Research Report

KSTS/RR-02/004
Jul. 17, 2002

Error term of prime geodesic theorem

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ERROR TERM OF PRIME GEODESIC THEOREM

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1. Introduction

For a $(d+1)$-dimensional hyperbolic manifold with finite volume, let Γ be the fundamental group. Then the prime geodesic theorem is

$$\pi_\Gamma(x) \sim \mathrm{li}(x^d) \tag{1.1}$$

where $\pi_\Gamma(x)$ is the number of prime geodesics P whose length $l(P)$ satisfies that $N(P) := e^{l(P)} \leq x$. Since the relation "$\sim$" means that the quotient of both sides goes to 1 as $x \to \infty$, the equation (1.1) can be written by

$$\pi_\Gamma(x) = \mathrm{li}(x^d) + (\text{error}). \tag{1.2}$$

The conjectural exponent of x for lower estimate of the error term in (1.2) is $\frac{d}{2}$. The chief concern of this paper is to give lower estimates of this error term.

Known cases are when $d = 1$ and $d = 2$:

Theorem 1.1. [5, p. 477, Theorem 3.8] *When $\Gamma \subset PSL(2, \mathbf{R})$ satisfies that*

$$\sum_{\gamma_n > 0} \frac{x^{\beta_n - \frac{1}{2}}}{{\gamma_n}^2} = O\left(\frac{1}{1 + (\log x)^2}\right), \tag{1.3}$$

it holds that

$$\pi_\Gamma(x) = \mathrm{li}(x) + \Omega_\pm\left(\frac{x^{\frac{1}{2}}(\log\log x)^{\frac{1}{2}}}{\log x}\right) \quad \text{as } x \to \infty, \tag{1.4}$$

where $\beta_n + i\gamma_n$ are poles of the scattering determinant.

Theorem 1.2. [7, Theorem 1.2] *When $\Gamma \subset PSL(2, \mathbf{C})$ satisfies that*

$$\sum_{\gamma_n > 0} \frac{x^{\beta_n - 1}}{{\gamma_n}^2} = O\left(\frac{1}{1 + (\log x)^3}\right), \tag{1.5}$$

it holds that

$$\pi_\Gamma(x) = \mathrm{li}(x^2) + \Omega_\pm\left(\frac{x(\log\log x)^{\frac{1}{3}}}{\log x}\right) \quad \text{as } x \to \infty, \tag{1.6}$$

where $\beta_n + i\gamma_n$ are poles of the scattering determinant.

Assumptions (1.3) and (1.5) mean we can omit the contribution of the continuous spectra. We will consider the meaning of these assumptions by using a generalization of Weyl's law by Selberg:

Proposition 1.3. [10, p. 15 (0.2)] *Let $\Gamma \subset PSL(2,\mathbf{R})$ be a cofinite group, λ_n be the eigenvalues of the Laplacian on $L^2(\Gamma\backslash\mathbf{H}^2)$ and $\varphi(s)$ is the determinant of the scattering matrix. We put*

$$N_\Gamma(T) := \#\left\{\lambda_n | \lambda_n < \frac{1}{4} + T^2\right\}.$$

Then

$$N_\Gamma(T) - \frac{1}{4\pi}\int_{-T}^{T} \frac{\varphi'}{\varphi}\left(\frac{1}{2}+it\right)dt \sim \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}^2)}{4\pi}T^2 \tag{1.7}$$

as $T \to \infty$.

The asymptotic formula for $d = 2$ is as follows:

Proposition 1.4. [1, p. 307 Theorem 5.4] *Let $\Gamma \subset PSL(2,\mathbf{C})$ be a cofinite group, λ_n be the eigenvalues of the Laplacian in $L^2(\Gamma\backslash\mathbf{H}^3)$ with and $\varphi(s)$ is the determinant of the scattering matrix. We put*

$$N_\Gamma(T) := \#\{\lambda_n | \lambda_n < 1 + T^2\}.$$

Then

$$N_\Gamma(T) - \frac{1}{4\pi}\int_{-T}^{T} \frac{\varphi'}{\varphi}(1+it)dt \sim \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}^3)}{6\pi^2}T^3 \tag{1.8}$$

as $T \to \infty$.

Each second terms on the left hand side of (1.7) and (1.8) measure the contribution of the continuous spectrum which are connected with the variation of the argument of $\varphi(\frac{1}{2}+it)$ and $\varphi(1+it)$ respectively, on the interval $-T \le t \le T$. The assumption (1.3) implies that

$$N_\Gamma(T) \sim \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}^2)}{4\pi}T^2$$

and assumption (1.5) implies that

$$N_\Gamma(T) \sim \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}^3)}{6\pi^2}T^3.$$

In this paper we deal with the opposite situation, where conribution of the discrete spectra can be ignored.

We have the following main theorems.

Theorem 1.5. *For $\Gamma \subset PSL(2,\mathbf{R})$, it satisfies that $N_\Gamma(T) = o(T^2)$ in (1.7), we have*

$$\pi_\Gamma(x) = \mathrm{li}(x) + \Omega(x^{\frac{1}{2}-\varepsilon}) \quad \text{as } x \to \infty,$$

where ε is any positive constant.

Theorem 1.6. *For $\Gamma \subset PSL(2, \mathbf{C})$, when it satisfies $N_\Gamma(T) = o(T^3)$ in (1.8), we have*

$$\pi_\Gamma(x) = \mathrm{li}(x^2) + \Omega(x^{1-\varepsilon}) \quad \text{as } x \to \infty,$$

where ε is any positive constant.

In those main theorems, we can't use the same method as in the proofs of Theorems 1.1 and 1.2. We overcame the difficulty by reffering to the method of Phillips and Rudnick [9] in which they treat the circle problem.

Acknowledgement: The author expresses her gratitude to Professor Peter Sarnak for his valuable suggestions and helpful comments.

Also she is thankful to Professor Shin-ya Koyama for his encouragement and patient support in all the process, who introduced her to the subject.

2. Preliminaries

Let $\mathbf{H}^{d+1} = \{(x, y) | x \in \mathbf{R}^d, y > 0 \in \mathbf{R}\}$ be the hyperbolic d-plane with the Riemannian metric

$$ds^2 = \frac{dx_1{}^2 + \cdots + dx_d{}^2 + dy^2}{y^2}.$$

The group Γ acts on $\mathbf{H}^d$ discontinuously. The volume measure is given by

$$\frac{dx_1 \cdots dx_d dy}{y^{d+1}}.$$

The Laplace-Beltrami operator is defined by

$$\Delta := -y^2 \left(\frac{\partial^2}{\partial x_1{}^2} + \cdots + \frac{\partial^2}{\partial x_d{}^2} + \frac{\partial^2}{\partial y^2} \right) + (d-1) y \frac{\partial}{\partial y}.$$

Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_M \leq \frac{d}{2} < \lambda_{M+1} \cdots$ be the eigenvalues of Δ and let $t_n = \sqrt{\lambda_n - \left(\frac{d}{2}\right)^2}$.

In this paper we restrict ourselves to $d = 1$ and $d = 2$.

The Selberg zeta functions for those cases are defined as follows, respectively. Here P_0 is a primitive element associated with P:

Definition 2.1. *For the case of $d = 1$, the Selberg zeta function is given by*

$$Z_1(s) := \prod_{\{P_0\}} \prod_{k=0}^{\infty} \{1 - N(P_0)^{-s-k}\}, \quad \mathrm{Re}(s) > 1,$$

where the product on $\{P_0\}$ is taken over all primitive hyperbolic conjugacy classes of Γ.

Definition 2.2. *For the case of $d = 2$, the Selberg zeta function is defined by*

$$Z_2(s) := \prod_{\{P_0\}} \prod_{(k,l)} (1 - a(P_0)^{-2k} \overline{a(P_0)}^{-2l} N(P_0)^{-s}),$$

where the product on $\{P_0\}$ is taken over all primitive hyperbolic or loxodromic conjugacy classes of Γ, (k, l) runs through all the pairs of positive integers satisfying the following *congruence relation:* $k \equiv l \pmod{m(P_0)}$ *with* $m(P)$ *the order of the torsion of the centralizer of* P, *and* $a(P) \in \mathbf{C}$ *is the eigenvalue of* $P \in G$ *such that* $|a(P)| > 1$.

For these Selberg zeta functions we have the following logarithmic derivatives.

Lemma 2.3. [4, p. 67, Proposition 4.2] *For* $\mathrm{Re}(s) > 1$, *we have*

$$\frac{Z_1{}'}{Z_1}(s) = \sum_{\{P\}} \frac{\log N(P_0)}{1 - N(P)^{-1}} N(P)^{-s},$$

where $\{P\}$ *runs through the hyperbolic conjugacy classes of* $\Gamma \subset PSL(2, \mathbf{R})$.

Lemma 2.4. [1, p. 208, Lemma 4.2] *For* $\mathrm{Re}(s) > 2$, *we have*

$$\frac{Z_2{}'}{Z_2}(s) = \sum_{\{P\}} \frac{N(P) \log N(P_0)}{m(P)|a(P) - a(P)^{-1}|^2} N(P)^{-s},$$

where $\{P\}$ *runs through the hyperbolic or loxodromic conjugacy classes of* $\Gamma \subset PSL(2, \mathbf{C})$.

Recall that the von-Mangoldt function $\Lambda(n)$ appears in the logarithmic derivative of the Riemann zeta function:

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}. \tag{2.1}$$

Comparing Lemma 2.3 or 2.4 and (2.1), we define Λ_d by the following:

$$\frac{Z_d{}'}{Z_d}(s) = \sum_{\{P\}} \Lambda_d(P) N(P)^{-s}. \tag{2.2}$$

By using this Λ_d, we define

$$\Psi_\Gamma(x) := \sum_{\substack{\{P\} \\ N(P) \le x}} \Lambda_d(P). \tag{2.3}$$

3. Theorem 1.5

In this section, we consider the case of $d = 1$ and give the proof of Theorem 1.5.

Throughout this section, let Γ be a cofinite subgroup of $PSL(2, \mathbf{R})$.

We denote poles of $\varphi(s)$ by $\rho_n = \beta_n + i\gamma_n$ and let $N_\Gamma(\sigma_0, T)$ be the number of ρ_n in the region $\beta_n \le \sigma_0$, $|\gamma_n| \le T$.

Selberg shows [10, p. 18, (0.15)]

$$-\frac{1}{4\pi} \int_{-T}^{T} \frac{\varphi'}{\varphi} \left(\frac{1}{2} + it\right) dt = N_\Gamma\left(\frac{1}{2}, T\right) + O(T). \tag{3.1}$$

We therefore rewrite (1.7) as

$$N_\Gamma(T) + N_\Gamma\left(\frac{1}{2}, T\right) \sim \frac{\mathrm{vol}(\Gamma \backslash \mathbf{H}^2)}{4\pi} T^2. \tag{3.2}$$

Our assumption in Theorem 1.5 says

$$N_\Gamma\left(\frac{1}{2},T\right) \sim \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}^2)}{4\pi}T^2. \tag{3.3}$$

Selberg also shows the following theorem.

Proposition 3.1. [10, p. 23, Theorem 1] *We have*

$$\sum_{\substack{0\le\gamma_n<T\\ \beta_n<\frac{1}{2}}}\left(\frac{1}{2}-\beta_n\right) = \frac{h_\Gamma T}{4\pi}\log\frac{T}{\pi} - \frac{1}{2\pi}\left(\frac{h_\Gamma}{2}+\log|a|\right)T + O(\log T), \tag{3.4}$$

where h_Γ is the number of cusps and a is some constant.

By applying this proposition we will show an Ω-result for Ψ_Γ:

Theorem 3.2. *Assume* (3.3). *Then for $\varepsilon > 0$, we have*

$$\Psi_\Gamma(x) = \Omega(x^{\frac{1}{2}-\varepsilon}).$$

Proof. Suppose that for some $\varepsilon > 0$, there exists $K > 0$ such that

$$\Psi_\Gamma(x) \le Kx^{\frac{1}{2}-\varepsilon} \tag{3.5}$$

for sufficiently large x.

Since we have from (2.2) and (2.3)

$$\frac{Z_1'}{Z_1}(s) = \int_0^\infty x^{-s}d\Psi_\Gamma(x),$$

integration by parts and applying the assumption (3.5) show that

$$\frac{Z_1'}{Z_1}(s) \ll \int_1^\infty |s|x^{-\sigma-\varepsilon-\frac{1}{2}}dx$$

where $\sigma = \mathrm{Re}(s)$. This gives an analytic continuation of $\frac{Z_1'}{Z_1}(s)$ in $\sigma > \frac{1}{2}-\varepsilon$ so that zeros of $Z_1(s)$ are only in $\mathrm{Re}(s) \le \frac{1}{2}-\varepsilon$. It means that the scattering matrix $\varphi(s)$ is holomorphic in $\mathrm{Re}(s) > \frac{1}{2}-\varepsilon$ except for poles on the real line. Therefore we have

$$\sum_{\substack{\rho_n\\ |\gamma_n|\le T}}\left(\frac{1}{2}-\beta_n\right) \ge \varepsilon N\left(\frac{1}{2},T\right).$$

From (3.3) this contradicts Proposition 3.1. □

Hejhal shows the relation between Ψ_Γ and π_Γ:

Proposition 3.3. [5, p. 477, Proof of Theorem 3.8(d)] *We have*

$$\pi_\Gamma(x) - \sum_{n=1}^{M}\mathrm{li}(x^{s_n}) = O\left(\frac{x^{\frac{1}{2}}}{\log x}\right) + \frac{1}{\log x}\left(\Psi_\Gamma(x) - \sum_{n=1}^{M}\frac{x^{s_n}}{s_n}\right).$$

Theorem 3.2 and Proposition 3.3 lead to the following theorem:

Theorem 3.4. *Assume* (3.3). *When $x \to \infty$, for $\varepsilon > 0$*

$$\pi_\Gamma(x) = \mathrm{li}(x^{\frac{1}{2}}) + \Omega\left(\frac{x^{\frac{1}{2}-\varepsilon}}{\log x}\right).$$

This theorem leads to Theorem 1.5.

4. Theorem 1.6

In this section, we consider the case of $d = 2$ and now let Γ be a cofinite subgoup of $PSL(2, \mathbf{C})$.

We again denote poles of $\varphi(s)$ by $\rho_n = \beta_n + i\gamma_n$ and let $N_\Gamma(\sigma_0, T)$ be the number of ρ_n in the region $\beta_n \le \sigma_0$, $|\gamma_n| \le T$.

We will generalize (3.1) and Proposition 3.1. We first show the following proposition and lemma.

Proposition 4.1. *Let Γ be fixed. Assume that Γ has $h_\Gamma > 0$ classes of cusps represented by $\zeta_1, \dots, \zeta_{h_\Gamma}$. Then the scattering determinant $\varphi(s)$ can be written in the following form for sufficiently large* $\mathrm{Re}(s)$*:*

$$\varphi(s) = \left(\frac{\pi}{\mathcal{D}(s-1)}\right)^{h_\Gamma} L(s),$$

where the following notation is used.

An absolutely convergent Dirichlet series is defined as for sufficiently large $\mathrm{Re}(s)$*:*

$$L(s) = \sum_{m=1}^{\infty} \frac{p_m}{{q_m}^{2s}}, \quad p_m \in \mathbf{C},\ p_1 \ne 0,\ 0 < q_1 < q_2 < \cdots,$$

and for a fixed imaginary quadratic field K whose class number is one, we denote its discriminant by $\mathcal{D}_K$ and put $\mathcal{D} = |\mathcal{D}_K|$.

Proof. By [1, p. 111 Theorem 4.1], when we choose $A \in G$ such that $A\zeta = \infty$, for a cusp $\zeta \in \mathbb{P}^1\mathbf{C} = \mathbf{C} \cup \{\infty\}$, the Eisenstein series of Γ at ζ is defined for $P \in \mathbf{H}^3$ as

$$E_A(P, s) := \sum_{M \in \Gamma'_\zeta \backslash \Gamma} y(AMP)^{1+s},$$

where $y(*)$ is the $(0,0,1)$-entry in $\mathbf{H}^3$. Γ_ζ is the stabilizer-group of ζ with its maximal unipotent subgoup Γ'_ζ. It converges for $\mathrm{Re}(s) > \sigma_0$. Let $\eta = B^{-1}\infty \in \mathbb{P}^1\mathbf{C}$ be another cusp of Γ, then $E_A(B^{-1}P,s)$ has the Fourier expansion;

$$\begin{aligned}E_A(B^{-1}P,s) =&\delta_{\eta,\zeta}[\Gamma_\zeta : \Gamma'_\zeta]|d_0|^{-2s}y^s \\ &+\frac{\pi}{\mathcal{D}(s-1)}\left(\sum_{\left(\begin{smallmatrix}* & * \\ c & d\end{smallmatrix}\right)\in\mathcal{R}}|c|^{-2s}\right)y^{2-s} \\ +\frac{2\pi^s}{\mathcal{D}\Gamma(s)}&\sum_{0\neq\mu\in O_K{}^*}|\mu|^{s-1}\left(\sum_{\left(\begin{smallmatrix}* & * \\ c & d\end{smallmatrix}\right)\in\mathcal{R}}\frac{e^{2\pi i\langle\mu,\frac{d}{c}\rangle}}{|c|^{2s}}\right)yK_{s-1}(2\pi|\mu|y,e^{2\pi i\langle\mu,z\rangle}),\end{aligned} \tag{4.1}$$

where the following notation is used. The Kronecker symbol is defined as

$$\delta_{\eta,\zeta} := \begin{cases}1, & (\text{if } \eta\equiv\zeta \mod \Gamma),\\ 0, & (\text{if } \eta\not\equiv\zeta \mod \Gamma),\end{cases}$$

and $\mathcal{R}$ denotes a system of representatives $\left(\begin{smallmatrix}* & * \\ c & d\end{smallmatrix}\right)$ of the double cosets in

$$A\Gamma'_\zeta A^{-1}\backslash A\Gamma B^{-1}/B\Gamma'_\eta$$

such that $c\neq 0$. If η and ζ are Γ-equivalent, let $L_0\in\Gamma$ be such that $L_0\eta=\zeta$ and let d_0 be defined by

$$AL_0B^{-1} = \begin{pmatrix}\cdot & \cdot \\ 0 & d_0\end{pmatrix}.$$

The multiplication of E_A written by $[\Gamma_\zeta : \Gamma'_\zeta]$ is restricted to the values $1,2,3,4,6$. The bracket $\langle\cdot,\cdot\rangle$ is the usual scalar product on $\mathbf{R}^2=\mathbf{C}$. Let O_K be the integer ring of K corresponding to Γ'_∞ and we denote $O_K/\sim$ by $O_K{}^*$, where $n\sim m$ means that they generate the same ideal in O_K. We use K_s for the usual Bessel function.

Now, when we write a Fourier expansion of $E_A(B^{-1}P,s)$ as

$$E_A(B^{-1}P,s) = \sum_{\mu\in O_K{}^*} a_\mu(y,s)e^{2\pi i\langle\mu,z\rangle} \quad (P=z+yj),$$

then from the equation (4.1), we have

$$a_0(y,s) = \delta_{\eta,\zeta}[\Gamma_\zeta,\Gamma'_\zeta]|d_0|^{-2s}y^s + \frac{\pi}{\mathcal{D}(s-1)}\left(\sum_{\left(\begin{smallmatrix}* & * \\ c & d\end{smallmatrix}\right)}|c|^{-2s}\right)y^{2-s}. \tag{4.2}$$

For the scattering matrix, we have the following properties from [1, p. 232]. We choose h_Γ representives, $c_1,\dots,c_{h_\Gamma}$, for the Γ-classes of cusps of Γ such that

$$\zeta_1 = c_1{}^{-1}\infty,\dots,\zeta_{h_\Gamma} = c_{h_\Gamma}{}^{-1}\infty \in \mathbb{P}^1\mathbf{C}.$$

We put for $\nu=1,\dots,h_\Gamma$ and $P\in\mathbf{H}^3$,

$$E_{c_\nu}(P,s) := \sum_{M\in\Gamma'_{\zeta_\nu}\backslash\Gamma} y(c_\nu MP)^s.$$

We further put

$$E_\nu(P,s) := \frac{1}{[\Gamma_{\zeta_\nu} : \Gamma'_{\zeta_\nu}]} E_{c_\nu}(P,s).$$

Then the $E_\nu(P,s)$ have Fourier expansion in the cusps of the form

$$E_\nu({c_\nu}^{-1}P,s) = y^s + \phi_{\nu\nu}(s)y^{2-s} + \dots$$

for $\nu = 1,\dots,h_\Gamma$ and in case $\nu \neq \mu$

$$E_\nu({c_\mu}^{-1}P,s) = \phi_{\nu\mu}(s)y^{2-s} + \dots.$$

The scattering matrix $\Phi(s)$ is defined as

$$\Phi(s) := (\phi_{\nu\mu}(s)). \tag{4.3}$$

By using (4.1), we have

$$\varphi(s) = \det(\Phi(s)) = \left(\frac{\pi}{\mathcal{D}(s-1)}\right)^{h_\Gamma} \sum_{\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)\in\mathcal{R}} \frac{k}{|c|^{2s}}$$

with $k = \sum_{\nu=1}^{h_\Gamma} \frac{1}{[\Gamma_\zeta:\Gamma'_{\zeta_\nu}]}$. □

Lemma 4.2. [10, Lemma 2] *Let $f(s)$ be holomorphic for $\sigma > \alpha$, except for at most a finite number of poles in this region, and let $f(s)$ have continuous boundary values on $\sigma = \alpha$. Further, assume that we have*

$$\sigma(f(s)-1) \to 0$$

as $\sigma \to \infty$ and

$$f(s)\overline{f(\overline{s})} = O(|t|^c),$$

for $|t| > t_0$ and $\sigma \geq \alpha$, with some positive constant c, then, for $T \leq 2$,

$$\sum_{\substack{|\gamma|<T\\ \beta>\alpha}} (T-|\gamma|)(\beta-\alpha) = \frac{1}{2\pi}\int_{-T}^{T}(T-|t|)\log|f(\alpha+it)|dt + T\sum_{\sigma_j>\alpha}(\sigma_j-\alpha) + O(\log T).$$

We now show the generalization of (3.1).

Proposition 4.3. *Let $\rho_n = \beta_n + i\gamma_n$ be poles of the scattering determinant. Then*

$$\sum_{0<\gamma_n<X} \beta_n = O(X\log X) \tag{4.4}$$

and we have

$$-\int_{-T}^{T}\frac{\varphi'}{\varphi}(1+it)dt = CN_\Gamma(1,T) + O(T^2) \tag{4.5}$$

with some positive constant C.

Proof. Let $|c_\Gamma|$ be the smallest absolute values of the non zero left-hand lower entries of the elements of Γ, where Γ is a cofinite subgroup having one class of cusps and let $0 < \sigma_1 < \sigma_2 \ldots \sigma_N = 1$ be the poles of $\varphi(s)$ in the segment (0, 1]. We define

$$\varphi^*(s) = |c_\Gamma|^{2(s-1)}\varphi(s)\prod_{i=1}^{N}\frac{s-\sigma_i}{s+\sigma_i} \tag{4.6}$$

for $s \in \mathbf{C}$, then we obtain from [1, p. 289]

$$\varphi^*(s) = e^{g(s)}\prod_{\rho_n\in P(\varphi^*)}\frac{s+\tilde{\rho}_n}{s-\rho_n},$$

where g is a polynomial of degree at most 4 and $P(\varphi^*)$ is the set of poles of φ^*. The growth properties ($|\varphi^*|$ is bounded for $Re(s) > 0$ and $\varphi^*(s) \to 0$ as $Re(s) \to \infty$) and symmetry of φ^* imply that e^g is constant. Hence we have

$$\varphi(s) = |c_\Gamma|^{-2(s-1)}e^{g(s)}\prod_{i=1}^{N}\frac{s-\sigma_i}{s+\sigma_i}\prod_{\rho_n\in P(\varphi^*)}\frac{s+\tilde{\rho}_n}{s-\rho_n}. \tag{4.7}$$

For positive constants c_j, we express

$$\varphi(s) \sim c_1|c_\Gamma|^{-2(s-1)}\prod_{\gamma_n>0}\left|\frac{s+\rho_n}{s-\rho_n}\right|^2. \tag{4.8}$$

From Proposition 4.1, we obtain

$$|\varphi(s)| \sim c_2 s^{-h_\Gamma}|c_\Gamma|^{-2s}. \tag{4.9}$$

Gathering together (4.8) and (4.9), we have

$$|c_\Gamma|^{-2(s-1)}\prod_{\gamma_n>0}\left|\frac{s+\rho_n}{s-\rho_n}\right|^2 \sim c_3 s^{-h_\Gamma}|c_\Gamma|^{-2s}.$$

It follows that

$$\prod_{\gamma_n>0}\left(1+\frac{4\beta_n s}{(s-\beta_n)^2+{\gamma_n}^2}\right) \sim c_3 s^{-h_\Gamma}.$$

Therefore, we see

$$\sum_{0<\gamma_n<X}\left(-\frac{4\beta_n X}{(X-\beta_n)^2+{\gamma_n}^2}\right) \sim c_4 \log X,$$

and it leads to (4.4)

We put

$$\omega(t) := 1 - \frac{\varphi'^*}{\varphi^*}(1+it). \tag{4.10}$$

Since

$$-\frac{\varphi'^*}{\varphi^*}(1+it) = \sum_{\rho_n \in P(\varphi^*)} \frac{-2\beta_n}{{\beta_n}^2 + (t-\gamma_n)^2}, \tag{4.11}$$

by [1, p. 289], it follows that

$$\omega(t) = O(\log t) + \sum_{|t-\gamma_n|\le 1} \frac{-2\beta_n}{{\beta_n}^2 + (t-\gamma_n)^2}.$$

Using $\int_{-T}^{T} \frac{-2\beta_n}{{\beta_n}^2+(t-\gamma_n)^2}dt \le \int_{-\infty}^{\infty} \frac{-2\beta_n}{{\beta_n}^2+(t-\gamma_n)^2}dt$, we have

$$\int_{-T}^{T} \omega(t)dt \le 2\pi N_\Gamma(1, 2T) + c_1 T \log T \tag{4.12}$$

with some constant c_1.

Whereas, there exists a constant c_2 satisfying the following inequality:

$$\int_{-T}^{T} \omega(t)dt \ge \sum_{|\gamma_n|\le \frac{1}{2}T} \int_1^T \frac{-2\beta_n}{{\beta_n}^2 + (t-\gamma_n)^2} dt_2 + c_2 T \log T.$$

Since $\int_1^T \frac{-2\beta_n}{{\beta_n}^2+(t-\gamma_n)^2}dt = (-2)\arctan\frac{T-\gamma_n}{\beta_n}$ leads to

$$\int_{-T}^{T} \omega(t)dt \ge \sum_{|\gamma_n|\le \frac{1}{2}T} (-2)\left(\frac{\pi}{2} + c_3\left(\frac{\beta_n}{T}\right)\right) + c_2 T \log T,$$

from (4.4) it follows

$$\int_{-T}^{T} \omega(t)dt \ge \pi N_\Gamma\left(1, \frac{T}{2}\right) + c_4 T \log T + c_5, \tag{4.13}$$

where c_3, c_4 and c_5 are some constants.

Therefore, gathering together (4.12) and (4.13), we have

$$\pi N_\Gamma\left(1, \frac{T}{2}\right) + c_4 T \log T \le \int_{-T}^{T} \omega(t)dt \le 2\pi N_\Gamma(1, 2T) + c_1 T \log T. \tag{4.14}$$

From (4.6) and (4.10), it leads to (4.5). □

From this proposition, our assumption in Theorem 1.6 means

$$\frac{C}{4\pi} N_\Gamma(1, T) \sim \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}^3)}{6\pi^2} T^3. \tag{4.15}$$

As the generalization of Proposition 3.1, we have the following proposition.

Proposition 4.4. *We have*

$$\sum_{\substack{0\le\gamma_n<T \\ \beta_n<1}} (1-\beta_n) = \frac{h_\Gamma T}{2\pi}\log\frac{T}{\pi} - \frac{1}{2\pi}\left(h_\Gamma \log \mathcal{D} - \log|a| - \frac{3}{2}\pi h_\Gamma + h_\Gamma\right)T + O(\log T), \tag{4.16}$$

where h_Γ is the number of cusps and a is some constant.

Proof. Since

$$\phi_{\nu\mu}(s) = \phi_{\mu\nu}(s)$$

in (4.3), we can conclude that $L(s)$ has real coefficients. We shall write

$$L(s) = ab^{1-s}L^*(s), \tag{4.17}$$

where $a \neq 0$ and real, b real and positive and

$$L^*(s) = 1 + \sum_{n=1}^{\infty} \frac{p_n}{{q_n}^{2s}},$$

where the q_n are greater than 1. Then $L^*(s)$ satisfies the assumption of Lemma 4.2 with some constant $c' > 0$. When we take $\alpha = 1$ in Lemma 4.2, it gives

$$\sum_{\substack{|\gamma_n|<T \\ \beta_n>1}} (T - |\gamma_n|)(\beta_n - 1) = \frac{1}{2\pi} \int_{-T}^{T} (T - |t|) \log |L^*(1+it)| dt + T \sum_{\sigma_j>1} (\sigma_j - 1) + O(\log T). \tag{4.18}$$

We have

$$\varphi(s)\varphi(2-s) = 1,$$

by [1, p. 232 Theorem 1.2] and in particular

$$|\varphi(1+it)| = 1. \tag{4.19}$$

From Proposition 4.1, (4.17) and (4.19) lead to

$$|L^*(1+it)| = \frac{1}{|a|} \left(\frac{\mathcal{D}|t|}{\pi} \right)^{h_\Gamma}.$$

Then we get

$$\begin{aligned}
&\int_{-T}^{T} (T - |t|) \log |L^*(1+it)| dt \\
&= 2(h_\Gamma \log \mathcal{D} - \log |a|) \int_0^T (T - t) dt + 2h_\Gamma \int_0^T (T - t) \log \frac{|t|}{\pi} dt \\
&= h_\Gamma T^2 \log \frac{T}{\pi} + \left(h_\Gamma \log \mathcal{D} - \log |a| - \frac{3}{2}\pi h_\Gamma \right) T^2.
\end{aligned} \tag{4.20}$$

Substituting (4.20) into (4.18) gives

$$\begin{aligned}
&\sum_{\substack{|\gamma_n|<T \\ \beta_n>1}} (T - |\gamma_n|)(\beta_n - 1) \\
&= \frac{1}{2\pi} \left\{ h_\Gamma T^2 \log \frac{T}{\pi} + \left(h_\Gamma \log \mathcal{D} - \log |a| - \frac{3}{2}\pi h_\Gamma \right) T^2 \right\} + T \sum_{\sigma_j>1} (\sigma_j - 1) + O(\log T).
\end{aligned}$$

Since the zeros are symmetric around the real axis,

$$\begin{aligned}\sum_{\substack{0\leq\gamma_n<T\\ \beta_n<1}}(T-|\gamma_n|)(1-\beta_n) &= \frac{1}{2}\left\{\sum_{\substack{|\gamma_n|<T\\ \beta_n>1}}(T-|\gamma_n|)(\beta_n-1)\right\}\\ &= \frac{1}{4\pi}h_\Gamma T^2\log\frac{T}{\pi}+\frac{1}{4\pi}\left(h_\Gamma\log\mathcal{D}-\log|a|-\frac{3}{2}\pi h_\Gamma\right)T^2\\ &\quad+\frac{1}{2}T\sum_{\sigma_j>1}(\sigma_j-1)+O(\log T). \qquad (4.21)\end{aligned}$$

If we denote the left hand side of (4.21) by $A(T)$ and the left hand side of (4.16) by $B(T)$, we have

$$A(T+1)-A(T)\leq B(T)\leq A(T)-A(T-1).$$

Thus, from (4.21) we have the proposition. □

Our next goal is to show an Ω-result for Ψ_Γ.

Theorem 4.5. *We assume* (4.15). *Then for $\varepsilon>0$, we have*

$$\Psi_\Gamma(x)=\Omega(x^{1-\varepsilon}).$$

Proof. Suppose now that for some $\varepsilon>0$, there exists $K>0$ such that

$$\Psi_\Gamma(x)\leq Kx^{1-\varepsilon}. \qquad (4.22)$$

By the same argument as in Theorem 3.2, it gives

$$\sum_{\substack{\rho_n\\ |\gamma_n|\leq T}}(1-\beta_n)\geq\varepsilon N_\Gamma(1,T).$$

From (4.15), it leads to contradiction for Theorem 4.4. □

We next introduce the following relation between Ψ_Γ and π_Γ.

Proposition 4.6. [8, Proof of Theorem 5.8] *We have*

$$\pi_\Gamma(x)-\sum_{n=1}^{M}\mathrm{li}(x^{s_n})=O\left(\frac{x}{\log x}\right)+\frac{1}{\log x}\left(\Psi_\Gamma(x)-\sum_{n=1}^{M}\frac{x^{s_n}}{s_n}\right).$$

Theorem 4.5 and Proposition 4.6 lead to the following theorem:

Theorem 4.7. *Assume* (4.15). *When $x\to\infty$, for $\varepsilon>0$*

$$\pi_\Gamma(x)=\mathrm{li}(x^2)+\Omega\left(\frac{x^{1-\varepsilon}}{\log x}\right).$$

This theorem implies Theorem 1.6.

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