Research Report

KSTS/RR-02/003
Jun. 19, 2002

Zetas and Normalized Multiple Sines

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Zetas and Normalized Multiple Sines

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Running title. Multiple sines

Abstract. By using normalized multiple sine functions we show expressions for special values of zeta functions and L-functions containing $\zeta(3)$, $\zeta(5)$, etc. Our result reveals the importance of division values of normalized multiple sine functions. Properties of multiple Hurwitz zeta functions are crucial for the proof.

1 Introduction

Let $\mathcal{S}_r(x;\underline{\omega}) = \mathcal{S}_r(x;(\omega_1,...,\omega_r))$ be the normalized multiple sine function constructed and studied in the previous paper [KK].

In this paper we study special values of these functions and their relations to zeta functions. Here we mainly use $\mathcal{S}_r(x) = \mathcal{S}_r(x;(1,...,1))$.

Our main results are as follows. The first result expresses the values of the Riemann zeta function at positive odd integers.

Theorem 1.1 *Let $n = 1,2,3,...,$ and for $k = 1,2,...,n$ put*

$$a(2n+1,k) = \sum_{l=1}^{k}(-1)^{k-l}l^{2n}\binom{2n+1}{k-l},$$

which is a positive integer. Then we have:

(1)

$$\zeta'(-2n) = -\log\left(\prod_{k=1}^{n}\mathcal{S}_{2n+1}(k)^{a(2n+1,k)}\right).$$

(2)

$$\zeta(2n+1) = \frac{(-1)^{n-1}2^{2n+1}\pi^{2n}}{(2n)!}\log\left(\prod_{k=1}^{n}\mathcal{S}_{2n+1}(k)^{a(2n+1,k)}\right).$$

Examples 1.2 We have

$$\begin{aligned}
\zeta(3) &= 4\pi^2 \log S_3(1), && (1.1)\\
\zeta(5) &= -\frac{4\pi^4}{3} \log(S_5(1)S_5(2)^{11}), && (1.2)\\
\zeta(7) &= \frac{8\pi^6}{45} \log(S_7(1)S_7(2)^{57}S_7(3)^{302}). && (1.3)
\end{aligned}$$

The above formula (1.1) was proved in [KK] and [KW] previously.

Remark 1.3 By the formula

$$S_r(k) = \prod_{l=0}^{k-1} S_{r-l}(1)^{\binom{k-1}{l}(-1)^l} \tag{1.4}$$

for $1 \square k < r$, we can also express $\zeta(2n+1)$ in terms of $S_l(1)$ $(2 \square l \square 2n+1)$:

$$\zeta(2n+1) = \frac{(-1)^{n-1}2^{2n+1}\pi^{2n}}{(2n)!} \log\left(\prod_{l=2}^{2n+1} S_l(1)^{b(2n+1,l)}\right)$$

with $b(2n+1,l) \in \mathbb{Z}$.

Example 1.4 Since $S_5(2) = S_5(1)S_4(1)^{-1}$,

$$\zeta(5) = -\frac{4\pi^4}{3} \log(S_5(1)^{12}S_4(1)^{-11}).$$

Next, let χ be a non-trivial primitive Dirichlet character modulo N, and

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s} \tag{1.5}$$

the Dirichlet L-function. Then the values $L(r,\chi)$ for $r = 1, 2, 3, \ldots$ are classified as

$$L(r,\chi) = \begin{cases} \pi^r \cdot (\chi\text{-Bernoulli number}) & \cdots \quad \chi(-1) = (-1)^r \\ \text{“difficult”} & \cdots \quad \chi(-1) = (-1)^{r+1}. \end{cases}$$

Here “difficult” means that these values have not been calculated explicitly yet except for the $r = 1$ case appearing in the socalled Dirichlet's class number formula: for even χ $(\chi(-1) = 1)$

$$L(1,\chi) = -\frac{\tau(\chi)}{N} \log\left(\prod_{k=1}^{N-1} S_1\left(\frac{k}{N}\right)^{\bar{\chi}(k)}\right)$$

and

$$L'(0,\chi) = -\frac{1}{2}\log\left(\prod_{k=1}^{N-1} S_1\left(\frac{k}{N}\right)^{\chi(k)}\right),$$

where

$$\tau(\chi) = \sum_{k=1}^{N-1} \chi(k) e^{2\pi i k/N}$$

is the Gauss sum. We note that $S_1(x) = 2\sin(\pi x)$.

We generalize Dirichlet's result to the difficult case.

Theorem 1.5 *Let χ be a primitive odd character modulo N. Then:*

(1)

$$L'(-1,\chi) = -\frac{1}{2}\log\prod_{k=1}^{N-1}\left(S_2\left(\frac{k}{N}\right)^N S_1\left(\frac{k}{N}\right)^k\right)^{\chi(k)}.$$

(2)

$$L(2,\chi) = \frac{2\pi i \tau(\chi)}{N^2}\log\prod_{k=1}^{N-1}\left(S_2\left(\frac{k}{N}\right)^N S_1\left(\frac{k}{N}\right)^k\right)^{\bar{\chi}(k)}.$$

Examples 1.6 We have

$$\begin{aligned}
L(2,\left(\frac{-4}{*}\right)) &= -\frac{\pi}{4}\log\left(S_2\left(\frac{1}{4}\right)^4 S_1\left(\frac{1}{4}\right) S_2\left(\frac{3}{4}\right)^{-4} S_1\left(\frac{3}{4}\right)^{-3}\right) \\
&= \frac{\pi}{4}\log\left(2^3 S_2\left(\frac{1}{4}\right)\right)^{-8} \\
L(2,\left(\frac{-3}{*}\right)) &= -\frac{2\sqrt{3}\pi}{9}\log\left(S_2\left(\frac{1}{3}\right)^3 S_1\left(\frac{1}{3}\right) S_2\left(\frac{2}{3}\right)^{-3} S_1\left(\frac{2}{3}\right)^{-2}\right) \\
&= \frac{4\sqrt{3}\pi}{9}\log\left(3S_2\left(\frac{1}{3}\right)^{-3}\right),
\end{aligned}$$

where we used

$$\begin{aligned}
S_2(1-x) &= S_2(1+x)^{-1} \\
&= \left(S_2(x)S_1(x)^{-1}\right)^{-1} \\
&= S_2(x)^{-1}S_1(x).
\end{aligned}$$

Theorem 1.7 *Let χ be a non-trivial primitive even character modulo N. Then:*

(1)

$$L'(-2,\chi)=-\frac{1}{2}\log\prod_{k=1}^{N-1}\left(S_3\left(\frac{k}{N}\right)^{2N^2}S_2\left(\frac{k}{N}\right)^{2Nk-3N^2}S_1\left(\frac{k}{N}\right)^{k^2}\right)^{\chi(k)}.$$

(2)

$$L(3,\chi)=\frac{2\pi^2\tau(\chi)}{N^3}\log\prod_{k=1}^{N-1}\left(S_3\left(\frac{k}{N}\right)^{2N^2}S_2\left(\frac{k}{N}\right)^{2Nk-3N^2}S_1\left(\frac{k}{N}\right)^{k^2}\right)^{\bar{\chi}(k)}.$$

Example 1.8

$$\begin{aligned}L(3,\left(\frac{12}{*}\right)) &= \frac{\sqrt{3}\pi^2}{432}\log\left(S_3\left(\frac{1}{12}\right)^{288}S_2\left(\frac{1}{12}\right)^{-408}S_1\left(\frac{1}{12}\right)S_3\left(\frac{5}{12}\right)^{-288}S_2\left(\frac{5}{12}\right)^{312}S_1\left(\frac{5}{12}\right)^{-25}\right.\\ &\left. S_3\left(\frac{7}{12}\right)^{-288}S_2\left(\frac{7}{12}\right)^{264}S_1\left(\frac{7}{12}\right)^{-49}S_3\left(\frac{11}{12}\right)^{288}S_2\left(\frac{11}{12}\right)^{-164}S_1\left(\frac{11}{12}\right)^{121}\right).\end{aligned}$$

Thus the values $S_r(a)$ for $a\in\mathbb{Q}$ satisfying $0<a<r$ are quite interesting in relation to zeta values. We formulate our conjecture as

Conjecture 1.9 $S_r(a)\in\overline{\mathbb{Q}}$ *for* $a\in\mathbb{Q}$ *satisfying* $0<a<r$.

The situation would become transparent when we generalize it as below:

Conjecture 1.10 $S_r(\frac{k_1\omega_1+\cdots+k_r\omega_r}{N};\underline{\omega})\in\overline{\mathbb{Q}}$ *for* $N=1,2,3,\ldots$ *and* $k_i=0,1,\ldots,N-1$.

It is easy to see that Conjecture 1.9 is contained in Conjecture 1.10 for $\underline{\omega}=(1,\ldots,1)$, and Conjecture 1.10 clearly indicates that we are studying division values of multiple sine functions.

Remark 1.11 A suitable restriction on the form of division points such as made in Conjecture 1.10 will be needed as the following example shows:

$$\frac{S_2(2,(1,\sqrt{2}))}{S_2(1,(1,\sqrt{2}))}\notin\overline{\mathbb{Q}}.$$

Hence, at least one of $S_2(2,(1,\sqrt{2}))$ and $S_2(1,(1,\sqrt{2}))$ is not an algebraic number. By this example, we must seriously look at $S_r(a_1\omega_1+\cdots+a_r\omega_r;(\omega_1,\cdots,\omega_r))$ for general $a_i\in\mathbb{Q}$. The proof of the above fact is given by

$$\frac{S_2(2,(1,\sqrt{2}))}{S_2(1,(1,\sqrt{2}))}=\frac{S_2(1+1,(1,\sqrt{2}))}{S_2(1,(1,\sqrt{2}))}=S_1(1,\sqrt{2})=2\sin\left(\frac{\pi}{\sqrt{2}}\right)\notin\overline{\mathbb{Q}},$$

where

$$\begin{aligned} 2\sin\left(\frac{\pi}{\sqrt{2}}\right) &= -i(e^{i\frac{\pi}{\sqrt{2}}} - e^{-i\frac{\pi}{\sqrt{2}}}) \\ &= -i((-1)^{1/\sqrt{2}} - ((-1)^{1/\sqrt{2}})^{-1}) \end{aligned}$$

and we used the transcendency result of Gelfond-Schneider $(-1)^{1/\sqrt{2}} \notin \overline{\mathbb{Q}}$.

Theorem 1.12 *(1) Conjectures 1.9 and 1.10 are valid for $r = 1$.*

(2) Conjectures 1.9 and 1.10 are valid for $r = 2$ with $N = 2$. Actually

$$S_2\left(\frac{\omega_1}{2}; \underline{\omega}\right) = S_2\left(\frac{\omega_2}{2}; \underline{\omega}\right) = \sqrt{2}$$

and

$$S_2\left(\frac{\omega_1 + \omega_2}{2}; \underline{\omega}\right) = 1.$$

2 The Riemann zeta function

We use the multiple Hurwitz zeta function due to Barnes

$$\zeta_r(s, x, \underline{\omega}) = \sum_{n_1, \ldots, n_r = 0}^{\infty} (n_1\omega_1 + \cdots + n_r\omega_r + x)^{-s}$$

for $\underline{\omega} = (\omega_1, \ldots, \omega_r)$ and the definitions of the multiple gamma and the multiple sine:

$$\Gamma_r(x, \underline{\omega}) = \exp\left(\frac{\partial}{\partial s}\zeta_r(s, x, \underline{\omega})\Big|_{s=0}\right),$$

$$S_r(x, \underline{\omega}) = \Gamma_r(x, \underline{\omega})^{-1}\Gamma_r(\omega_1 + \cdots + \omega_r - x, \underline{\omega})^{(-1)^r}.$$

When $\underline{\omega} = (1, \ldots, 1)$ we simplify the notation:

$$\zeta_r(s, x) = \zeta_r(s, x, (1, \ldots, 1)) = \sum_{n_1, \ldots, n_r = 0}^{\infty} (n_1 + \cdots + n_r + x)^{-s} = \sum_{n=0}^{\infty} {}_nH_r(n + x)^{-s},$$

where ${}_nH_r = \binom{n+r-1}{r-1}$ and

$$\Gamma_r(x) = \Gamma_r(x, (1, \ldots, 1)),$$

$$S_r(x) = \Gamma_r(x)^{-1}\Gamma_r(r - x)^{(-1)^r}.$$

Lemma 2.1 *There exist uniquely determined integers* $a(r,k)$ *such that*

$$x^{r-1} = \sum_{k=1}^{r-1} a(r,k)\ {}_rH_{x-k} \tag{2.1}$$

with ${}_rH_{x-k} = \frac{(x-k+r-1)\cdots(x-k+1)}{(r-1)!}$ *for an indeterminate* x*. Indeed* $a(r,k)$ *are given as follows:*

$$a(r,k) = \sum_{l=1}^{k} (-1)^{k-l} l^{r-1} \binom{r}{k-l}. \tag{2.2}$$

Moreover,

$$a(r, r-k) = a(r,k). \tag{2.3}$$

Proof. The existence of $a(r,k)$ follows from the fact that the $(r-1)$ polynomials ${}_rH_{x-k}$ $(k = 1, ..., r-1)$ are linearly independent over $\mathbb{Q}$. By putting $x = k$ in (2.1), we have

$$k^{r-1} = a(r,1)\binom{k+r-2}{r-1} + a(r,2)\binom{k+r-3}{r-1} + \cdots + a(r,k)\cdot 1.$$

This leads to

$$a(r,k) = k^{r-1} - \sum_{j=1}^{k-1} a(r,j)\binom{k+r-1-j}{r-1}.$$

Thus (2.2) is proved by induction on k. Next, from (2.1)

$$(-x)^{r-1} = \sum_{k=1}^{r-1} a(r,k)\ {}_rH_{-x-k}$$

and

$${}_rH_{-x-k} = \frac{(-x-k+r-1)\cdots(-x-k+1)}{(r-1)!} = (-1)^{r-1}\ {}_rH_{x-(r-k)},$$

so

$$x^{r-1} = \sum_{k=1}^{r-1} a(r,k)\ {}_rH_{x-(r-k)} = \sum_{k=1}^{r-1} a(r,r-k)\ {}_rH_{x-k},$$

Hence, by the uniquenes of $a(r,k)$ we have $a(r,r-k) = a(r,k)$. ∎

Examples 2.2 For $x = n \in \mathbb{Z}$ and $r = 2,3,4,5$ we have

$$\begin{aligned} n &= {}_2H_{n-1}, \\ n^2 &= {}_3H_{n-1} + {}_3H_{n-2}, \\ n^3 &= {}_4H_{n-1} + 4\ {}_4H_{n-2} + {}_4H_{n-3}, \\ n^4 &= {}_5H_{n-1} + 11\ {}_5H_{n-2} + 11\ {}_5H_{n-3} + {}_5H_{n-4}. \end{aligned}$$

Proof of Theorem 1.1:
For $r \geq 2$ we have by Lemma 2.1

$$\begin{aligned} \zeta(s+1-r) &= \sum_{n=1}^{\infty} \frac{n^{r-1}}{n^s} \\ &= \sum_{k=1}^{r-1} a(r,k) \sum_{n=1}^{\infty} \frac{{}_rH_{n-k}}{n^s} \\ &= \sum_{k=1}^{r-1} a(r,k)\zeta_r(s,k), \end{aligned}$$

where $\zeta_r(s,k)$ is the multiple Hurwitz zeta function

$$\zeta_r(s,k) = \sum_{n=0}^{\infty} \frac{{}_rH_n}{(n+k)^s}.$$

Thus we have

$$\zeta'(1-r) = \sum_{k=1}^{r-1} a(r,k) \log \Gamma_r(k).$$

In case $r = 2n+1$, it follows that

$$\begin{aligned} \zeta'(-2n) &= \sum_{k=1}^{2n} a(2n+1,k) \log \Gamma_{2n+1}(k) \\ &= -\sum_{k=1}^{n} a(2n+1,k) \log S_{2n+1}(k) \\ &= -\log \left(\prod_{k=1}^{n} S_{2n+1}(k)^{a(2n+1,k)} \right), \end{aligned}$$

where we used $S_{2n+1}(k) = \Gamma_{2n+1}(k)^{-1}\Gamma_{2n+1}(2n+1-k)^{-1}$ and $a(2n+1, 2n+1-k) = a(2n+1,k)$.∎

Examples 2.3 We saw in [KK] Theorem 3.8(c) (and [KW] also) that

$$\zeta(3) = 4\pi^2 \log S_3(1).$$

Combining this with the fact that

$$S_3(1) = \sqrt{2} S_3 \left(\frac{1}{2}\right)^{-4/3},$$

which can be obtained by the facts

$$\begin{aligned} S_3(1) &= S_3(2\cdot\frac{1}{2}) \\ &= S_3(\frac{1}{2})S_3(1)^3S_3(\frac{3}{2})^3S_3(2) \\ &= S_3(1)^4S_3(\frac{1}{2})^4S_2(\frac{1}{2})^{-3} \end{aligned}$$

and that $S_2(\frac{1}{2})=\sqrt{2}$, we have

$$\zeta(3)=\frac{16\pi^2}{3}\log\left(S_3\left(\frac{1}{2}\right)^{-1}2^{\frac{3}{8}}\right)$$

as [KK] Theorem 3.8(b).

3 Dirichlet L-functions for odd characters

We prove the formula for $L(2,\chi)$ for odd characters. Since our method follows a proof for Dirichlet's result on $L(1,\chi)$ for even characters, we first recall it. We show the formula for $L'(0,\chi)$. Then the result on $L(1,\chi)$ follows via the functional equation.

Let χ be a non-trivial primitive Dirichlet character modulo N. We have

$$\begin{aligned} L(s,\chi) &= \sum_{k=1}^{N-1}\chi(k)\sum_{m=0}^{\infty}\frac{1}{(mN+k)^s} \\ &= N^{-s}\sum_{k=1}^{N-1}\chi(k)\zeta(s,\frac{k}{N}), \end{aligned}$$

where

$$\zeta(s,x)=\sum_{m=0}^{\infty}\frac{1}{(m+x)^s}$$

is the Hurwitz zeta function. Hence

$$L(0,\chi)=\sum_{k=1}^{N-1}\chi(k)\zeta(0,\frac{k}{N})$$

and

$$\begin{aligned} L'(0,\chi) &= \sum_{k=1}^{N-1}\chi(k)\zeta'(0,\frac{k}{N})-(\log N)\sum_{k=1}^{N-1}\chi(k)\zeta(0,\frac{k}{N}) \\ &= \sum_{k=1}^{N-1}\chi(k)\zeta'(0,\frac{k}{N})-(\log N)L(0,\chi). \end{aligned}$$

When χ is even, it holds that $L(0,\chi)=0$ (this is the reason of "difficult"), so we have

$$
\begin{aligned}
L'(0,\chi) &= \sum_{k=1}^{N-1}\chi(k)\zeta'(0,\frac{k}{N}) \\
&= \sum_{k=1}^{N-1}\chi(k)\log\Gamma_1\left(\frac{k}{N}\right) \\
&= \frac{1}{2}\sum_{k=1}^{N-1}\chi(k)\left(\log\Gamma_1\left(\frac{k}{N}\right)+\log\Gamma_1\left(\frac{N-k}{N}\right)\right) \\
&= -\frac{1}{2}\sum_{k=1}^{N-1}\chi(k)\log S_1\left(\frac{k}{N}\right).
\end{aligned}
$$

This gives the Dirichlet's result.

Proof of Theorem 1.5

We prove (1), then (2) is obtained via the functional equation. Since

$$
\zeta(s-1,x)=\sum_{n=0}^{\infty}\frac{n+x}{(n+x)^s}=\sum_{n=0}^{\infty}\frac{n+1}{(n+x)^s}+(x-1)\sum_{n=0}^{\infty}\frac{1}{(n+x)^s}=\zeta_2(s,x)+(x-1)\zeta_1(s,x),
$$

we have

$$
\zeta'(-1,x)=\zeta_2'(0,x)+(x-1)\zeta_1'(0,x),
$$

as $\zeta_1(s,x)=\zeta(s,x)$. Now that χ is odd and that $L(-1,\chi)=0$, we compute

$$
\begin{aligned}
L'(-1,\chi) &= N\sum_{k=1}^{N-1}\chi(k)\zeta'(-1,\tfrac{k}{N}) \\
&= N\sum_{k=1}^{N-1}\chi(k)\zeta_2'(0,\tfrac{k}{N}) + N\sum_{k=1}^{N-1}\chi(k)(\tfrac{k}{N}-1)\zeta_1'(0,\tfrac{k}{N}) \\
&= N\sum_{k=1}^{N-1}\chi(k)\log\Gamma_2(\tfrac{k}{N}) + N\sum_{k=1}^{N-1}\chi(k)(\tfrac{k}{N}-1)\log\Gamma_1'(\tfrac{k}{N}) \\
&= N\sum_{k=1}^{N-1}\chi(k)\log\left(\Gamma_2(\tfrac{k}{N})\Gamma_1(\tfrac{k}{N})^{\frac{k}{N}-1}\right) \\
&= \frac{N}{2}\sum_{k=1}^{N-1}\chi(k)\log\left(\frac{\Gamma_2(\frac{k}{N})}{\Gamma_2(1-\frac{k}{N})}\frac{\Gamma_1(\frac{k}{N})^{\frac{k}{N}-1}}{\Gamma_1(1-\frac{k}{N})^{-\frac{k}{N}}}\right) \\
&= \frac{N}{2}\sum_{k=1}^{N-1}\chi(k)\log\left(\frac{\Gamma_2(\frac{k}{N})}{\Gamma_2(2-\frac{k}{N})}\left(\Gamma_1(\tfrac{k}{N})\Gamma_1(1-\tfrac{k}{N})\right)^{\frac{k}{N}-1}\right) \\
&= -\frac{N}{2}\sum_{k=1}^{N-1}\chi(k)\log\left(S_2(\tfrac{k}{N})S_1(\tfrac{k}{N})^{\frac{k}{N}-1}\right) \\
&= -\frac{N}{2}\sum_{k=1}^{N-1}\chi(k)\log\left(S_2(\tfrac{k}{N})S_1(\tfrac{k}{N})^{\frac{k}{N}}\right),
\end{aligned}
$$

where we used the fact $S_1(\frac{k}{N})=S_1(\frac{N-k}{N})$ with $\chi(N-k)=-\chi(k)$. ∎

4 Dirichlet L-functions for even characters

Proof of Theorem 1.7

We again show (1), then (2) is obtained via the functional equation. Since

$$(n+x)^2 = 2\ {}_3H_n + (2x-3)\ {}_2H_n + (x-1)^2\ {}_1H_n,$$

we have

$$\zeta(s-2,x)=\sum_{n=0}^{\infty}\frac{(n+x)^2}{(n+x)^s}=2\zeta_3(s,x)+(2x-3)\zeta_2(s,x)+(x-1)^2\zeta_1(s,x).$$

Therefore we have

$$\zeta'(-2,x)=2\zeta_3'(0,x)+(2x-3)\zeta_2'(0,x)+(x-1)^2\zeta_1'(0,x).$$

Now that χ is even and that $L(-2,\chi)=0$, we compute

$$\begin{aligned} L'(-2,\chi) &= N^2\sum_{k=1}^{N-1}\chi(k)\zeta'(-2,\tfrac{k}{N}) \\ &= N^2\sum_{k=1}^{N-1}\chi(k)\left(2\zeta_3'(0,\tfrac{k}{N})+(2\tfrac{k}{N}-3)\zeta_2'(0,\tfrac{k}{N})+(\tfrac{k}{N}-1)^2\zeta_1'(0,\tfrac{k}{N})\right) \\ &= N^2\sum_{k=1}^{N-1}\chi(k)\log\left(\Gamma_3(\tfrac{k}{N})^2\Gamma_2(\tfrac{k}{N})^{2\frac{k}{N}-3}\Gamma_1(\tfrac{k}{N})^{(\frac{k}{N}-1)^2}\right) \\ &= -\tfrac{1}{2}\log\prod_{k=1}^{N-1}\left(S_3(\tfrac{k}{N})^{2N^2}S_2(\tfrac{k}{N})^{2Nk-3N^2}S_1(\tfrac{k}{N})^{k^2}\right)^{\chi(k)}. \blacksquare \end{aligned}$$

5 Division values of normalized multiple sines

Proof of Theorem 1.12

Since $S_1(x,\omega)=2\sin(\frac{\pi x}{\omega})$ by [KK, §2], we have

$$S_1(\tfrac{k\omega}{N},\omega)=2\sin(\tfrac{k\pi}{N})=-i(e^{i\pi k/N}-e^{-i\pi k/N})\in\overline{\mathbb{Q}},$$

which leads to (1).

Recall that

$$S_2(x,(\omega_1,\omega_2))=\frac{\Gamma_2(\omega_1+\omega_2-x,(\omega_1,\omega_2))}{\Gamma_2(x,(\omega_1,\omega_2))}.$$

First

$$S_2(\tfrac{\omega_1+\omega_2}{2},(\omega_1,\omega_2))=\frac{\Gamma_2(\frac{\omega_1+\omega_2}{2},(\omega_1,\omega_2))}{\Gamma_2(\frac{\omega_1+\omega_2}{2},(\omega_1,\omega_2))}=1.$$

Secondly

$$S_2(\tfrac{\omega_1}{2},(\omega_1,\omega_2))=\frac{\Gamma_2(\frac{\omega_1}{2}+\omega_2,(\omega_1,\omega_2))}{\Gamma_2(\frac{\omega_1}{2},(\omega_1,\omega_2))}.$$

Here we use ([KK, §2])

$$\Gamma_2(x+\omega_2,(\omega_1,\omega_2))=\Gamma_2(x,(\omega_1,\omega_2))\Gamma_1(x,\omega_1)^{-1}.$$

Then

$$\Gamma_2(\tfrac{\omega_1}{2}+\omega_2,(\omega_1,\omega_2))=\Gamma_2(\tfrac{\omega_1}{2},(\omega_1,\omega_2))\Gamma_1(\tfrac{\omega_1}{2},\omega_1)^{-1}.$$

Hence

$$S_2(\tfrac{\omega_1}{2},(\omega_1,\omega_2))=\Gamma_1(\tfrac{\omega_1}{2},\omega_1)^{-1}.$$

Now ([KK, §2])

$$\Gamma_1(x,\omega) = \frac{\Gamma(\frac{x}{\omega})}{\sqrt{2\pi}}\omega^{\frac{x}{\omega}-\frac{1}{2}},$$

so

$$\Gamma_1(\tfrac{\omega_1}{2},\omega_1) = \frac{\Gamma(\frac{1}{2})}{\sqrt{2\pi}} = \frac{1}{\sqrt{2}}.$$

Thus

$$S_2(\tfrac{\omega_1}{2},(\omega_1,\omega_2)) = \sqrt{2}. \blacksquare$$

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