^v \frac{\mathcal{P}(e^u)}{e^u}du \quad \text{for} \quad v \geq 1,$$

where $\beta_1 \in \mathbf{C}$ is unspecified temporarily. By changing of variables with $x = e^u$, we have

$$F(v) = \beta_1 + \int_e^{e^v} \frac{\mathcal{P}(x)}{x^2}dx.$$

By integration by parts and (3.26), $F(v)$ is written with a constant b_4 as

$$F(v) = \beta_1 + b_4 + \frac{\mathcal{P}_1(e^v)}{e^{2v}} + 2\int_e^{e^v} \frac{\mathcal{P}_1(x)}{x^3}dx.$$

Integrating by parts again, and from (3.27) it follows that

$$F(v) = \beta_1 + b_5 + \frac{\mathcal{P}_1(e^v)}{e^{2v}} + 2\frac{\mathcal{P}_2(e^v)}{e^{3v}} + 6\int_e^{e^v} \frac{\mathcal{P}_2(x)}{x^4}dx \tag{3.28}$$

with some constant b_5.

Since applying Theorem 3.9 leads to

$$\mathcal{P}_1(x) = \sum_{t_n>0} \frac{x^{s_n+1}}{s_n(s_n+1)} + \sum_{t_n>0} \frac{x^{\tilde{s}_n+1}}{\tilde{s}_n(\tilde{s}_n+1)},$$

we have

$$\frac{\mathcal{P}_1(e^v)}{e^{2v}} = \sum_{t_n>0} \left(\frac{e^{(s_n-1)v}}{s_n(s_n+1)} + \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n+1)} \right).$$

From (3.21) and (3.27), we obtain

$$\frac{\mathcal{P}_2(e^v)}{e^{3v}} = \sum_{t_n>0} \left(\frac{e^{(s_n-1)v}}{s_n(s_n+1)(s_n+2)} + \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)} \right)$$

and

$$\int_e^{e^v} \frac{\mathcal{P}_2(x)}{x^4}dx = \sum_{t_n>0} \left(\frac{e^{(s_n-1)v}}{s_n(s_n+1)(s_n+2)(s_n-1)} + \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)(\tilde{s}_n-1)} \right).$$

Inserting these calculations into (3.28) gives

$$\begin{aligned} F(v) &= \beta_1 + b_5 + \sum_{t_n>0} \left(\frac{e^{(s_n-1)v}}{s_n(s_n+1)} + \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n+1)} \right) \\ &\quad + 2\sum_{t_n>0} \left(\frac{e^{(s_n-1)v}}{s_n(s_n+1)(s_n+2)} + \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)} \right) \\ &\quad + 6\sum_{t_n>0} \left(\frac{e^{(s_n-1)v}}{s_n(s_n+1)(s_n+2)(s_n-1)} + \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)(\tilde{s}_n-1)} \right) \\ &= \beta_1 + b_5 + \sum_{t_n>0} \left(\frac{e^{(s_n-1)v}}{s_n(s_n-1)} + \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)} \right). \end{aligned}$$

By taking $\beta_1 = -b_5 =: b_3$, we have the lemma. □

In what follows we put

$$F(v) := b_3 + \int_1^v \frac{\mathcal{P}(e^u)}{e^u}du. \tag{3.29}$$

Similarly, we find the following property:

Lemma 3.12. *There exists $b_6 \in \mathbf{C}$ such that*

$$b_6 + \int_1^v F(u)du = \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)^2} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)^2}.$$

In what follows we put

$$G(v) := b_6 + \int_1^v F(u)du. \tag{3.30}$$

Further for $G(v)$ we have

Lemma 3.13. *There exists $b_8 \in \mathbf{C}$ such that*

$$b_8 + \int_1^v G(u)du = \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)^3} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)^3}.$$

In what follows we put

$$H(v) := b_8 + \int_1^v G(u)du. \tag{3.31}$$

Lemma 3.12 and 3.13 are proved similarly to Lemma 3.11. We note $H(v)$ is uniformly convergent for all $v \in \mathbf{R}$. Therefore it is possible to extend the definition of $H(v)$ to all $\mathbf{R}$ by using the series representation.

We here introduce the following lemma.

Lemma 3.14. *Let* $k(x) := (\frac{\sin \pi x}{\pi x})^2$. *Then*

a) $k(x)$ *is a* C^∞*-function on* $\mathbf{R}$,

b) $k(x), k'(x), k''(x), k'''(x)$ *are all* $O(x^{-2})$ *when* $|x| \to \infty$,

c) $\int_{-\infty}^{\infty} k(x)e^{iux}dx = \max[0, 1 - \frac{|u|}{2\pi}]$.

Proof. Every statement except for $k'''(x)$ in b) is proved by Hejhal [5, p. 264, Lemma 16.9]. The relevant property of $k'''(x)$ is also deduced by the same method. □

We will express

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv$$

with large r, A and N:

Lemma 3.15. *Let* A *be a positive constant. We have*

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv$$
$$= -\frac{2}{N} \sum_{0<t_n \le 2\pi N} \frac{\sin(t_n r)}{t_n} \left(1 - \frac{t_n}{2\pi N}\right) + O\left(\frac{1}{A^3}\right) + O\left(\frac{1}{r^3}\right) + O(1).$$

Proof. For convenience, we assume A and N are integers. From (3.29) and the property of $k(x)$, it follows that

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv = O\left(\frac{1}{N^2 r^2}\right) - N \int_1^{r+A} F(v)k'(N(v-r))dv. \tag{3.32}$$

From (3.30), the integral in the right hand side of (3.32) can be written as

$$O\left(\frac{1}{Nr^2}\right) + N^2 \int_1^{r+A} G(v)k''(N(v-r))dv.$$

Therefore using (3.31) yields

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv = O\left(\frac{1}{Nr^2}\right) - N^3 \int_1^{r+A} H(v)k'''(N(v-r))dv. \tag{3.33}$$

The function $H(v)$ has been defined for all $v \in \mathbf{R}$. Since $H(v)$ is uniformly bounded, we can estimate as follows:

$$N^3 \int_{r+A}^{\infty} |H(v)k'''(N(v-r))|dv = O\left(\frac{1}{A^3}\right),$$

and

$$N^3 \int_{-\infty}^{1} |H(v)k'''(N(v-r))|dv = O\left(\frac{1}{r^3}\right).$$

Hence (3.33) becomes

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv = O\left(\frac{1}{A^3}\right) + O\left(\frac{1}{r^3}\right) - N^3 \int_{-\infty}^{\infty} H(v)k'''(N(v-r))dv. \tag{3.34}$$

Since the series representation for $H(v)$ converges uniformly on $\mathbf{R}$, we can substitute

$$H(v) = \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)^3} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)^3}$$

into (3.34) and change the order of integration and summation. After integrating term-by-term, we obtain

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv$$
$$= O\left(\frac{1}{A^3}\right) + O\left(\frac{1}{r^3}\right) - \sum_{t_n>0} \int_{-\infty}^{\infty} \frac{1}{e^v} \left(\frac{e^{s_n v}}{s_n} + \frac{e^{\tilde{s}_n v}}{\tilde{s}_n}\right) k(N(v-r))dv. \quad (3.35)$$

By considering $s_n = 1 + it_n$, $\tilde{s}_n = 1 - it_n$ and changing of variables with $X := N(v-r)$, we have

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv$$
$$= O\left(\frac{1}{A^3}\right) + O\left(\frac{1}{r^3}\right) - \frac{2}{N} \sum_{t_n>0} \int_{-\infty}^{\infty} \mathrm{Re}\left(\frac{e^{it_n r} e^{i\left(\frac{t_n}{N}\right)X}}{1+it_n}\right) k(X)dX.$$

Let $\tilde{k}(u) := \int_{-\infty}^{\infty} k(x)e^{iux}dx$. It holds

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv = O\left(\frac{1}{A^3}\right) + O\left(\frac{1}{r^3}\right) - \frac{2}{N} \sum_{t_n>0} \mathrm{Re}\left(\frac{e^{it_n r}}{1+it_n}\right) \tilde{k}\left(\frac{t_n}{N}\right).$$

From Lemma 3.14 (c), the part of the last sum corresponding to $t_n > 0$ is

$$\sum_{0<t_n\le 2\pi N} \frac{\cos(t_n r) + t_n \sin(t_n r)}{1+{t_n}^2} \left(1 - \frac{t_n}{2\pi N}\right).$$

Since Proposition 3.3 gives

$$\sum_{0<t_n\le R} \frac{1}{|s_n|^2} = O(R)$$

and we have

$$\frac{t_n}{1+{t_n}^2} = \frac{1}{t_n} - \frac{1}{t_n({t_n}^2+1)},$$

it becomes

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv$$
$$= -\frac{2}{N} \sum_{0<t_n\le 2\pi N} \frac{\sin(t_n r)}{t_n} \left(1 - \frac{t_n}{2\pi N}\right) + O\left(\frac{1}{A^3}\right) + O\left(\frac{1}{r^3}\right) + O(1).$$

This completes the proof of the lemma. □

We next consider t_n in the range of $(0, 2\pi N]$.

Lemma 3.16. *For N large, there exist some constants c_1 and c_2 which satisfy*

$$e^{c_1 N^3} \le \prod_{0<t_n\le 2\pi N} \left(1+\frac{100\pi N}{t_n}\right) \le e^{c_2 N^3}.$$

Proof. From an obvious estimate

$$\prod_{0<t_n\le 2\pi N} \left(\frac{100\pi N}{t_n}\right) \le \prod_{0<t_n\le 2\pi N} \left(1+\frac{100\pi N}{t_n}\right) \le \prod_{0<t_n\le 2\pi N} \left(\frac{200\pi N}{t_n}\right),$$

we find c_3 and c_4, which satisfy

$$e^{c_3 N^3} \prod_{0<t_n\le 2\pi N} \left(\frac{2\pi N}{t_n}\right) \le \prod_{0<t_n\le 2\pi N} \left(1+\frac{100\pi N}{t_n}\right) \le e^{c_4 N^3} \prod_{0<t_n\le 2\pi N} \left(\frac{2\pi N}{t_n}\right). \tag{3.36}$$

For the logarithm of the product $\prod \left(\frac{2\pi N}{t_n}\right)$, integration by parts gives

$$\sum_{0<t_n\le 2\pi N} \log\left(\frac{2\pi N}{t_n}\right) = O(\log N) + \int_1^{2\pi N} \log\left(\frac{2\pi N}{x}\right) dN(x) = O(N^3). \tag{3.37}$$

From (3.36) and (3.37), the lemma follows. □

Lemma 3.17. [5, p. 266 Lemma 16.10] *Let $a_1, \ldots, a_n$ be real numbers. Suppose that $T_0, \delta_1, \ldots, \delta_n$ are positive numbers. There will then exist integers $x_1, \ldots, x_n$ and a number r such that:*

$$|ra_k - x_k| \le \delta_k \quad \textit{for} \quad 1 \le k \le n$$

$$T_0 \le r \le T_0 \prod_{k=1}^{n} \left(1+\frac{1}{\delta_k}\right). \tag{3.38}$$

Applying Lemma 3.17 to Lemma 3.16 leads us to the property of t_n.

Lemma 3.18. *There exists r_0 such that*

$$r_0 t_n = 2\pi I + \varepsilon_n \quad \textit{for} \quad 0 < t_n \le 2\pi N, \tag{3.39}$$

where I is an integer and $|\varepsilon_n| \le \frac{t_n}{50N}$, and

$$e^{c_5 N^3} \le r_0 \le e^{2c_5 N^3}. \tag{3.40}$$

Proof. By applying Lemma 3.17 with $k = n$, $a_n = \frac{t_n}{2\pi}$, $\delta_n = \frac{t_n}{100\pi N}$, and with x_k an integer we obtain (3.39).

Set $T_0 = e^{c_5 N^3}$ in (3.38), and Lemma 3.16 yields (3.40). □

From Lemmas 3.15 and 3.18, we obtain the Ω-result for $\mathcal{P}(x)$, which gives the following result.

Theorem 3.19. *We have*

$$P(x) = \Omega_\pm \left(x(\log\log x)^{\frac{1}{3}}\right).$$

Proof. We define

$$r_1 = r_0 - \frac{1}{4\pi N}.$$

From (3.39) in Lemma 3.18, it holds

$$r_1 t_n = 2\pi I + \varepsilon_n - \frac{t_n}{4\pi N},$$

where $-\frac{t_n}{2\pi N} \leq \varepsilon_n - \frac{t_n}{4\pi N} \leq \left(\frac{1}{50N} - \frac{1}{4\pi N}\right) t_n$.

For N large, there exists $c_6 > 0$ which satisfies $-\sin(r_1 t_n) \geq c_6 \frac{t_n}{N}$. Therefore we find $c_7 > 0$ such that

$$-\frac{2}{N} \sum_{0<t_n \leq 2\pi N} \frac{\sin(t_n r_1)}{t_n} \left(1 - \frac{t_n}{2\pi N}\right) \geq \frac{c_6}{N} \sum_{0<t_n \leq 2\pi N} 1 \geq c_7.$$

Referring back to Lemma 3.15, we obtain

$$\int_1^{r_1+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r_1))dv \geq c_8 \tag{3.41}$$

with some $c_8 > 0$, where A and N are kept sufficiently large. The number A is independent of N. Let

$$\mathcal{M} := \sup \left\{ \frac{\mathcal{P}(e^v)}{e^v} \middle| 1 \leq v \leq A + r_1 \right\}.$$

From (3.41), we immediately deduce that

$$\mathcal{M} \int_{-\infty}^{\infty} k(N(v-r_1))dv \geq \mathcal{M} \int_1^{r_1+A} k(N(v-r))dv \geq c_8.$$

It follows from $\int_{-\infty}^{\infty} k(N(v-r_1))dv = O(\frac{1}{N})$ that

$$\mathcal{M} \geq c_8' N \tag{3.42}$$

with some $c_8' > 0$.

Since (3.40) in Lemma 3.18 leads us to that

$$e^{c_5 N^3} - \frac{1}{4\pi N} \leq r_1 \leq e^{2c_5 N^3},$$

we find c_9 and c_{10} such that

$$c_9 N \leq (\log r_1)^{\frac{1}{3}} \leq c_{10} N. \tag{3.43}$$

By (3.42) and (3.43), there exists $c_8'' > 0$ such that

$$\sup_{1 \leq v \leq r_1 + A} \frac{\mathcal{P}(e^v)}{e^v} \geq c_8' N \geq c_8'' (\log r_1)^{\frac{1}{3}}.$$

Since it shows that

$$\overline{\lim_{v\to\infty}} \frac{\mathcal{P}(e^v)}{e^v (\log v)^{\frac{1}{3}}} \geq c_8'',$$

by putting $x = e^v$, we obtain

$$\overline{\lim_{x\to\infty}} \frac{\mathcal{P}(x)}{x(\log\log x)^{\frac{1}{3}}} \geq c_8''.$$

It can be expressed that $\mathcal{P}(x) = \Omega_+ \left(x(\log\log x)^{\frac{1}{3}}\right)$.

The Ω_--result is proved similarly by using $r_2 = r_0 + \frac{1}{4\pi N}$. Then we have

$$\mathcal{P}(x) = \Omega_\pm \left(x(\log\log x)^{\frac{1}{3}}\right).$$

From the definition of $\mathcal{P}(x)$ in (3.25), this concludes the proof of this theorem. □

We now show the relation between $\pi_\Gamma(x)$ and $\Psi_\Gamma(x)$.

Theorem 3.20. [2, p. 224 Lemma 7.1] *For $x \to \infty$,*

$$\sum_{N(P_0)\leq x} \frac{\log N(P_0)}{N(P_0)} - \sum_{N(P)\leq x} \frac{\log N(P_0)}{m(P)|a(P) - a(P)^{-1}|^2} = O(\log x).$$

From Definition 2.5 and Theorem 3.20, we have

$$\sum_{N(P_0)\leq x} \log N(P_0) - \Psi_\Gamma(x) = O(x\log x).$$

Let

$$P_0(x) := \sum_{N(P_0)\leq x} \log N(P_0) - \sum_{n=0}^{M} \frac{x^{s_n}}{s_n} - \sum_{n=0}^{M} \frac{x^{\tilde{s}_n}}{\tilde{s}_n}.$$

From (3.22), we express

$$P(x) = P_0(x) + O(x\log x). \tag{3.44}$$

Whereas it holds

$$\begin{aligned}
\int_2^x \frac{dP_0(t)}{\log t} &= \sum_{N(P_0)\leq x} 1 - \int_2^x \left(\sum_{n=0}^{M} \frac{t^{s_n-1}}{\log t} + \sum_{n=0}^{M} \frac{t^{\tilde{s}_n-1}}{\log t}\right) dt \\
&= \sum_{N(P_0)\leq x} 1 - \left(\sum_{n=0}^{M} \int_2^x \frac{t^{s_n-1}}{\log t} dt + \sum_{n=0}^{M} \int_2^x \frac{t^{\tilde{s}_n-1}}{\log t} dt\right) \\
&= \pi_\Gamma(x) - \left(\sum_{n=0}^{M} \mathrm{li}(x^{s_n}) + \sum_{n=0}^{M} \mathrm{li}(x^{\tilde{s}_n})\right) + O(1).
\end{aligned}$$

Putting

$$Q(x) := \pi_\Gamma(x) - \left(\sum_{n=0}^{M} \mathrm{li}(x^{s_n}) + \sum_{n=0}^{M} \mathrm{li}(x^{\tilde{s}_n})\right),$$

from (3.44) we have

$$Q(x) = \int_2^x \frac{dP(t)}{\log t} + O\left(\int_2^x \frac{\log t + 1}{\log t} dt\right).$$

By using the definition (3.23), it can be

$$Q(x) = \frac{P(x)}{\log x} - \left[\frac{P_1(t)}{t(\log t)^2}\right]_2^x + \int_2^x \frac{P_1(t)}{t^2(\log t)^2}dt + O(x). \tag{3.45}$$

On the other hand, by (3.22), (3.23) and (3.24),we have

$$P_2(x) = \sum_{t_n \geq 0} \frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)} + \sum_{t_n \geq 0} \frac{x^{\tilde{s}_n+2}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)}. \tag{3.46}$$

Since $\int_0^R \frac{1}{x^3} dN_\Gamma(x) = \log R$ by Proposition 3.3, we have

$$\sum_{0<t_n \leq R} \frac{1}{|s_n|^3} = O(\log R).$$

Applying it to (3.46) gives

$$P_2(x) = O(x^3 \log x). \tag{3.47}$$

Since (3.47) leads to $P_1(x) = O(x^2 \log x)$, the equation (3.45) is expressed with

$$Q(x) = \frac{P(x)}{\log x} + O\left(\frac{x}{\log x}\right), \tag{3.48}$$

which shows the relation between $\pi_\Gamma(x)$ and $\Psi_\Gamma(x)$.

Inserting the result in Theorem 3.19 into (3.48), we reached our main theorem for cocompact subgroup Γ.

Theorem 3.21. *When $\Gamma \subset PSL(2, \mathbf{C})$ is a cocompct subgroup, we have*

$$\pi_\Gamma(x) = \mathrm{li}(x^2) + \sum_{n=1}^{M} \mathrm{li}(x^{s_n}) + \Omega_\pm\left(\frac{x(\log\log x)^{\frac{1}{3}}}{\log x}\right) \qquad x \to \infty.$$

4. Ω-result for cofinite groups with some assumption

In this and next sections, we obtain the Ω-result of $\pi_\Gamma(x)$, where Γ is a cofinite subgroup of G. We have to consider the contribution from the elliptic classes, the parabolic classes and the continuous spectra. In this section, we consider the generalization of Hejhal's result (Theorem 1.1), which is Theorem 1.2 and its extension Theorem 1.4.

At first we treat the scattering determinant $\varphi(s)$, which is regarded as the contribution from the continuous spectra. We start off with the Eisenstein series.

By [2, p. 111 Theorem 4.1], when we choose $A \in G$ such that $A\zeta = \infty$, for a cusp $\zeta \in \mathbb{P}^1\mathbf{C} = \mathbf{C} \cup \{\infty\}$, the Eisenstein series of Γ at ζ is defined for $v \in \mathbf{H}^3$ as

$$E_A(v, s) := \sum_{M \in \Gamma'_\zeta \backslash \Gamma} y(AMv)^{1+s},$$

where $y(*)$ is the $(0,0,1)$-entry in $\mathbf{H}^3$. Γ_ζ is the stabilizer-group of ζ with its maximal unipotent subgoup Γ'_ζ. It converges for $\mathrm{Re}(s) > \sigma_0$. Let $\eta = B^{-1}\infty \in \mathbb{P}^1\mathbf{C}$ be another cusp of Γ, then $E_A(B^{-1}v, s)$ has the Fourier expansion;

$$E_A(B^{-1}v,s) = \delta_{\eta,\zeta}[\Gamma_\zeta : \Gamma'_\zeta]|d_0|^{-2s}y^s + \frac{\pi}{\mathcal{D}(s-1)}\left(\sum_{\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)\in\mathcal{R}} |c|^{-2s}\right) y^{2-s}$$
$$+ \frac{2\pi^s}{\mathcal{D}\Gamma(s)} \sum_{0\neq\mu\in\Lambda^*} |\mu|^{s-1}\left(\sum_{\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)\in\mathcal{R}} \frac{e^{2\pi i\langle\mu,\frac{d}{c}\rangle}}{|c|^{2s}}\right) yK_{s-1}(2\pi|\mu|y, e^{2\pi i\langle\mu,z\rangle}), \quad (4.1)$$

where the following notation is used. The Kronecker symbol is defined as

$$\delta_{\eta,\zeta} := \begin{cases} 1, & (\text{if } \eta \equiv \zeta \mod \Gamma), \\ 0, & (\text{if } \eta \not\equiv \zeta \mod \Gamma). \end{cases}$$

If η and ζ are Γ-equivalent, let $L_0 \in \Gamma$ be such that $L_0\eta = \zeta$ and let d_0 be defined by

$$AL_0B^{-1} = \begin{pmatrix} \cdot & \cdot \\ 0 & d_0 \end{pmatrix}.$$

We denote the multiplication of E_A by $[\Gamma_\zeta : \Gamma'_\zeta]$. For $\Lambda \subset \mathbf{C}$ being the lattice corresponding to the maximal unipotent subgroup $\Gamma'_\infty \subset \Gamma_\infty$, put the Euclidean area of the fundamental paralleogram of Λ for $\zeta = \infty$ and $A = I$ to be $\mathcal{D}$:

$$\mathcal{D} := \text{area}\left\{\alpha_1\omega_1 + \alpha_2\omega_2 \,\middle|\, \alpha_1, \alpha_2 \in \mathbf{R}, |\alpha_1| \le \frac{1}{2}, |\alpha_2| \le \frac{1}{2}\right\}, \quad (4.2)$$

where $0 \neq \omega_1 \in \Lambda$ is such that $|\omega_1|$ is minimal and where $\omega_2 \in \Lambda \backslash \mathbf{Z}\omega_1$ is such that $|\omega_2|$ is minimal. The bracket $\langle\cdot,\cdot\rangle$ is the usual scalar product on $\mathbf{R}^2 = \mathbf{C}$. By $\mathcal{R}$ we denote a system of representatives $\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)$ of the double cosets in $A\Gamma'_\zeta A^{-1}\backslash A\Gamma B^{-1}/B\Gamma'_\eta$ such that $c \neq 0$. The lattice dual to Λ is written by Λ^* and K_s is the usual Bessel function.

When we write a Fourier expansion of $E_A(B^{-1}v,s)$ as

$$E_A(B^{-1}v,s) = \sum_{\mu\in\Lambda^*} a_\mu(y,s)e^{2\pi i\langle\mu,z\rangle} \quad (v = z + yj),$$

then from the equation (4.1), we have

$$a_0(y,s) = \delta_{\eta,\zeta}[\Gamma_\zeta, \Gamma'_\zeta]|d_0|^{-2s}y^s + \frac{\pi}{\mathcal{D}(s-1)}\left(\sum_{\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)} |c|^{-2s}\right) y^{2-s}. \quad (4.3)$$

For the scattering matrix, we have the following properties which is in [2, p. 232]. We choose h_Γ representives, $c_1, \ldots, c_{h_\Gamma}$, for the Γ-classes of cusps of Γ such that

$$\zeta_1 = {c_1}^{-1}\infty, \ldots, \zeta_{h_\Gamma} = {c_{h_\Gamma}}^{-1}\infty \in \mathbb{P}^1\mathbf{C}.$$

We put for $\nu = 1, \dots, h_\Gamma$ and $v \in \mathbf{H}^3$,

$$E_{c_\nu}(v,s) := \sum_{M \in \Gamma'_{\zeta_\nu} \backslash \Gamma} y(c_\nu M v)^s$$

and

$$E_\nu(v,s) := \frac{1}{[\Gamma_{\zeta_\nu} : \Gamma'_{\zeta_\nu}]} E_{c_\nu}(v,s).$$

Then the $E_\nu(v,s)$ have Fourier expansion in the cusps of the form

$$E_\nu({c_\nu}^{-1}v, s) = y^s + \phi_{\nu\nu}(s) y^{2-s} + \cdots$$

and in case $\nu \neq \mu$

$$E_\nu({c_\mu}^{-1}v, s) = \phi_{\nu\mu}(s) y^{2-s} + \cdots .$$

We define the scattering matrix $\Phi(s)$ as

$$\Phi(s) := (\phi_{\nu\mu}(s)). \tag{4.4}$$

Compared with (4.1), we have the following proposition for $\varphi(s) = \det(\Phi(s))$:

Proposition 4.1. *Let Γ be fixed. Assume that Γ has $h_\Gamma > 0$ classes of cusps represented by $\zeta_1, \dots, \zeta_{h_\Gamma}$. Then the scattering determinant $\varphi(s)$ can be written in the following form for sufficiently large* $\mathrm{Re}(s)$*:*

$$\varphi(s) = \left(\frac{\pi}{\mathcal{D}(s-1)}\right)^{h_\Gamma} L(s),$$

where the following notation is used.

An absolutely convergent Dirichlet series is defined as for sufficiently large $\mathrm{Re}(s)$*:*

$$L(s) = \sum_{m=1}^{\infty} \frac{p_m}{{q_m}^{2s}}, \quad p_m \in \mathbf{C},\ p_1 \neq 0,\ 0 < q_1 < q_2 < \cdots ,$$

and $\mathcal{D}$ is defined by the above notation (4.2).

Let $\rho_n = \beta_n + i\gamma_n$ be poles of the scattering determinant. We estimate the sum of β_n:

Proposition 4.2. *It holds*

$$\sum_{0<\gamma_n<X} \beta_n = O(X \log X).$$

Proof. Let $|c_\Gamma|$ be the smallest absolute values of the non zero left-hand lower entries of the elements of Γ. Let $0 < \sigma_1 < \sigma_2 \dots \sigma_N = 1$ be the poles of $\varphi(s)$ in the segment (0, 1]. From [2, p. 289], when we define

$$\varphi^*(s) = |c_\Gamma|^{2(s-1)} \varphi(s) \prod_{i=1}^{N} \frac{s-\sigma_i}{s+\sigma_i} \tag{4.5}$$

for $s \in \mathbf{C}$, then we obtain

$$\varphi^*(s) = e^{g(s)} \prod_{\rho_n \in P(\varphi^*)} \frac{s+\tilde{\rho}_n}{s-\rho_n},$$

where g is a polynomial of degree at most 4 and $P(\varphi^*)$ is the set of poles of φ^*. The growth properties ($|\varphi^*|$ is bounded for $Re(s) > 0$ and $\varphi^*(s) \to 0$ as $Re(s) \to \infty$) and symmetry of φ^* imply that e^g is constant. Hence we have

$$\varphi(s) = |c_\Gamma|^{-2(s-1)} e^{g(s)} \prod_{i=1}^{N} \frac{s-\sigma_i}{s+\sigma_i} \prod_{\rho_n \in P(\varphi^*)} \frac{s+\rho_n}{s-\rho_n}. \tag{4.6}$$

For positive constants c_1, we express

$$\varphi(s) \sim c_1 |c_\Gamma|^{-2(s-1)} \prod_{\gamma_n > 0} \left| \frac{s+\rho_n}{s-\rho_n} \right|^2. \tag{4.7}$$

From Proposition 4.1, we obtain

$$|\varphi(s)| \sim c_2 s^{-h_\Gamma} |c_\Gamma|^{-2s} \tag{4.8}$$

for positive constant c_2. Gathering together (4.7) and (4.8), there exists $c_3 > 0$ such that

$$|c_\Gamma|^{-2(s-1)} \prod_{\gamma_n > 0} \left| \frac{s+\rho_n}{s-\rho_n} \right|^2 \sim c_3 s^{-h_\Gamma} |c_\Gamma|^{-2s}.$$

It follows that

$$\prod_{\gamma_n > 0} \left(1 + \frac{4\beta_n s}{(s-\beta_n)^2 + {\gamma_n}^2} \right) \sim c_3 s^{-h_\Gamma}.$$

Therefore, we see

$$\sum_{0<\gamma_n<X} \left(- \frac{4\beta_n X}{(X-\beta_n)^2 + {\gamma_n}^2} \right) \sim c_4 \log X,$$

and it leads to Proposition 4.2 □

From Propositions 4.1 and 4.2, we have the following theorem, which shows the assumption in Theorem 1.2 includes that in Theorem 1.4:

Theorem 4.3. *Suppose that*

$$\sum_{\gamma_n > 0} \frac{x^{\beta_n - 1}}{{\gamma_n}^2} = O\left(\frac{1}{1 + (\log x)^3} \right),$$

as $x \to \infty$, where $\rho_n = \beta_n + i\gamma_n$ are poles of the scattering determinant. Then it holds

$$N_\Gamma(T) \sim \frac{\mathrm{vol}(\Gamma)}{6\pi^2} T^3.$$

as $T \to \infty$.

Proof. Let

$$\omega(t) := 1 - \frac{\varphi'^*}{\varphi^*}(1+it) \tag{4.9}$$

with φ^* in (4.5). Since

$$-\frac{\varphi'^*}{\varphi^*}(1+it) = \sum_{\rho_n \in P(\varphi^*)} \frac{-2\beta_n}{{\beta_n}^2 + (t-\gamma_n)^2}, \tag{4.10}$$

by [2, p. 289] it follows that

$$\omega(t) = O(\log t) + \sum_{|t-\gamma_n| \le 1} \frac{-2\beta_n}{{\beta_n}^2 + (t-\gamma_n)^2}.$$

Let $N_\Gamma(\sigma_0, T)$ be the number of ρ_n in the region $\beta_n \le \sigma_0$, $|\gamma_n| \le T$. Using $\int_{-T}^{T} \frac{-2\beta_n}{{\beta_n}^2+(t-\gamma_n)^2} dt \le \int_{-\infty}^{\infty} \frac{-2\beta_n}{{\beta_n}^2+(t-\gamma_n)^2} dt$, we have

$$\int_{-T}^{T} \omega(t) dt \le 2\pi N_\Gamma(1, 2T) + c_1 T \log T \tag{4.11}$$

with some constant c_1.

Whereas, there exists a constant c_2 satisfying the following inequality:

$$\int_{-T}^{T} \omega(t) dt \ge \sum_{|\gamma_n| \le \frac{1}{2}T} \int_{1}^{T} \frac{-2\beta_n}{{\beta_n}^2 + (t-\gamma_n)^2} dt + c_2 T \log T.$$

Since $\int_{1}^{T} \frac{-2\beta_n}{{\beta_n}^2+(t-\gamma_n)^2} dt = (-2) \arctan \frac{T-\gamma_n}{\beta_n}$ leads to

$$\int_{-T}^{T} \omega(t) dt \ge \sum_{|\gamma_n| \le \frac{1}{2}T} (-2) \left(\frac{\pi}{2} + c_3 \left(\frac{\beta_n}{T} \right) \right) + c_2 T \log T,$$

It follows from Proposition 4.2 that

$$\int_{-T}^{T} \omega(t) dt \ge \pi N_\Gamma \left(1, \frac{T}{2}\right) + c_4 T \log T + c_5, \tag{4.12}$$

where c_3, c_4 and c_5 are some constants.

Gathering together (4.11) and (4.12), we have

$$\pi N_\Gamma \left(1, \frac{T}{2}\right) + c_4 T \log T \le \int_{-T}^{T} \omega(t) dt \le 2\pi N_\Gamma(1, 2T) + c_1 T \log T. \tag{4.13}$$

Since the assumption shows that $\sum_{\gamma_n > 0} \frac{1}{{\gamma_n}^2}$ converges, we see that

$$N_\Gamma(1, T) = o(T^2). \tag{4.14}$$

It shows

$$\int_{-T}^{T} \omega(t) dt = o(T^2).$$

The definitions of φ^* and ω, which are (4.5) and (4.9) respectively, lead to

$$\int_{-\sqrt{T}}^{\sqrt{T}} \frac{\varphi'}{\varphi}(1+it) dt = o(T).$$

By applying this to Proposition 1.3, we have the theorem. □

This theorem shows that Theorem 1.2 implies Theorem 1.4. In the case of cofinite subgroup Γ, it is enough to show the case of Theorem 1.4. Hence We restrict ourself to it from now.

The Selberg trace formula is explicitly written by Elstrodt-Grunewald-Mennicke as follows:

Theorem 4.4. [2, p. 297 (5.4)] *We use the notation in Proposition 4.1. Put h to be a function holomorphic in a strip of width strictly greater than 2 around the real axis satisfying the growth condition $h(1+z^2) = O((1+|z|^2)^{\frac{3}{2}-\varepsilon})$ for $|z| \to \infty$ uniformly in the strip. Let g be the cosine transform*

$$g(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} h(1+t^2)e^{-itx}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(1+t^2)\cos(xt)dt.$$

Then there is for each cusp ζ_i a number $l_i \in \mathbf{N}$ $(i = 1, \dots, h_\Gamma)$ and constants

$$c_\Gamma,\ \tilde{c}_\Gamma,\ d_\Gamma,\ d(i,j),\ \alpha(i,j) > 0 \quad (i = 1, \dots, h_\Gamma,\ j = 1, \dots, l_i)$$

so that the following identity holds with all sums being absolutely convergent:

$$\begin{aligned}
\sum_{n=0}^{\infty} h(\lambda_n) =& \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}^3)}{4\pi^2}\int_{-\infty}^{\infty} h(1+t^2)t^2 dt \\
&+ \sum_{\{R\}} \frac{\pi \log N(P_0)}{m(R)\sin^2\left(\frac{\pi k}{m(R)}\right)} g(0) + \sum_{\{P\}} \frac{4\pi g(\log N(P))\log N(P_0)}{m(P)|a(P)-a(P)^{-1}|^2} \\
&+ c_\Gamma g(0) + \tilde{c}_\Gamma h(1) - \frac{\mathrm{tr}\Phi(1)h(1)}{4} \\
&+ \frac{1}{4\pi}\int_{-\infty}^{\infty} h(1+t^2)\frac{\varphi'}{\varphi}(1+it)dt - d_\Gamma \int_{-\infty}^{\infty} h(1+t^2)\frac{\Gamma'}{\Gamma}(1+it)dt \\
&+ \sum_{i=1}^{h_\Gamma}\sum_{j=1}^{l_i} d(i,j)\int_0^{\infty} g(x)\frac{\sinh x}{\cosh x - 1 + \alpha(i,j)}dx.
\end{aligned}$$

The first sum in the second line extends over all Γ-conjugacy classes of elliptic elements in Γ which do not stabilize a cusp. The second sum extends over all hyperbolic or loxodromic conjugacy classes.

If the stabilizer Γ_{ζ_i} of the cusp ζ_i is torsion free then $d(i,j) = 0$ for $j = 1, \dots, l_i$.

When we add the assumption (1.5), we have the following proposition.

Proposition 4.5. *Assume* (1.5). *Then we have for* $\mathrm{Re}(s) \geq 1$,

$$\frac{Z'}{Z}(s) = \frac{1}{s-2} + \sum_{|s-s_n|<1} \frac{1}{s-s_n} + \sum_{|s-\tilde{s}_n|<1} \frac{1}{s-\tilde{s}_n} + O(|s|^2+1), \tag{4.15}$$

where $s_n = 1 + it_n$ and $\tilde{s}_n = 1 - it_n$ run over the zeros of $Z(s)$ on $\mathrm{Re}(s) = 1$.

Proof. In Theorem 4.4, we take the test function

$$h(1+t^2) := \frac{1}{t^2+1+s(s-2)} - \frac{1}{t^2+\beta^2},$$

and

$$g(x) = \frac{1}{2s-2}e^{-(s-1)|x|} - \frac{1}{2\beta}e^{-\beta|x|},$$

where $\beta > 1$ is fixed and $2 < \mathrm{Re}(s) < \beta + 1$. Then

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{1}{{t_n}^2+(s-1)^2} - \frac{1}{{t_n}^2+\beta^2}\right) \\
&= \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}^3)}{8\pi}(\beta - s + 1) \\
&+ \sum_{\{R\}} \frac{\pi \log N(P_0)}{m(R)\sin^2\left(\frac{\pi k}{m(R)}\right)}\left(\frac{1}{2s-2} - \frac{1}{2\beta}\right) + \frac{2\pi}{s-1}\frac{Z'}{Z}(s) - \frac{2\pi}{\beta}\frac{Z'}{Z}(\beta+1) \\
&+ c_\Gamma\left(\frac{1}{2s-2} - \frac{1}{2\beta}\right) + \tilde{c}_\Gamma\left(\frac{1}{(s-1)^2} - \frac{1}{\beta^2}\right) - \frac{\mathrm{tr}\Phi(1)}{4}\left(\frac{1}{(s-1)^2} - \frac{1}{\beta^2}\right) \\
&+ \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\frac{1}{t^2+(s-1)^2} - \frac{1}{t^2+\beta^2}\right)\frac{\varphi'}{\varphi}(1+it)dt \\
&- d_\Gamma\int_{-\infty}^{\infty}\left(\frac{1}{t^2-(s-1)^2} - \frac{1}{t^2+\beta^2}\right)\frac{\Gamma'}{\Gamma}(1+it)dt \\
&+ \sum_{i=1}^{\kappa}\sum_{j=1}^{l_i} d(i,j)\int_0^{\infty} g(x)\frac{\sinh x}{\cosh x - 1 + \alpha(i,j)}dx. \qquad (4.16)
\end{aligned}$$

Since the assumption (1.5) means

$$\int_{-T}^{T}\frac{\varphi'}{\varphi}(1+it)dt = o(T^3),$$

it leads to

$$\frac{\varphi'}{\varphi}(1+it) = o(t^2). \qquad (4.17)$$

Inserting $\frac{\Gamma'}{\Gamma}(1+it) = O(\log t)$[4, 8.362.2] and (4.17) into (4.16), the proposition follows from (3.6). $\square$

Gangolli-Warner [3, Theorem 4.4] shows the functional equation:

$$\begin{aligned}
Z(2-s) = Z(s)&\left(\frac{\Gamma(2-s)}{\Gamma(s)}\right)^{4\kappa h_\Gamma}[\varphi(1-s)]^{4\kappa} \\
&\prod_{k=1}^{l}\left(\frac{s-1-q_k}{1-s-q_k}\right)^{4\kappa b_k}\exp\left[\int_0^{s-1}4\pi\kappa\mathrm{vol}(\Gamma\backslash\mathbf{H}^3)t^2dt + \kappa_1(s-1)\right], \qquad (4.18)
\end{aligned}$$

where κ and κ_1 are constants defined through the process described in [3], and q_k $(1 \le k \le l)$ are the finitely many poles of φ in the interval $(0, 1]$ with order b_k. It leads to the following lemma:

Lemma 4.6. *Assume* (1.5). *Then the Selberg zeta function satisfies*

$$\frac{Z'}{Z}(s)+\frac{Z'}{Z}(2-s)=O(|s|^2).$$

Proof. By the logarithmic derivative of (4.18), we have

$$\frac{Z'}{Z}(s)+\frac{Z'}{Z}(2-s)=4\kappa h_\Gamma\left(\frac{\Gamma'}{\Gamma}(2-s)+\frac{\Gamma'}{\Gamma}(s)\right)+4\kappa\frac{\varphi'}{\varphi}(1-s)$$
$$-\sum_{k=1}^{l}(4\kappa b_k)\left(\frac{1}{s-1-q_k}+\frac{1}{1-s-q_k}\right)-4\pi\kappa\mathrm{vol}(\Gamma\backslash\mathbf{H}^3)(s-1)^2+\kappa_1. \quad (4.19)$$

From the property of the Γ-function [4, 8.362.2], it holds that

$$\frac{\Gamma'}{\Gamma}(2-s)+\frac{\Gamma'}{\Gamma}(s)=O(\log s). \quad (4.20)$$

The lemma follows from inserting (4.17) and (4.20) into (4.19). □

Taking Proposition 4.5 and Lemma 4.6 into account, we see that the proof of Lemma 3.5 goes through in this case. For the expression of $\Psi_\Gamma(x)$, we have the same arguments as in Theorem 3.9 except for the residue. We notice from (4.16) that the residues for the poles of $\varphi(s)$ are added to the right hand side of (3.20):

$$\mathrm{Res}_{s=\rho_n}\left(\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)\right)=\frac{\mu_1(n)x^{\rho_n+2}}{\rho_n(\rho_n+1)(\rho_n+2)} \quad (n\ge 0),$$
$$\mathrm{Res}_{s=\tilde\rho_n}\left(\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)\right)=\frac{\mu_1(n)x^{\tilde\rho_n+2}}{\tilde\rho_n(\tilde\rho_n+1)(\tilde\rho_n+2)} \quad (n\ge 0),$$

where $\mu_1(n)$ is the multiplicity of ρ_n.

Then we have the following theorem:

Theorem 4.7. *Suppose* (1.5), *then for constants* α, β, α_1, *we have*

$$\Psi_1(x)=\alpha x+\beta x\log x+\alpha_1+\sum_{n=0}^{M}\frac{x^{s_n+1}}{s_n(s_n+1)}+\sum_{n=0}^{M}\frac{x^{\tilde s_n+1}}{\tilde s_n(\tilde s_n+1)}$$
$$+\sum_{t_n\ge 0}\frac{x^{s_n+1}}{s_n(s_n+1)}+\sum_{t_n\ge 0}\frac{x^{\tilde s_n+1}}{\tilde s_n(\tilde s_n+1)}+\sum_{\gamma_n\ge 0}\frac{x^{\rho_n+1}}{\rho_n(\rho_n+1)}+\sum_{\gamma_n>0}\frac{x^{\tilde\rho_n+1}}{\tilde\rho_n(\tilde\rho_n+1)},$$

where $s_n=1+it_n$ *are the zeros of* $Z(s)$, *and* $\rho_n=\beta_n+i\gamma_n$ *are poles of the scattering determinant.* $\tilde s_n$ *and* $\tilde\rho_n$ *are the conjugacy elements for* s_n *and* ρ_n, *respectively.*

We now need to show an Ω-result for $P(x)$ defined by (3.22).

Appeal to P_i, $\mathcal{P}$ and $\mathcal{P}_i$ (i=1, 2) as in Definition 3.10. The following facts are obtained by the same arguments as in Lemmas 3.11, 3.12 and 3.13.

Lemma 4.8. *There exists* $d_1 \in \mathbf{C}$ *such that*

$$d_1 + \int_1^v \frac{\mathcal{P}(e^u)}{e^u} du$$
$$= \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)} + \sum_{\gamma_n>0} \frac{e^{(\rho_n-1)v}}{\rho_n(\rho_n-1)} + \sum_{\gamma_n>0} \frac{e^{(\tilde{\rho}_n-1)v}}{\tilde{\rho}_n(\tilde{\rho}_n-1)}.$$

When we put

$$F(v) = d_1 + \int_1^v \frac{\mathcal{P}(e^u)}{e^u} du,$$

there exists $d_2 \in \mathbf{C}$ *such that*

$$d_2 + \int_1^v F(u) du$$
$$= \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)^2} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)^2} + \sum_{\gamma_n>0} \frac{e^{(\rho_n-1)v}}{\rho_n(\rho_n-1)^2} + \sum_{\gamma_n>0} \frac{e^{(\tilde{\rho}_n-1)v}}{\tilde{\rho}_n(\tilde{\rho}_n-1)^2}.$$

Further, when we put

$$G(v) = d_2 + \int_1^v F(u) du,$$

there exists $d_3 \in \mathbf{C}$ *such that*

$$d_3 + \int_1^v G(u) du$$
$$= \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)^3} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)^3} + \sum_{\gamma_n>0} \frac{e^{(\rho_n-1)v}}{\rho_n(\rho_n-1)^3} + \sum_{\gamma_n>0} \frac{e^{(\tilde{\rho}_n-1)v}}{\tilde{\rho}_n(\tilde{\rho}_n-1)^3}.$$

In what follows we put $F(v)$ and $G(v)$ to be as in Lemma 4.8 and

$$H(v) = d_3 + \int_1^v G(u) du.$$

The equation (4.17) shows the degree of the number of ρ_n is less than 3. From $\beta_n - 1 < 0$, we find a constant e_1 such that

$$\sum_{\gamma_n>0} \frac{e^{(\rho_n-1)v}}{\rho_n(\rho_n-1)^3} + \sum_{\gamma_n>0} \frac{e^{(\tilde{\rho}_n-1)v}}{\tilde{\rho}_n(\tilde{\rho}_n-1)^3} \leq \frac{e_1}{1+v}. \tag{4.21}$$

It leads us to express $H(v)$ under (1.5) as

$$H(v) = \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)^3} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)^3} + O\left(\frac{1}{1+v}\right).$$

This estimate shows that

$$G(v) = \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)^2} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)^2} + O\left(\frac{1}{1+v^2}\right),$$
$$F(v) = \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)} + O\left(\frac{1}{1+v^3}\right).$$

Since all O-terms in $H(v)$, $G(v)$, $F(v)$ are $O(1)$ as $v \to \infty$, Lemma 3.15 is established and we include this case in that of cocompact groups.

Through the same arguments as in Lemma 3.16 and 3.18, we now have the following theorem, which is the same estimate in Theorem 3.19.

Theorem 4.9. *When we suppose* (1.5), *we have*

$$P(x) = \Omega_{\pm}\left(x(\log\log x)^{\frac{1}{3}}\right).$$

Recall Definition 3.10, then the estimate in Theorem 4.7 leads us to

$$P_2(x) = \sum_{t_n \geq 0} \frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)} + \sum_{t_n \geq 0} \frac{x^{\tilde{s}_n+2}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)} + \sum_{\gamma_n \geq 0} \frac{x^{\rho_n+2}}{\rho_n(\rho_n+1)(\rho_n+2)} + \sum_{\gamma_n > 0} \frac{x^{\tilde{\rho}_n+2}}{\tilde{\rho}_n(\tilde{\rho}_n+1)(\tilde{\rho}_n+2)}.$$

Because of the same reason for (4.21), we find a constant e_2 such that

$$\sum_{\gamma_n > 0} \frac{1}{\rho_n(\rho_n+1)(\rho_n+2)} \leq \frac{e_2}{1+v}. \tag{4.22}$$

It follows that

$$P_2(x) = \sum_{t_n \geq 0} \frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)} + \sum_{t_n \geq 0} \frac{x^{\tilde{s}_n+2}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)} + O\left(\frac{x^3}{1+\log x}\right).$$

As the analogue of (3.47), we now obtain

$$P_2(x) = O(x^3 \log x),$$

from which we have

$$P_1(x) = O(x^2 \log x).$$

By this estimate for $P_1(x)$, we can use the equation (3.48), so that Theorem 4.9 leads us to our main theorem:

Theorem 4.10. *Assume* (1.5). *When $x \to \infty$, we have*

$$\pi_\Gamma(x) = \mathrm{li}(x^2) + \sum_{n=1}^{M} \mathrm{li}(x^{s_n}) + \Omega_{\pm}\left(\frac{x(\log\log x)^{\frac{1}{3}}}{\log x}\right).$$

Remark *In the case of Theorem 1.2, from* (4.5) *and* (4.10) *we have*

$$\frac{\varphi'}{\varphi}(1+it) \ll \sum_{\gamma_n > 0} \frac{1}{(t-\gamma_n)^2} \ll \sum_{\gamma_n > 0} \frac{1}{{\gamma_n}^2}.$$

The equation (4.14) *shows*

$$\frac{\varphi'}{\varphi}(1+it) = O(1). \tag{4.23}$$

Because of (4.23), *we follow the whole proof more easily under the assumption* (1.3). *In particular, we find* e_1 *and* e_2 *without a difficulty which satisfy* (4.21) *and* (4.22), *respectively.* The example which satisfies the assumtion (1.3) is as follows:

Example 4.11. *When* Γ *is the Bianchi group associated to an imaginary quadratic number field* $K = \mathbf{Q}(\sqrt{-D})$ $(D \neq 1, 3)$, *i.e.*

$$\Gamma = \Gamma_D = PSL(2, O_K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a, b, c, d \in O_K, ad - bc = 1 \right\} / \{\pm 1\},$$

where O_K *is the ring of integers of* K, *the assumption* (1.3) *is satisfied.*

We will deduce it. By Efrat-Sarnak [1],

Theorem 4.12. [1, p. 817 Theorem 1] *For* Γ_D, *let*

$$\xi_H(s) = \left({d_H}^{1/2} / (2\pi)^{h_\Gamma} \right)^s \Gamma(s)^{h_\Gamma} \zeta_H(s),$$

where H *is the Hilbert class field of* K, d_H *is the absolute value of the discriminant of* H, $\zeta_H(s)$ *is the Dedekind zeta function of* H *and* h_Γ *is the class number of* K. *Then,*

$$\varphi(s) = (-1)^{(h_\Gamma - 2^{t-1})/2} {w_K}^{2s-2} \frac{\xi_H(s-1)}{\xi_H(s)},$$

where $w_K = \sqrt{2} / {d_K}^{\frac{1}{4}}$ *with* d_K *being the absolute value of the discriminant of* K *and* t *is the number of prime divisors of* d_K.

By Suetsuna [15], $\zeta_H(s)$ has no zeros in the region

$$\sigma > 1 - \frac{a}{\log(|t| + 2)} \quad (a > 0). \tag{4.24}$$

From Theorem 4.12 and (4.24), we have

$$\beta_n < 1 - \frac{a}{\log(|\gamma_n| + 2)}.$$

Since it shows

$$\frac{1}{(1 - \beta_n)^3} < \left(\frac{\log(|\gamma_n| + 2)}{a} \right)^3,$$

we have

$$\sum_{\gamma_n > 0} \frac{1}{{\gamma_n}^2 (1 - \beta_n)^3} = O(1). \tag{4.25}$$

On noting that

$$\sum_{\gamma_n > 0} \frac{e^{(\beta_n - 1)v}}{{\gamma_n}^2} \leq \left(\sum_{\gamma_n > 0} \frac{6}{{\gamma_n}^2 (1 - \beta_n)^3} \right) \frac{1}{v^3},$$

from (4.25) we have

$$\sum_{\gamma_n>0} \frac{e^{(\beta_n-1)v}}{{\gamma_n}^2} = O\left(\frac{1}{v^3}\right).$$

By substituting $x = e^v$, the assumption (1.3) is satisfied.

5. GENERAL COFINITE CASES

In this section, we consider the Ω-result for the case of that (1.5) doesn't hold. This case means that the second term in the left hand side becomes the main term in (1.4) or that the first and second tems have same order in it. By combining the results of those cases with Theorem 1.4, we have Theorem 1.5.

We start off with an extension of Proposition 4.5. The following equation holds:

Proposition 5.1. *Let $s_n = 1+it_n$ be the zeros of $Z(s)$ on $\mathrm{Re}(s)=1$, and $\rho_n = \beta_n + i\gamma_n$ the poles of the scattering determinant. Put $\tilde{s}_n$ and $\tilde{\rho}_n$ to be the conjugate element for s_n and ρ_n, respectively. Then we have for $\mathrm{Re}(s) \geq 1$,*

$$\frac{Z'}{Z}(s) = \frac{1}{s-2} + \sum_{|s-s_n|<1} \frac{1}{s-s_n} + \sum_{|s-\tilde{s_n}|<1} \frac{1}{s-\tilde{s_n}} + \sum_{|s-\rho_n|<1} \frac{1}{s-\rho_n} + \sum_{|s-\tilde{\rho_n}|<1} \frac{1}{s-\tilde{\rho_n}} + O(|s|^3+1).$$

Proof. From (4.6), it is expressed that

$$\frac{\varphi'}{\varphi}(s) = \sum_{|s-\rho_n|<1} \frac{1}{s-\rho_n} + \sum_{|s-\tilde{\rho}_n|<1} \frac{1}{s-\tilde{\rho}_n} + O(|s|^3+1). \tag{5.1}$$

Applying this to (4.16), the proposition follows. □

Since (4.5), (4.9) and (4.13) shows

$$-\int_{-T}^{T} \frac{\varphi'}{\varphi}(1+it)dt = CN_\Gamma(1,T) + O(T^2)$$

with some positive constant C, we assume there exists some constant C' such that

$$\frac{C}{4\pi} N_\Gamma(1,T) \sim C'T^3 \qquad \left(0 < C' \leq \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}^3)}{6\pi^2}\right). \tag{5.2}$$

From Proposition 5.1 and (5.2), Lemma 3.4 now becomes as follows:

Lemma 5.2. *For $\varepsilon > 0$, we get*

$$\frac{Z'}{Z}(1+\varepsilon+it) \ll \frac{|t|^2}{\varepsilon} + |t|^3 \quad (|t| \geq 2) \tag{5.3}$$

,

$$\frac{Z'}{Z}(2+\varepsilon+it) \ll \frac{1}{\varepsilon}, \tag{5.4}$$

$$\frac{Z'}{Z}(-\varepsilon + it) \ll |t|^3 + 1, \tag{5.5}$$

$$\left|\frac{Z'}{Z}(s)\right| \ll |t|^{3\max(0,2-\sigma)} \log|t| \;\; (s = \sigma + it,\ \sigma > 1 + \frac{1}{\log|t|}, |t| \geq 2). \tag{5.6}$$

Moreover, for any T there exists τ in $[T, T+1]$ such that

$$\int_0^2 \left|\frac{Z'}{Z}(\sigma + i\tau)\right| d\sigma \ll T^2 \log T. \tag{5.7}$$

Proof. By a direct calculation with Proposition 5.1, we have

$$\begin{aligned}\frac{Z'}{Z}(1+\varepsilon+it) = \frac{1}{\varepsilon+it-1} &+ \sum_{|s-s_n|<1} \frac{1}{\varepsilon + i(t-t_n)} + \sum_{|s-\tilde{s_n}|<1} \frac{1}{\varepsilon + i(t+t_n)} \\ &+ \sum_{|s-\rho_n|<1} \frac{1}{1+\varepsilon+it-\rho_n} + \sum_{|s-\tilde{\rho_n}|<1} \frac{1}{1+\varepsilon+it-\tilde{\rho_n}} + O(|t|^3).\end{aligned}$$

Considering the order of $N_\Gamma(T)$ and (5.2), the equation (5.3) is obtained.

We deduce (5.4), (5.5) and (5.6) by imitating the arguments as in (3.8), (3.9) and (3.10), respectively.

To see (5.7), we now take $\mathcal{T}$ as follows instead of (3.12):

$$\mathcal{T} := \{\tau \mid T < \tau < T+1,\ |\tau - t_n| \geq T^{-3}, |\tau - \gamma_n| \geq T^{-3}\}.$$

After taking the same way as in (3.11), it completes the proof. □

Inserting the estimate (5.1) into (4.19) leads us to the following functional equaion:

Lemma 5.3. *We have*

$$\frac{Z'}{Z}(s) + \frac{Z'}{Z}(2-s) = O(|s|^3).$$

Now we are ready to consider about Ψ_Γ. In this case, we need $\Psi_3(x) := \int_1^x \Psi_2(t)dt$. The following result holds:

Theorem 5.4. *For constants α_i $(i = 0,1,2,3)$ and β_0, we have*

$$\begin{aligned}\Psi_3(x) = \alpha_0 x^3 &+ \beta_0 x^3 \log x + \alpha_1 x^2 + \alpha_2 x + \alpha_3 \\ &+ 6\left\{\sum_{n=0}^{M} \frac{x^{s_n+3}}{s_n(s_n+1)(s_n+2)(s_n+3)} + \sum_{n=0}^{M} \frac{x^{\tilde{s_n}+3}}{\tilde{s_n}(\tilde{s_n}+1)(\tilde{s_n}+2)(\tilde{s_n}+3)}\right. \\ &+ \sum_{t_n \geq 0} \frac{x^{s_n+3}}{s_n(s_n+1)(s_n+2)(s_n+3)} + \sum_{t_n \geq 0} \frac{x^{\tilde{s_n}+3}}{\tilde{s_n}(\tilde{s_n}+1)(\tilde{s_n}+2)(\tilde{s_n}+3)} \\ &\left. + \sum_{\gamma_n \geq 0} \frac{x^{\rho_n+3}}{\rho_n(\rho_n+1)(\rho_n+2)(\rho_n+3)} + \sum_{\gamma_n \geq 0} \frac{x^{\tilde{\rho_n}+3}}{\tilde{\rho_n}(\tilde{\rho_n}+1)(\tilde{\rho_n}+2)(\tilde{\rho_n}+3)}\right\},\end{aligned}$$

where $s_n = 1 + it_n$ are the zeros of the discrete spectrum for $Z(s)$, and $\rho_n = \beta_n + i\gamma_n$ are that of continuous spectrum. Put $\tilde{s_n}$ and $\tilde{\rho_n}$ be the conjugacy elements for s_n and ρ_n, respectively.

Proof. We have

$$6\Psi_3(x) = \sum_{N(P)\le x} (x - N(P))^3 \Lambda_\Gamma(P),$$

which comes from Theorem 3.6 by taking $n = P$, $\lambda_n = N(n)$, $\psi(x) = (x - N(n))^3$ and $c_n = \Lambda_\Gamma(n)$. By using Theorem 3.8, it holds

$$\Psi_3(x) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s+3}}{s(s+1)(s+2)(s+3)} \frac{Z'}{Z}(s)ds.$$

Taking $c = 3$ leads us to

$$\Psi_3(x) = \frac{1}{2\pi i}\int_{3-iT}^{3+iT} \frac{x^{s+3}}{s(s+1)(s+2)(s+3)} \frac{Z'}{Z}(s)ds + O\left(\frac{x^6}{T^3}\right). \tag{5.8}$$

We will imitate the argument as in Theorem 3.9. Suppose $T \ge 1000$, let $A := N + \frac{1}{2}$ where N is a positive integer. By Cauchy's residue theorem, we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_{3-iT}^{3+iT} \frac{x^{s+3}}{s(s+1)(s+2)(s+3)} \frac{Z'}{Z}(s)ds \qquad (5.9)\\ &= \frac{1}{2\pi i}\left(\int_{-A-iT}^{-A+iT} + \int_{-A+iT}^{3+iT} - \int_{-A-iT}^{3-iT}\right) \frac{x^{s+3}}{s(s+1)(s+2)(s+3)} \frac{Z'}{Z}(s)ds \\ &\quad + \sum_{z\in R(A,T)} \mathrm{Res}_{s=z}\left(\frac{x^{s+3}}{s(s+1)(s+2)(s+3)} \frac{Z'}{Z}(s)\right), \end{aligned}$$

where $R(A,T) := \{z \in \mathbf{C} | -A \le \mathrm{Re}(z) \le 3, -T \le \mathrm{Im}(z) \le T\}$.

We denote each integral in the right hand side of (5.9) by I_1', I_2' and I_3'. Then by using Lemma 5.3 we obtain

$$\begin{aligned} I_1' &= O(x^{3-A}), \\ I_2' &= O\left(\frac{x^3(1-x^{-A})}{T\log x}\right) + O\left(\int_{1-iT}^{3-iT} \frac{x^{\sigma+3}}{T^4}\left|\frac{Z'}{Z}(s)\right| |ds|\right) \\ &\quad + O\left(\frac{x^3(x-1)}{T\log x}\right) + O\left(\int_{1+iT}^{3+iT} \frac{x^{\sigma+3}}{T^4}\left|\frac{Z'}{Z}(s)\right| |ds|\right). \end{aligned}$$

From Lemma 5.2, we have

$$\int_1^3 \frac{x^{\sigma+3}}{T^4}\left|\frac{Z'}{Z}(\sigma+iT)\right| d\sigma \ll \frac{x^6}{T\log x - 3\log T},$$

with which I_3' and the terms of integral of I_2' are expressed. The calculations of residues lead

$$\begin{aligned}
&\sum_{z\in R(A,T)} \mathrm{Res}_{s=z}\left(\frac{x^{s+3}}{s(s+1)(s+2)(s+3)}\frac{Z'}{Z}(s)\right)\\
&= \sum_{n=0}^{M}\frac{x^{s_n+3}}{(s_n(s_n+1)(s_n+2)(s_n+3)} + \sum_{n=0}^{M}\frac{x^{\tilde{s_n}+3}}{(\tilde{s_n}(\tilde{s_n}+1)(\tilde{s_n}+2)(\tilde{s_n}+3)}\\
&+ \sum_{t_n\geq 0}\frac{x^{s_n+3}}{(s_n(s_n+1)(s_n+2)(s_n+3)} + \sum_{t_n\geq 0}\frac{x^{\tilde{s_n}+3}}{(\tilde{s_n}(\tilde{s_n}+1)(\tilde{s_n}+2)(\tilde{s_n}+3)}\\
&+ \sum_{\gamma_n\geq 0}\frac{x^{\rho_n+3}}{\rho_n(\rho_n+1)(\rho_n+2)(\rho_n+3)} + \sum_{\gamma_n\geq 0}\frac{x^{\tilde{\rho_n}+3}}{\tilde{\rho_n}(\tilde{\rho_n}+1)(\tilde{\rho_n}+2)(\tilde{\rho_n}+3)}\\
&+ \alpha_0 x^3 + \beta_0 x^3\log x + \alpha_1 x^2 + \alpha_2 x + \alpha_3.
\end{aligned}$$

with some constants α_i $(i = 0, 1, 2, 3)$ and β_0. Gathering together (5.8) and above estimates, the equation (5.9) becomes

$$\begin{aligned}
&\Psi_3(x) + O\left(\frac{x^6}{T^3}\right) = O(x^{3-A}) + O\left(\frac{x^4}{T\log x}\right) + O\left(\frac{x^6}{T\log x - 3\log T}\right)\\
&+ \sum_{n=0}^{M}\frac{x^{s_n+3}}{(s_n(s_n+1)(s_n+2)(s_n+3)} + \sum_{n=0}^{M}\frac{x^{\tilde{s_n}+3}}{(\tilde{s_n}(\tilde{s_n}+1)(\tilde{s_n}+2)(\tilde{s_n}+3)}\\
&+ \sum_{t_n\geq 0}\frac{x^{s_n+3}}{s_n(s_n+1)(s_n+2)(s_n+3)} + \sum_{t_n\geq 0}\frac{x^{\tilde{s_n}+3}}{\tilde{s_n}(\tilde{s_n}+1)(\tilde{s_n}+2)(\tilde{s_n}+3)}\\
&+ \sum_{\gamma_n\geq 0}\frac{x^{\rho_n+3}}{\rho_n(\rho_n+1)(\rho_n+2)(\rho_n+3)} + \sum_{\gamma_n\geq 0}\frac{x^{\tilde{\rho_n}+3}}{\tilde{\rho_n}(\tilde{\rho_n}+1)(\tilde{\rho_n}+2)(\tilde{\rho_n}+3)}\\
&+ \alpha_0 x^3 + \beta_0 x^3\log x + \alpha_1 x^2 + \alpha_2 x + \alpha_3. \qquad (5.10)
\end{aligned}$$

As both A and T go to ∞ in (5.10), we obtain the theorem. □

Recall the definitions of $\Psi_i(x)$ $(i = 1, 2, 3)$, and we obtain

$$\begin{aligned}
\Psi_1(x) &= \alpha_0{}''x + \beta_0{}''x\log x + \alpha_1{}'' + \sum_{n=0}^{M}\frac{x^{s_n+1}}{s_n(s_n+1)} + \sum_{n=0}^{M}\frac{x^{\tilde{s_n}+1}}{\tilde{s_n}(\tilde{s_n}+1)}\\
&+ \sum_{t_n\geq 0}\frac{x^{s_n+1}}{s_n(s_n+1)} + \sum_{t_n\geq 0}\frac{x^{\tilde{s_n}+1}}{\tilde{s_n}(\tilde{s_n}+1)} + \sum_{\gamma_n\geq 0}\frac{x^{\rho_n+1}}{\rho_n(\rho_n+1)} + \sum_{\gamma_n\geq 0}\frac{x^{\tilde{\rho_n}+1}}{\tilde{\rho_n}(\tilde{\rho_n}+1)},
\end{aligned}$$

with some constants $\alpha_0{}''$, $\alpha_1{}''$ and $\beta_0{}''$.

Before estimating for

$$P(x) := \Psi_\Gamma(x) - \left(\alpha_0{}'' + \beta_0{}''\log x + \beta_0{}'' + \sum_{n=0}^{M}\frac{x^{s_n}}{s_n} + \sum_{n=1}^{M}\frac{x^{\tilde{s_n}}}{\tilde{s_n}}\right),$$

we introduce the property for $\rho_n = \beta_n + i\gamma_n$, which are poles of the scattering determinant.

Lemma 5.5. [14, Lemma 2] *Let $f(s)$ be holomorphic for $\sigma > \alpha$, except for at most a finite number of poles in this region, and let $f(s)$ have continuous boundary values on $\sigma = \alpha$. Further, assume that we have*

$$\sigma(f(s) - 1) \to 0$$

as $\sigma \to \infty$ and

$$f(s)\overline{f(\overline{s})} = O(|t|^c),$$

for $|t| > t_0$ and $\sigma \geq \alpha$, with some positive constant c, then, for $T \leq 2$,

$$\sum_{\substack{|\gamma|<T \\ \beta>\alpha}} (T - |\gamma|)(\beta - \alpha) = \frac{1}{2\pi} \int_{-T}^{T} (T - |t|) \log |f(\alpha + it)| dt + T \sum_{\sigma_j > \alpha} (\sigma_j - \alpha) + O(\log T).$$

From this lemma, we obtain

Proposition 5.6. *We have*

$$\sum_{\substack{0 \leq \gamma_n < T \\ \beta_n < 1}} (1 - \beta_n) = \frac{h_\Gamma T}{2\pi} \log \frac{T}{\pi} - \frac{1}{2\pi}(h_\Gamma \log |\mathcal{D}| - \log |a| - \frac{3}{2}\pi h_\Gamma + h_\Gamma) T + O(\log T), \tag{5.11}$$

where h_Γ and $\mathcal{D}$ are defined in Section 4 and a is some constant.

Proof. In the definition for scattering matrix (4.4), we note

$$\phi_{\nu\mu}(s) = \phi_{\mu\nu}(s).$$

It leads that we can conclude that $L(s)$ has real coefficients. We shall write

$$L(s) = ab^{1-s} L^*(s), \tag{5.12}$$

with $a \neq 0$ and real, b real and positive, and

$$L^*(s) = 1 + \sum_{n=1}^{\infty} \frac{p_n}{{q_n}^{2s}},$$

where the q_n are greater than 1. Since $L^*(s)$ satisfies the assumption of Lemma 5.5 with some constant $c > 0$, taking $\alpha = 1$ gives

$$\sum_{\substack{|\gamma_n|<T \\ \beta_n>1}} (T - |\gamma_n|)(\beta_n - 1) = \frac{1}{2\pi} \int_{-T}^{T} (T - |t|) \log |L^*(1 + it)| dt + T \sum_{\sigma_j > 1} (\sigma_j - 1) + O(\log T). \tag{5.13}$$

By [2, p. 232 Theorem 1.2] we have

$$\varphi(s)\varphi(2 - s) = 1,$$

and in particular

$$|\varphi(1 + it)| = 1. \tag{5.14}$$

From Proposition 4.1, (5.12) and (5.14) lead to

$$|L^*(1+it)| = \frac{1}{|a|}\left(\frac{\mathcal{D}|t|}{\pi}\right)^{h_\Gamma}.$$

Then we get

$$\begin{aligned}
&\int_{-T}^{T}(T-|t|)\log|L^*(1+it)|dt \\
&\quad = 2(h_\Gamma \log\mathcal{D} - \log|a|)\int_0^T (T-t)dt + 2h_\Gamma\int_0^T (T-t)\log\frac{|t|}{\pi}dt \\
&\quad = h_\Gamma T^2 \log\frac{T}{\pi} + \left(h_\Gamma\log\mathcal{D} - \log|a| - \frac{3}{2}\pi h_\Gamma\right)T^2. \qquad (5.15)
\end{aligned}$$

Substituting (5.15) into (5.13) gives

$$\begin{aligned}
&\sum_{\substack{|\gamma_n|<T \\ \beta_n>1}} (T-|\gamma_n|)(\beta_n-1) \\
&\quad = \frac{1}{2\pi}\left\{h_\Gamma T^2\log\frac{T}{\pi} + \left(h_\Gamma\log\mathcal{D} - \log|a| - \frac{3}{2}\pi h_\Gamma\right)T^2\right\} + T\sum_{\sigma_j>1}(\sigma_j-1) + O(\log T).
\end{aligned}$$

Since the zeros are symmetric around the real axis, it follows

$$\begin{aligned}
\sum_{\substack{0\le\gamma_n<T \\ \beta_n<1}} (T-|\gamma_n|)(1-\beta_n) &= \frac{1}{2}\sum_{\substack{|\gamma_n|<T \\ \beta_n>1}} (T-|\gamma_n|)(\beta_n-1) \\
&= \frac{1}{4\pi}h_\Gamma T^2\log\frac{T}{\pi} + \frac{1}{4\pi}\left(h_\Gamma\log\mathcal{D} - \log|a| - \frac{3}{2}\pi h_\Gamma\right)T^2 \\
&\quad + \frac{1}{2}T\sum_{\sigma_j>1}(\sigma_j-1) + O(\log T). \qquad (5.16)
\end{aligned}$$

We denote the left hand side of (5.16) by $A(T)$ and the left hand side of (5.11) by $B(T)$. Then we have

$$A(T+1) - A(T) \le B(T) \le A(T) - A(T-1).$$

From (5.16) we have the proposition. □

This proposition leads us to the desired estimate for $P(x)$.

Theorem 5.7. *We assume* (5.2). *For* $\varepsilon > 0$, *we have*

$$P(x) = \Omega(x^{1-\varepsilon}).$$

Proof. Suppose that for some $\varepsilon > 0$, there exists $K > 0$ such that

$$P(x) \le Kx^{1-\varepsilon}. \qquad (5.17)$$

From (2.2), we have for $\mathrm{Re}(s) > 2$

$$\frac{Z'}{Z}(s) = \int_0^\infty x^{-s} d\Psi_\Gamma(x).$$

Since the definition of $P(x)$ leads to

$$\frac{Z'}{Z}(s) \ll \int_1^\infty x^{-s} dP(x),$$

the integration by parts and applying the assumption (5.17) show that

$$\frac{Z'}{Z}(s) \ll \int_1^\infty |s| x^{-\sigma-\varepsilon} dx$$

for $\sigma = \mathrm{Re}(s)$. This gives an analytic continuation of $\frac{Z'}{Z}(s)$ in $\sigma > 1 - \varepsilon$ so that zeros of $Z(s)$ are only in $\mathrm{Re}(s) \le 1 - \varepsilon$. It means the scattering determinant $\varphi(s)$ is holomorphic in $\mathrm{Re}(s) > 1 - \varepsilon$ except for poles on the real line. Therefore we have

$$\sum_{\substack{\rho_n \\ 1 \le \gamma_n \le T}} (1 - \beta_n) \ge \varepsilon N_\Gamma(1, T).$$

From (5.2) this contradicts Proposition 5.6. □

The point of the consideration in the proof of Theorem 5.7 is similar to that in [11]. The author is indebted to Professor Sarnak who suggested this point.

We define P_1 and P_2 as in Definition 3.10. Theorem 5.4 shows

$$P_2(x) = \sum_{t_n \ge 0} \frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)} + \sum_{t_n > 0} \frac{x^{\tilde{s_n}+2}}{\tilde{s_n}(\tilde{s_n}+1)(\tilde{s_n}+2)} + \sum_{\gamma_n \ge 0} \frac{x^{\rho_n+2}}{\rho_n(\rho_n+1)(\rho_n+2)} + \sum_{\gamma_n > 0} \frac{x^{\tilde{\rho_n}+2}}{\tilde{\rho_n}(\tilde{\rho_n}+1)(\tilde{\rho_n}+2)}.$$

We shall write

$$n(T) := N_\Gamma(T) + N_\Gamma(1, T).$$

Gathering together (4.5), (4.9), (4.10) and (4.13), we have $\int_0^R \frac{1}{x^3} dn(x) = \log R$ by Proposition 1.3. It leads us to (3.47). Further, it gives us (3.48). Now we have the following theorem.

Theorem 5.8. *Assume* (5.2). *When $x \to \infty$, for $\varepsilon > 0$*

$$\pi_\Gamma(x) = \mathrm{li}(x^2) + \sum_{n=1}^M \mathrm{li}(x^{s_n}) + \Omega\left(\frac{x^{1-\varepsilon}}{\log x}\right).$$

This theorem implies

$$\pi_\Gamma(x) = \mathrm{li}(x^2) + \sum_{n=1}^M \mathrm{li}(x^{s_n}) + \Omega(x^{1-\varepsilon}) \qquad \text{as} \quad x \to \infty.$$

Taking Theorem 4.10 into consideration, we reach our main theorem, Theorem 1.5.

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