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Star Exponential Functions for Quadratic Forms and Polar Elements

by

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Strange phenomena related to ordering problems in quantizations

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Abstract. In this paper, we introduce an object which looks like an extended notion of covering spaces. Such an object is requested to understand the "group" generated by the exponential functions of quadratic forms in the Weyl algebra. This is in some sense a complexification of the metaplectic group.

1. Introduction

Quantum picture is basically set up by the Weyl algebra. It is derived from the differential calculus via correspondence principle: Let u be the multiplication operator $x\cdot$ by the coordinate function x on $\mathbb{R}$ acting on the space of all C^∞ functions on $\mathbb{R}$, and let v be the differential operator $i\hbar\partial_x$. u and v generate an algebra $W_\hbar$, called the *Weyl algebra*. Thus, the Weyl algebra is an associative algebra generated over $\mathbb{C}$ by u, v with the fundamental relation $[u, v] = -i\hbar$.

However, the correspondence principle, $u \leftrightarrow x\cdot$, $v \leftrightarrow i\hbar\partial_x$, causes a lot of mathematical questions. We meet immediately the ordering problem (see §1). Such ordering problem occurs mainly in Schrödinger quantization procedure which assigns a differential operator defined on a configuration space to every classical observable.

Apart from configuration spaces, Heisenberg procedure for quantum mechanics is a formalism to set up von Neumann algebras or C^* algebras (cf. [Co]). In this formalism, the ordering problem means how to express an element of an algebra in the unique way. Whenever the expression is fixed, it makes us possible to put a topology on the algebra and to take the topological completion (cf.§1). However in this formalism, it is difficult to know how the quantum world relates to the classical world. It is very heavy to treat everything in the theory of selfadjoint operators.

In this paper, we first make several topological completions of the Weyl algebra. Here, we are not restricted within C^*-algebras or operator algebras, but we want only to treat $\hbar$ as a deformation parameter (a positive real parameter) by

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following the notion of deformation quantization initiated by [BFFLS]. Remarking that many laws in physics are expressed as evolution equations, we will consider the evolution equation $\frac{d}{dt}f_t = i\hbar p_*(u,v)*f_t$ in $W_\hbar$. Thus, solving the evolution equation, we have to know the individual phenomenon.

In the theory where $\hbar$ is treated as a formal parameter, which is called formal deformation quantization, no problem occurs to solve evolution equations. In the theory where $\hbar$ is treated as a positive real parameter, the existence of solutions of evolution equations is not so obvious. However, it is easy to see the uniqueness of real analytic solution if it exists. To obtain solutions, we have to make some topological completion of the Weyl algebra, in which one can define the exponential function $e_*^{itp_*(u,v)}$.

Along the direction of Schödinger, which uses configuration spaces, Hörmander made the notion of pseudo-differential operators (ψDO) on any manifold, and of Fourier integral operators [Hö1] to treat $e_*^{itp_*(u,v)}$ such that $p(u,v)$ is in the symbol class of order one with respect to v.

Furthermore, Hörmander [Hö2] proposed the Weyl calculus on $\mathbb{R}^{2n}$ by using extended notion of ψDO's where u and v have the same weight. In these theories, the essential self-adjointness of $p_*(u,v)$ is crucial, because the evolution equations for $e_*^{itp_*(u,v)}$ are treated as partial differential equations.

On the other hand, there is another classical way of treating such evolution equations. This is indeed the method of Lie theory, which treats such evolution equations within the system of ordinary differential equations. In order to use this method, we restrict our attention to the linear hull over $\mathbb{C}$ of $*$-exponential functions of polynomials of degree ≤ 2.

However, in this restricted objects, we have met some pathological phenomena: A typical phenomenon is that the area where the product is defined depends on the way of ordering expressions. (See Lemma10.) In spite of this, one can obtain some product formula by collecting various ordering expressions all together. Moreover, it happens that an element has *two different inverses*. Since this causes associativity breaking (see §1), we can not treat such a system as an associative algebra.

Motivated by such pathological phenomena, we should know more precisely what sort of difficulties will happen in such objects. To extend products, we have to treat intertwiners between several ordering expressions. It happens however, that intertwiners are defined only 2-to-2 mappings on the space of exponential functions of quadratic forms, because of the ambiguity of $\sqrt{\ }$ in the calculation (see § 6), and such ambiguity can not be eliminated by treating an appropriate double covering spaces. (See § 4.)

Think about the serious meaning of such a pathological phenomenon. This forces us to treat the notion of manifolds which do not form a point set. We propose in this paper the idea of two-valued elements.

Besides of such strange phenomena, we have another motivation to treat $\hbar$ as a genuine parameter. The deformation quantization of [BFFLS] made us free from operator theory. Especially, if we treat the deformation parameter $\hbar$ as a formal parameter and consider everything in the category of formal power series of $\hbar$ (formal deformation), then the quantization problem goes very smooth. Kontsevich [K] showed every Poisson algebra on a manifold is formally deformation quantizable. However it is apparent that formal deformation quantization plays only a probe for the quantum world with exact physical significance.

After Kontsevich's result, deformation theory is coming up the next generation, called the exact deformation theory. We have to pay a lot of effort to make the deformation theory more close to the theories where C^* algebras or von Neumann algebras are explicitly used (cf. Connes [Co]). Actually, Rieffel [R] proposed a notion of such a deformation theory, called strictly deformation quantization, and pointed out many serious problems.

In this paper we attempt to point out several serious problems are still involved in the theory of classical ordering problems.

This paper is organized as follows:

In §2, we give several basic facts, several orderings, and product formulas. We explain also several pathological phenomena, and how such strange phenomena appear naturally in the exact deformation quantization theory. But no problem occurs as for exponential functions of linear functions of generators. (See Theorem 3, and the equation (30).)

Thus, in §3, we restrict our attention to the space of exponential functions of quadratic forms. Infinitesimal action of quadratic forms are computed in Weyl ordering and normal ordering, and these define involutive distributions on the space of exponential functions. We easily obtain maximal integral submanifolds.

In §4, we give the explicit formula of $*$-exponential functions in the Weyl ordering and in normal ordering. Via these explicit expressions, we find an "element" ε_{00}, called *polar element*, having a very strange property that one must say this is a "two valued" element, although such notion have never appeared in ordinary mathematics.

In spite of this, ε_{00} is very useful in the computation. We give in §5 several product formulas, and show that $*$-exponential functions of quadratic forms generates a group-like object, which looks like a double cover of $SL_{\mathbb{C}}(2)$. In spite of this, technicality is involved in a standard classical Lie theory. To understand why such strange element appears, we define in §6 the notion of intertwiners between several ordering expressions. We see that such strange phenomena is caused by the ambiguity of $\frac{1}{\sqrt{}}$ of intertwiners. Because of this ambiguity, intertwiners are defined only as "2-to-2 diffeomorphisms" on the set of exponential functions of quadratic forms.

Hence in §7, we will try to understand how the glued object looks like. We happen to know that a similar phenomena occurred in the magnetic monopole theory, and to manage this phenomena mathematically, Brylinski [Br] uses gerbes of Giraud. (See last chapter of [Br].) But we prefer to use the notion of two-valued element, because this is very simple and intuitive. To make these more clear, we propose the notion of blurred $\mathbb{C}_*$-bundles.

Our conclusion in this paper is that *$*$-exponential functions of quadratic forms generate a group-like object which is not a point set, but is understood as a non-trivial double cover of $SL_{\mathbb{C}}(2)$. In spite of this, this object contains the non trivial double covers of $SL_{\mathbb{R}}(2)$ and $SU(1,1)$. Hence this object may be understood as a complexification of metaplectic group $Mp(2,\mathbb{R})$ [GiS]. It is known that there is no complexification of such groups as genuine Lie groups.*

2. The Weyl algebra and extensions

We consider the Weyl algebra $W_\hbar$ generated by u, v over $\mathbb{C}$ with the fundamental relation $[u,v]\,(= u{*}v - v{*}u) = -i\hbar$ where $\hbar$ is a positive constant. The pair (u,v) of generators is called a *canonical conjugate pair*.

2.1. Orderings and product formulas.

To express elements of the Weyl algebra $W_\hbar$, we introduce an ordering. Namely, we choose the typical orderings in $W_\hbar$; normal ordering, anti-normal ordering, and Weyl ordering, respectively. The normal ordering (resp. the anti-normal ordering) is the way of writing elements of the form $\sum a_{m,n}u^m{*}v^n$ (resp. $\sum a_{m,n}v^m{*}u^n$) by arranging u to the left (resp. right) hand side in each term. The Weyl ordering is the way of writing elements in the form $\sum a_{m,n}u^m{\odot}v^n$ defined by using the symmetric product $\cdot$ given by $u{\cdot}v = \frac{1}{2}(u{*}v + v{*}u)$. (See [OMY] §1.2). But we have no need to know about the symmetric product.

Using such orderings, one can identify the Weyl $W_\hbar$ algebra with the space $\mathbb{C}[u,v]$ of all polynomials on $\mathbb{C}^2$ with the coordinates u, v. Thus, the Weyl algebra $W_\hbar$ can be viewed as a non commutative associative product structure defined on the space $\mathbb{C}[u,v]$ for fixing an ordering of $W_\hbar$. According to the (normal, anti-normal, Weyl) orderings of $W_\hbar$, we have the noncommutative product on $\mathbb{C}[u,v]$, and denote them by $*_N, *_{\tilde{N}}, *_M$, respectively.

Product formula. Let $f(u,v), g(u,v) \in \mathbb{C}[u,v]$. We use these notations for the ordinary continuous products and denote them by $\circ, \bullet, \cdot$ to explain the orderings of $W_\hbar$.

- *The normal ordering* : the product $*$ of the Weyl algebra is given by the ΨDO-*product formula* as follows: (Noting this coincides with the product formula of ΨDO's,)

$$f(u,v) *_N g(u,v) = f\exp\{i\hbar(\overleftarrow{\partial_v}\circ\overrightarrow{\partial_u})\}g. \tag{1}$$

- *The anti-normal ordering* : the product $*$ of the Weyl algebra is given by the $\overline{\Psi}$DO-*product formula* as follows:

$$f(u,v) *_{\bar{N}} g(u,v) = f\exp\{-i\hbar(\overleftarrow{\partial_u}\bullet\overrightarrow{\partial_v})\}g. \tag{2}$$

- *The Weyl ordering* : the product $*$ of the Weyl algebra is given by the *Moyal product formula* as follows:

$$f(u,v) *_M g(u,v) = f\exp\frac{i\hbar}{2}\{\overleftarrow{\partial_v}\overset{\cdot}{\wedge}\overrightarrow{\partial_u}\}g \tag{3}$$

where $\overleftarrow{\partial_v}\overset{\cdot}{\wedge}\overrightarrow{\partial_u} = \overleftarrow{\partial_v}\cdot\overrightarrow{\partial_u} - \overleftarrow{\partial_u}\cdot\overrightarrow{\partial_v}$, and

$$f(\overleftarrow{\partial_v}\cdot\overrightarrow{\partial_u} - \overleftarrow{\partial_u}\cdot\overrightarrow{\partial_v})g = \partial_v f\cdot\partial_u g - \partial_u g\cdot\partial_v f.$$

Every product formula yields $u * v - v * u = -i\hbar$, and recovers the Weyl algebra $W_\hbar$. We notice that commutative products $\circ$, $\bullet$, $\cdot$ play only supplementary roles to express elements in the unique way. We distinguish these to indicate what ordering expression is used.

On the contrary, in the Weyl algebra $W_\hbar$, $\frac{1}{i\hbar}\mathrm{ad}(v)$ and $-\frac{1}{i\hbar}\mathrm{ad}(v)$ form mutually commutative pair of derivations. These derivations also reproduce commutative products $\circ, \bullet, \cdot$ from the $*$-product by reversing formulas (cf.[OMY]). Such inverse expressions ensure that there is *no other relation imported unexpectedly via such ordering expressions.*

For elements $p_*(u,v)$, $q_*(u,v) \in W_\hbar$, we have various expressions according to the orderings. The product is given as follows:

$$p_*(u,v) * q_*(u,v) = f_.(u,v) *_M g_.(u,v) = f_\circ(u,v) *_N g_\circ(u,v) = f_\bullet(u,v) *_{\bar N} g_\bullet(u,v).$$

If no confusion is suspected, then we omit the suffix $M, N, \bar N$ in the $*$-product.

Let $Hol(\mathbb{C}^2)$ be the space of all entire functions on $\mathbb{C}^2$ with the compact open topology. The product formula (1), (2), (3) have the following properties.

Then, obviously we see

Proposition 1. *(1) $f*g$ is defined if one of f,g is a polynomial.*
(2) For every polynomial $p=p(u,v)$, the left-(resp. right-) multiplication $p*$ (resp. $*p$) is a continuous linear mapping of $Hol(\mathbb{C}^2)$ into itself under the compact open topology.

We call such a system $Hol(\mathbb{C}^2), \mathbb{C}[u,v], *)$ a $(\mathbb{C}[u,v]; *)$-*bimodule.*

By the polynomial approximation theorem, the associativity

$$f*(g*h) = (f*g)*h$$

holds if two of f,g,h are polynomials. We call this *2-p-associativity.*

2.2. Canonical conjugate pairs.

For every $A \in SL_{\mathbb{C}}(2)$, we change the generators

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}.$$

It is obvious that $[u',v']_* = -i\hbar$, and hence u',v' may be viewed as generators. The replacement (pull-back) A^* of u,v by u',v' gives an algebra isomorphism of $W_\hbar$. Thus, we may consider the ordering problem by using u',v' instead of u,v.

The following is the most useful property of Moyal product formula (3):

Proposition 2. *For every $A \in SL_{\mathbb{C}}(2)$ and $\binom{\alpha}{\beta} \in \mathbb{C}^2$, let Φ^* be the replacement (pull-back) of u,v into u',v' by the combination of the linear transformation by the matrix A and the parallel displacement $\binom{\alpha}{\beta}$:*

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \qquad A \in SL_{\mathbb{C}}(2), \quad (\alpha,\beta) \in \mathbb{C}^2.$$

Then, Φ^* is an isomorphism both on $(\mathbb{C}[u,v], \cdot)$ and $(\mathbb{C}[u,v], *)$.

Remark that normal, anti-normal orderings do not have such a property. It is easily seen that

$$(au+bv)^m_* = (au+bv)^m_., \quad \text{but} \quad (au+bv)^m_* \neq (au+bv)^m_\circ \quad \text{for} \quad ab \neq 0.$$

For the proof of Proposition 2, we have only to remark the following identity:

$$\overleftarrow{\partial_v} \stackrel{\cdot}{\wedge} \overrightarrow{\partial_u} = \overleftarrow{\partial_{v'}} \stackrel{\cdot}{\wedge} \overrightarrow{\partial_{u'}}.$$

2.3. Evolution equations.

In the $(\mathbb{C}[u,v],*)$-bimodule $(Hol(\mathbb{C}), \mathbb{C}[u,v],*)$, we consider the evolution equation

$$\frac{d}{dt} f_t = p_*(u,v) * f_t, \quad f_0 = f(u,v) \tag{4}$$

for every polynomial $p_*(u,v)$, whenever we do not mind about the existence of solutions. However, the real analytic solution in t is unique if it exists. The solution, if it exists for the initial function $f_0 = 1$, will be denoted by $e_*^{tp_*(u,v)}$. If the infinite series $\sum \frac{t^k}{k!} p(u,v)_*^k$ converges, then it must be the solution of (4). Since $\sum \frac{t^k}{k!}(\alpha u + \beta v)_*^k$ converges, we use the $*$-exponential function $e_*^{t(\alpha u+\beta v)}$ to define the intertwiners between different orderings. (See §6.)

2.4. Extensions of product formula.

Starting from $(\mathbb{C}[u,v];*)$, we extend the $*$-product to a wider class of functions. For every positive real number p, we set

$$\mathcal{E}_p(\mathbb{C}^2) = \{f \in Hol(\mathbb{C}^2) \mid \|f\|_{p,s} = \sup |f|\, e^{-s|\xi|^p} < \infty, \ \forall s > 0\} \tag{5}$$

where $|\xi| = (|u|^2 + |v|^2)^{1/2}$. The family of seminorms $\{\|\cdot\|_{p,s}\}_{s>0}$ induces a topology on $\mathcal{E}_p(\mathbb{C}^2)$ and $(\mathcal{E}_p(\mathbb{C}^2), \cdot)$ is an associative commutative Fréchet algebra, where the dott $\cdot$ is the ordinary product for functions in $\mathcal{E}_p(\mathbb{C}^2)$. The product $\cdot$ may be replaced by $\circ$ or $\bullet$ to indicate the ordering. It is easily seen that for $0 < p < p'$, there is a continuous embedding

$$\mathcal{E}_p(\mathbb{C}^2) \subset \mathcal{E}_{p'}(\mathbb{C}^2) \tag{6}$$

as commutative Fréchet algebras (cf.[GS]), and that $\mathcal{E}_p(\mathbb{C}^2)$ is $SL_{\mathbb{C}}(2)$-invariant.

It is obvious that every polynomial is contained in $\mathcal{E}_p(\mathbb{C}^2)$ and $\mathbb{C}[u,v]$ is dense in $\mathcal{E}_p(\mathbb{C}^2)$ for any $p > 0$ in the Fréchet topology defined by the family of seminorms $\{\|\ \|_{p,s}\}_{s>0}$.

We remind that every exponential function $e_\cdot^{\alpha u+\beta v}$ is contained in $\mathcal{E}_p(\mathbb{C}^2)$ for any $p > 1$, but not in $\mathcal{E}_1(\mathbb{C}^2)$, and functions such as $e_\cdot^{au^2+bv^2+2cuv}$ are contained in $\mathcal{E}_p(\mathbb{C}^2)$ for any $p > 2$, but not in $\mathcal{E}_2(\mathbb{C}^2)$. Functions such as $\sum \frac{1}{(k!)^{\frac{1}{p}}} u^k$ is contained in $\mathcal{E}_q(\mathbb{C}^2)$ for any $q > p$, but not in $\mathcal{E}_p(\mathbb{C}^2)$. $Hol(\mathbb{C}^2)$ is a complete topological linear space under the compact open topology.

The following theorem is the main result of [OMMY1]: [2]

Theorem 3. *The product formulas* (1), (2), (3) *extend to give the following:*
(i) For $0 < p \le 2$, the space $(\mathcal{E}_p(\mathbb{C}^2), *)$ forms a complete topological associative algebra.
(ii) For $p > 2$, every product formula gives continuous bi-linear mappings of

$$\mathcal{E}_p(\mathbb{C}^2) \times \mathcal{E}_{p'}(\mathbb{C}^2) \to \mathcal{E}_p(\mathbb{C}^2), \quad \mathcal{E}_{p'}(\mathbb{C}^2) \times \mathcal{E}_p(\mathbb{C}^2) \to \mathcal{E}_p(\mathbb{C}^2), \tag{7}$$

for every p' such that $\frac{1}{p} + \frac{1}{p'} \ge 1$.

[2] In [OMMY1], the proof is given in the case of Weyl ordering, but the same proof works for other orderings.

Let $\mathcal{E}_{2+}(\mathbb{C}^2)=\bigcap_{p>2}\mathcal{E}_p(\mathbb{C}^2)$. Thus, $\mathcal{E}_{2+}(\mathbb{C}^2)$ is a Fréchet space under the natural intersection topology. Note that $e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$ is continuous in $\mathcal{E}_{2+}(\mathbb{C}^2)$ in $(a,b,c)\in\mathbb{C}^3$.

By Theorem 3, it is easy to handle $(\mathcal{E}_p(\mathbb{C}^2),*)$ for $0<p\leq 2$. We now focus to the space $\mathcal{E}_{2+}(\mathbb{C}^2)$. As we mention in §2.5, the extended space $\mathcal{E}_{2+}(\mathbb{C}^2)$ has several strange phenomena.

2.5. Vacuums, half-inverses and the break down the associativity.

A direct calculation using the Moyal product formula (3) shows that the coordinate function v has a right inverse and a left inverse $v^\circ=\frac{1}{v}(1-e^{\frac{2i}{\hbar}uv})$ and $v^\bullet=\frac{1}{v}(1-e^{-\frac{2i}{\hbar}uv})$ respectively in $\mathcal{E}_{2+}(\mathbb{C}^2)$, i.e,

$$v*v^\circ=1=v^\bullet*v,\quad v^\circ*v=1-2e^{\frac{2i}{\hbar}uv},\quad v*v^\bullet=1-2e^{-\frac{2i}{\hbar}uv},$$

where uv means $u{\cdot}v$ in precise. The $\cdot$-sign is occasionally omitted in the expression of the Weyl ordering.

If the associativity holds in $\mathcal{E}_{2+}(\mathbb{C}^2)$, then v° should coincide with $v^\bullet$. Hence $\frac{1}{v}\sin\frac{2}{\hbar}uv=0$, which is a contradiction (cf. [OMMY1]). Then, we loose the associativity in $\mathcal{E}_{2+}(\mathbb{C}^2)$. This is one of the typical phenomena which shows the lack of the associativity. That is, *coordinate functions have both left- and right-inverses.*

By the Moyal product formula (3), we also have

$$v*e^{\frac{2i}{\hbar}uv}=0=e^{\frac{2i}{\hbar}uv}*u,\quad u*e^{-\frac{2i}{\hbar}uv}=0=e^{-\frac{2i}{\hbar}uv}*v.$$

We denote by $\varpi_{00}=2e^{\frac{2i}{\hbar}uv}$, $\bar{\varpi}_{00}=2e^{-\frac{2i}{\hbar}uv}$ which are called a *vacuum*, a *bar-vacuum* respectively. Using the Moyal product formula and the 2-p-associativity, we easily have

$$(uv-\tfrac{i\hbar}{2})*e^{\frac{2i}{\hbar}uv}=u*v*e^{\frac{2i}{\hbar}uv}=0. \tag{8}$$

In Lemma 4 in §4, we show that $e_*^{\frac{it}{\hbar}uv}=\frac{1}{\cosh\frac{t}{2}}e^{\frac{i}{\hbar}(\tanh\frac{t}{2})2uv}$ in the Weyl ordering. Remark that $\int_{-\infty}^{\infty}\frac{1}{\cosh\frac{t}{2}}e^{\frac{i}{\hbar}(\tanh\frac{t}{2})2uv}dt<\infty$ in the space $\mathcal{E}_{2+}(\mathbb{C}^2)$. Setting

$$(uv)_{+i0}^{-1}=-i\hbar\int_0^\infty e_*^{\frac{it}{\hbar}uv}dt,\quad (uv)_{-i0}^{-1}=i\hbar\int_{-\infty}^0 e_*^{\frac{it}{\hbar}uv}dt,$$

we have uv has *two different* inverses, since the difference is given as

$$(uv)_{+i0}^{-1}-(uv)_{-i0}^{-1}=-i\hbar\int_{-\infty}^\infty e_*^{\frac{it}{\hbar}uv}dt. \tag{9}$$

The r.h.s. of (9) has the expression as follows by using Hansen-Bessel formula:

$$\int_{-\infty}^\infty e_*^{\frac{it}{\hbar}uv}dt=\int_{-\infty}^\infty\frac{1}{\cosh\frac{t}{2}}e^{\frac{i}{\hbar}(\tanh\frac{t}{2})2uv}dt=\frac{\pi}{2}J_0(\frac{2}{\hbar}uv),$$

where J_0 is the Bessel function. This is obviously $\neq 0$. This causes another break down of the associativity. Thus, it is impossible to treat $(uv)_{+i0}^{-1}$ and $(uv)_{-i0}^{-1}$ in the same associative algebra.

Since the r.h.s. of (9) can be viewed as the $*$-Fourier transform of the constant function 1, it may be regarded as the $*$-*delta function* $-i\delta_*(uv)$ (cf. [OMMY1]). This is actually expressed as the difference of two holomorphic functions and several nice relation to Sato's hyper functions are observed [O],[OMMY1]. (See also [M].)

Hence, the $*$-delta function $\delta_*(uv)$ is expressed as an entire function in terms of the Weyl ordering. We are much interested in such phenomena, since these may be useful in nano-technology.

3. Quadratic forms

These strange phenomena as in §2 are deeply related to $*$-exponential functions, such as $e_*^{\frac{t}{\hbar}u\cdot v}$, defined by the evolution equation (4) of quadratic forms.

It is easy to see that the set of all quadratic forms in $W_\hbar$ is closed under the commutator bracket $[a,b]=a{*}b-b{*}a$. Set $X=\frac{1}{\hbar\sqrt{8}}u^2, Y=\frac{1}{\sqrt{8}\hbar}v^2, H=\frac{i}{2\hbar}uv$, where $uv=u*v+\frac{i\hbar}{2}$. Then, they form a basis of the Lie algebra $\mathfrak{sl}_{\mathbb{C}}(2)$: We see

$$[H,X]=-X,\quad [H,Y]=Y,\quad [X,Y]=-H,$$

and $\{X,Y,H\}$ generate an associative algebra in the space $\mathbb{C}[u,v]$, which is an enveloping algebra of $\mathfrak{sl}_{\mathbb{C}}(2)$. Setting $\mathrm{ad}(W)V=[W,V]$, we see

$$\mathrm{ad}(\frac{i}{2\hbar}(au^2+bv^2+2cuv))\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}-c & -b\\ a & c\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}. \tag{10}$$

Thus, $\mathrm{ad}(\frac{i}{2\hbar}(au^2+bv^2+2cuv))$ generates the complex Lie group $SL_{\mathbb{C}}(2)$, which is a useful fact to fix the product formula involving $*$-exponential functions of quadratic forms (see (30)). In a $(\mathbb{C}[u,v];*)$-bimodule $(Hol(\mathbb{C}^2),\mathbb{C}[u,v],*)$ with an ordering expression as in §2, we consider the evolution equation (4) for every polynomial $p(u,v)$ with the initial function f. If $p(u,v)=u^2+(\frac{i}{\hbar}v)^2$, the equation corresponds to that of standard harmonic oscillator. For a complex parameter t, the evolution equation (4) may not be necessarily solved for arbitrary initial function. However a real analytic solution for (4) in t is unique if it exists. If the real analytic solution of (4) exists, then we denote it by $e_*^{tp(u,v)}*f(u,v)$, where $e_*^{tp(u,v)}$ is the solution with initial condition 1. Thus, following standard method of Lie theory, we change a partial differential equation to a system of ordinary differential equations.

3.1. Singular distributions in Weyl ordering.

In what follows, we identify $(a,b,c;s)\in\mathbb{C}^3\times\mathbb{C}_*$ with

$$se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\in\mathcal{E}_{2+}(\mathbb{C}^2),\ \text{i.e.}\ (a,b,c;s)\Longleftrightarrow se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)},$$

if no confusion is suspected. s and $\frac{1}{\hbar}(au^2+bv^2+2cuv)$ are called the *amplitude* and the *phase* respectively. The function $e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$ is called the *phase part.*

For every point $(a,b,c;s)$ in $\mathbb{C}^4$, we consider a curve $s(t)e^{\frac{1}{\hbar}(a(t)u^2+b(t)v^2+2c(t)uv)}$ starting at $se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$. The tangent vector of this curve at $t=0$ is given as

$$\big(\frac{1}{\hbar}(a'u^2+b'v^2+2c'uv)s+s'\big)e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}.$$

On the other hand, consider the $*$-product $e^{\frac{t}{\hbar}(a'u^2+b'v^2+2c'uv)}*se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$. We now compute the derivative of this quantity at $t=0$. By using the Moyal product formula, we have

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} & e^{\frac{t}{\hbar}(a'u^2+b'v^2+2c'uv)}*se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\\ &=\frac{1}{\hbar}(a'u^2+b'v^2+2c'uv)*se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\\ &=\frac{1}{\hbar}(a'u^2+b'v^2+2c'uv)se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\\ &+\frac{2i}{\hbar}\{(b'v{+}c'u)(au{+}cv)-(a'u{+}c'v)(bv{+}cu)\}se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\\ &-\frac{1}{2\hbar}\{b'(\hbar a+2(au{+}cv)^2)-2c'(\hbar c+2(au{+}cv)(bv{+}cu))\\ &\qquad\qquad +a'(\hbar b+2(bv{+}cu)^2)\}se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}.\end{aligned} \tag{11}$$

Then, (11) is written as

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} & e^{\frac{t}{\hbar}(a'u^2+b'v^2+2c'uv)}*se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\\ &=\frac{1}{\hbar}(a',b',c')M(a,b,c;s)\begin{bmatrix}u^2\\ v^2\\ 2uv\\ \hbar\end{bmatrix}se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)},\end{aligned} \tag{12}$$

where

$$M(a,b,c;s)=\begin{bmatrix}-(c+i)^2, & -b^2, & -b(c+i); & -\frac{b}{2}\\ -a^2, & -(c-i)^2, & -a(c-i); & -\frac{a}{2}\\ 2a(c+i), & 2b(c-i), & 1+ab+c^2; & c\end{bmatrix} \tag{13}$$

We denote by $M(a,b,c)$ the submatrix of first three columns of $M(a,b,c;s)$. Remark that

$$\det M(a,b,c)=(c^2{-}ab{+}1)^3. \tag{14}$$

It is seen that every radial direction is the eigen vector of $M(a,b,c)$:

$$(a,b,c)M(\tau a,\tau b,\tau c)=(1+(c^2{-}ab)\tau^2)(a,b,c). \tag{15}$$

If $c^2{-}ab{+}1=0$, then we write as

$$au^2+bv^2+2cuv=2i(\alpha u+\beta v)(\gamma u+\delta v),\quad \alpha\delta-\beta\gamma=1.$$

Clearly, $[\alpha u{+}\beta v,\gamma u{+}\delta v]=-i\hbar$. By setting $u'=\alpha u{+}\beta v$, $v'=\gamma u{+}\delta v$, (u',v') is a canonical conjugate pair. Applying (3) for (u',v'), we easily see that

$$(\gamma u+\delta v)*e^{\frac{2i}{\hbar}(\alpha u+\beta v)(\gamma u+\delta v)}=0,\quad \text{for}\quad \alpha\delta-\beta\gamma=1. \tag{16}$$

It follows by 2-p-associativity that

$$\begin{aligned}&(\gamma u+\delta v)^2_*\,*e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}=0,\\ &(\alpha u+\beta v)*(\gamma u+\delta v)*e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}=0.\end{aligned} \tag{17}$$

The second identity of (17) yields $(a,b,c)M(a,b,c)=0$, if $c^2-ab+1=0$, which corresponds to (15), and the first identity of (17) yields

$$(\gamma^2,\delta^2,\gamma\delta)M(a,b,c)=0, \qquad c^2-ab+1=0.$$

Hence $M(a,b,c)$ has rank 1 at the point $c^2-ab+1=0$, but the rank of $M(a,b,c;s)$ is 2 at there. Setting $u'=\alpha u+\beta v$, $v'=\gamma u+\delta v$, we call $2e^{\frac{2i}{\hbar}u'v'}$ the *vacuum* w.r.t. (u',v'). Thus, one can allow to call the bar-vacuum $2e^{-\frac{2i}{\hbar}u'v'}$ the *vacuum* w.r.t. $(-v',u')$.

We consider a holomorphic singular distribution $\mathcal{D}_\mu$ on $\mathbb{C}^3\times\mathbb{C}_*$ given by

$$\mathcal{D}_\mu(a,b,c;s)=\{(a',b',c')M(a,b,c;s)|\ (a',b',c')\in\mathbb{C}^3\}.$$

Let $\pi:\mathbb{C}^3\times\mathbb{C}_*\to\mathbb{C}^3$ be the natural projection. Set

$$V_\mu=\{(a,b,c);c^2-ab+1=0\} \quad \text{(phase part of vacuums)}. \tag{18}$$

Then, $2e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}, (a,b,c)\in V_\mu$ is a vacuum. Though $\mathcal{D}_\mu$ is singular on the submanifold $V_\mu\times\mathbb{C}_*$, this gives an ordinary involutive distribution on $(\mathbb{C}^3 – V_\mu)\times\mathbb{C}_*$. Hence, there is the 3-dimensional maximal integral holomorphic submanifold M^3 of $\mathcal{D}_\mu$ through the origin $(0,0,0;1)$. Since

$$M(a,b,c)^{-1}=\frac{1}{(1+c^2-ab)^2}\begin{bmatrix}-(c-i)^2, & -b^2, & -b(c-i)\\ -a^2, & -(c+i)^2, & -a(c+i)\\ 2a(c-i), & 2b(c+i), & c^2+ab+1\end{bmatrix},$$

the distribution $\mathcal{D}_\mu$ on $(\mathbb{C}^3-V_\mu)\times\mathbb{C}_*$ is given by

$$\begin{bmatrix}1, & 0, & 0; & \frac{1}{2}\partial_a\log(1+c^2-ab)\\ 0, & 1, & 0; & \frac{1}{2}\partial_b\log(1+c^2-ab)\\ 0, & 0, & 1; & \frac{1}{2}\partial_c\log(1+c^2-ab)\end{bmatrix}.$$

Hence M^3 is given by

$$(a,b,c;\sqrt{1+c^2-ab})\Longleftrightarrow\sqrt{1+c^2-ab}\ e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}, \quad (a,b,c)\in\mathbb{C}^3-V_\mu. \tag{19}$$

Since $\sqrt{\ }$ is two-valued function, M^3 is in fact a non-trivial double cover of $\mathbb{C}^3 – V_\mu$. (See also Proposition 5 below.)

3.2. Singular distributions in the normal ordering.

Since $uv=u{\circ}v+\frac{i\hbar}{2}$, we have $au^2+2cuv+bv^2=au^2+2cu{\circ}v+bv^2+\hbar ci$. In this subsection, we compute $e_*^{\frac{t}{\hbar}(au^2+bv^2+2cuv)}=e^{cit}e_*^{\frac{t}{\hbar}(au^2+bv^2+2cu\circ v)}$ by ΨDO-product formula (1). Setting

$$e_*^{\frac{t}{\hbar}(au^2+bv^2+2cuv)}=s(t)e_\circ^{\frac{1}{\hbar}(a(t)u^2+b(t)v^2+2c(t)u\circ v)}, \tag{20}$$

as in the similar computations in §3.1, we have (20) as follows:

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}&e^{\frac{t}{\hbar}(a'u^2+b'v^2+2c'uv)}*se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\\ =&\frac{1}{\hbar}(a'u^2+b'v^2+2c'(u{\circ}v+\frac{i\hbar}{2}))*se_\circ^{\frac{1}{\hbar}(au^2+bv^2+2cu\circ v)}\\ =&\{\frac{1}{\hbar}(a'u^2+b'v^2+2c'(u{\circ}v+\frac{i\hbar}{2}))+\frac{i}{\hbar}(2b'v+2c'u){\circ}(2au+2cv)\\ &+\frac{-1}{\hbar}\frac{1}{2}(2b')((2au+2cv)^2+2a\hbar)\}{\circ}se_\circ^{\frac{1}{\hbar}(au^2+bv^2+2cu\circ v)}.\end{aligned} \tag{21}$$

This is written as

$$= \frac{1}{\hbar}(a', b', c')N(a, b, c; s) \begin{bmatrix} u^2 \\ v^2 \\ 2u{\circ}v \\ \hbar \end{bmatrix} \circ s e_{\circ}^{\frac{1}{\hbar}(au^2+bv^2+2cu\circ v)}, \tag{22}$$

where

$$N(a, b, c; s) = \begin{bmatrix} 1, & 0, & 0; & 0 \\ -4a^2, & (1+2ci)^2, & 2ai(1+2ci); & -2a \\ 4ai, & 0, & 1+2ci; & i \end{bmatrix} \tag{23}$$

We denote by $N(a, b, c)$ the submatrix of first three columns of $N(a, b, c; s)$. The determinant of $N(a, b, c)$ is $(1 + 2ci)^3$, i.e. $= 0$ occurs at $e_{\circ}^{\frac{1}{\hbar}(au^2+bv^2+iu\circ v)}$. This is in fact a phase part of a vacuum computed in the normal ordering w.r.t. a certain canonical conjugate pair. (See Proposition 22 below.) Let $\mathcal{D}_\nu$ be the the singular distribution given by $N(a, b, c; s)$. Let

$$V_\nu = \{(a, b, c); 1 + 2ci = 0\} \quad \text{(phase part of vacuums)}. \tag{24}$$

Since

$$N(a, b, c)^{-1} = \frac{1}{(1+2ci)^2} \begin{bmatrix} (1+2ci)^2, & 0, & 0, \\ -4a^2, & 1, & -2ai \\ -4ai(1+2ci), & 0, & 1+2ci \end{bmatrix},$$

$\mathcal{D}_\nu$ is an ordinary involutive distribution on $(\mathbb{C}^3 - V_\nu) \times \mathbb{C}_*$ given by

$$\mathcal{D}_\nu(a, b, c) = \{(a', b', c'; \frac{c'i}{1+2ci}); (a', b', c') \in \mathbb{C}^3\}.$$

The maximal integral holomorphic submanifold N^3 of $\mathcal{D}_\nu$ through the origin $(0, 0, 0; 1)$ is given by

$$(a, b, c; \sqrt{1+2ci}) \Longleftrightarrow \sqrt{1+2ci}\; e_{\circ}^{\frac{1}{\hbar}(au^2+bv^2+2cu\circ v)}. \tag{25}$$

Since $\sqrt{}$ is a two valued function, N^3 is the non-trivial double cover of $\mathbb{C}^3 - V_\nu$.

4. $*$-exponential functions and vacuums

We now consider the evolution equation (4) for every quadratic form as integral curves of the distributions mentioned in §3. To define the $*$-exponential function $e_*^{t(au^2+bv^2+2cuv)}$, we set $e_*^{t(au^2+bv^2+2cuv)} = F(t, u, v)$, and consider the evolution equation

$$\frac{\partial}{\partial t}F(t, u, v) = (au^2+bv^2+2cuv)*F(t, u, v), \quad F(0, u, v) = 1 \tag{26}$$

First, we compute the r.h.s. of (26) by the Moyal product formula (3). Minding a real analytic solution of (26) in t is unique, if it exists, we assume that $F(t, u, v)$ has the form $s(t)e^{a(t)u^2+b(t)v^2+2c(t)uv}$. Then, we solve the system of ordinary differential equations:

$$\begin{aligned} &(a'(t), b'(t), c'(t); s'(t)/s(t)) = (a, b, c)M(a(t), b(t), c(t); s(t)), \\ &(a(0),\ b(0), c(0); s(0)) = (0, 0, 0; 1). \end{aligned} \tag{27}$$

Lemma 4. (Cf.[B], [MS]) *The solution of* (26) *is given by*

$$f_t(x) = \frac{1}{\cosh(\hbar\sqrt{ab-c^2}t)} \exp \frac{x}{\hbar\sqrt{ab-c^2}} \Big\{\tanh(\hbar\sqrt{ab-c^2}\ t)\Big\},$$

where $x = au^2 + bv^2 + 2cuv$.

Thus, Lemma 4 holds for the case $ab-c^2 = 0$. We may set

$$\frac{1}{\hbar\sqrt{ab-c^2}} \tanh(\hbar\sqrt{ab-c^2}\ t) = t.$$

via Taylor expansion.

By Lemma 4, we have

$$\begin{aligned} e_*^{\frac{t}{\hbar}(au^2+bv^2+2cuv)} &= \frac{1}{\cosh(\sqrt{ab-c^2}\ t)} e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)\Big(\frac{1}{\sqrt{ab-c^2}}\tanh(\sqrt{ab-c^2}\ t)\Big)} \\ &= \frac{1}{\cos(\sqrt{c^2-ab}\ t)} e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)\Big(\frac{1}{\sqrt{c^2-ab}}\tan(\sqrt{c^2-ab}\ t)\Big)} \end{aligned} \tag{28}$$

The ambiguity of $\pm\sqrt{ab-c^2}$ makes no difference for the result.

By (28), we have in particular, if $c^2 \neq ab$, then $e_*^{\frac{\pi}{\hbar\sqrt{c^2-ab}}(au^2+bv^2+2cuv)} = -1$, but $e_*^{\frac{\pi}{2\hbar\sqrt{c^2-ab}}(au^2+bv^2+2cuv)}$ diverges in the Weyl ordering. Let Π_μ be the subset of $\mathbb{C}^3$ where $e_*^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$ is singular in the Weyl ordering:

$$\Pi_\mu = \{(a,b,c) \in \mathbb{C}^3; \sqrt{c^2-ab} = \pi(\mathbb{Z}+\frac{1}{2})\}.$$

The $*$-exponential mapping $\exp_*$ is a holomorphic mapping of $\mathbb{C}^3 - \Pi_\mu$ into M^3. Using (19) and Lemma 4, we have

Proposition 5. M^3 *is a non-trivial double cover of* $\mathbb{C}^3 - V_\mu$, *and*

$$M^3 = \{\pm\sqrt{c^2-ab+1}\ e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)};\ c^2-ab+1 \neq 0\}.$$

$\{e_*^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\}$ *covers the open dense subset*

$$M^3 - \{-e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)};\ c^2-ab = 0, (a,b,c) \neq (0,0,0)\}$$

of M^3.

Proof. Suppose $Q \in M^3$. Set $\pi Q = (a,b,c)$. Then, $c^2-ab+1 \neq 0$. Since the exceptional values of $\tan z$ are $\pm i$, there is θ such that $\tan\theta = \sqrt{c^2-ab}$. By (28), we have

$$\sqrt{c^2-ab+1}e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)} = e_*^{\frac{\theta}{\hbar\sqrt{c^2-ab}}(au^2+bv^2+2cuv)}$$

Remind $\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1$. For $c^2-ab = 0$, $\frac{1}{\sqrt{c^2-ab}}\theta$ should be read as 1.

Remark that $e_*^{t(au^2+bv^2+2cuv)} \in M^3$, whenever this is defined. The difference of the periodicity of cosine and tangent gives that if $c^2-ab\neq 0$, then

$$\pi^{-1}\pi\{e_*^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}; (a,b,c)\notin \Pi_\mu\} = \{\pm e_*^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}; (a,b,c)\notin \Pi_\mu\}. \qquad (29)$$

However, we have to set $\sqrt{1}=1$ in the case $c^2-ab=0$ to get the initial value 1 at $t=0$. Thus we can not get $-e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$ by the exponential function, if $(a,b,c)\neq(0,0,0)$. This proves the last assertion.

Remark that $e_*^{\frac{2\pi}{\hbar}uv} = -1 \in M^3$ means the integral submanifold through $(0,0,0;-1)$ is in M^3. These arguments together with (19) give the first and second assertions. □

In what follows we denote by M^3_* the set of elements of M^3 expressed in the form of $*$-exponential functions:

$$M^3_* = \{\pm e_*^{\frac{1}{\hbar}(au^2+bv^2+c(u*v+v*u))};\ \text{it's Weyl ordering} \in M^3\}.$$

Similarly, we denote for each canonical conjugate pair (u,v),

$$N^3_* = \{\pm e_*^{\frac{1}{\hbar}(au^2+bv^2+c(u*v+v*u))};\ \text{it's normal ordering} \in N^3\}.$$

By the uniqueness of analytic solution of the evolution equation (4), the exponential law $e_*^{isx} * e_*^{itx} = e_*^{i(s+t)x}$ for a quadratic function in x holds where both sides are defined. Using this, we have

Lemma 6. *For $s,\sigma\in\mathbb{C}$ such that $1+s\sigma(ab-c^2)\neq 0$, we have*

$$e^{\frac{s}{\hbar}(au^2+bv^2+2cuv)} * e^{\frac{\sigma}{\hbar}(au^2+bv^2+2cuv)} = \frac{1}{1+s\sigma(ab-c^2)} e^{\frac{s+\sigma}{\hbar(1+s\sigma(ab-c^2))}(au^2+bv^2+2cuv)}$$

In particular, we have an idempotent element

$$2e^{\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)} * 2e^{\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)} = 2e^{\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)}.$$

Recall $2e^{\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)}$ is a vacuum defined in §2.

Corollary 7. *Vacuums are obtained as the limit point of $*$-exponential functions; i.e.*

$$2e^{\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)} = \lim_{t\to\infty} e^t e_*^{\frac{t}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)}$$

is a vacuum for every (a,b,c) such that $c^2-ab\neq 0$.

This shows that vacuums may be regarded as certain equilibrium states (cf. [BL]).
Remarks. Let $\mathrm{Ad}(g)h = g{*}h{*}g^{-1}$. Using (10) and the uniqueness of the solution, we see that

$$\mathrm{Ad}(\pm e_*^{\frac{it}{2\hbar}(au^2+bv^2+2cuv)}) \begin{bmatrix} u \\ v\end{bmatrix} = \Big(\exp t\begin{bmatrix} -c & -b \\ a & c\end{bmatrix}\Big)\begin{bmatrix} u \\ v\end{bmatrix}. \qquad (30)$$

Remark that $\mathrm{Ad}(\pm e_*^{\frac{it}{2\hbar}(au^2+bv^2+2cuv)})$ has no singularity in t, and $\pm e_*^{\frac{it}{2\hbar}(au^2+bv^2+2cuv)}$ makes no difference. Hence, the "group" generated by the $*$-exponential functions of quadratic forms looks like a "double covering group" of $SL_{\mathbb{C}}(2)$, which is known to be simply connected.

Moreover, (30) is useful to make the product formula involving elements f,g of $\mathcal{E}_p(\mathbb{C}^2)$, $p<2$. We compute as follows:

$$(f*e_*^{p(u,v)})*(g*e_*^{q(u,v)})=(f*(e^{\mathrm{ad}(p(u,v))}g))*(e_*^{p(u,v)}*e_*^{q(u,v)})$$

This is welldefined whenever $e_*^{p(u,v)}*e_*^{q(u,v)}$ is welldefined. Hence, we have only to care about the product formula $e_*^{p(u,v)}*e_*^{q(u,v)}$.

The following lemma is useful to make transcendental formula, and is proved by that both quantities satisfy the same partial differential equation with the same initial condition, but intuitively this is given by the trivial identity $v*(u*v)^m=(v*u)^m*v$, which explain the name of the next lemma:

Lemma 8. (Bumping lemma)

$$v*e_*^{itu*v}=e_*^{itv*u}*v, \quad e_*^{itu*v}*u=u*e_*^{itv*u}.$$

4.1. $*$-exponential functions by the normal ordering.

Although $e_*^{\pm\frac{\pi}{\hbar}uv}$ diverge in the Weyl ordering, we prove in this subsection that such elements make sense in the normal ordering. We now consider the evolution equation (26) in the normal ordering. Assuming that

$$e_*^{\frac{t}{\hbar}(au^2+bv^2+2cu*v)}=\psi(t)e_\circ^{\phi_1(t)u^2+\phi_2(t)v^2+2\phi_3(t)u\circ v},$$

we solve the system of ordinary differential equations:

$$\begin{cases}\phi_1'(t)=\frac{1}{\hbar}a+4ic\phi_1(t)-4\hbar b\phi_1(t)^2\\ \phi_2'(t)=\frac{1}{\hbar}b+4ib\phi_3(t)-4\hbar b\phi_3(t)^2\\ \phi_3'(t)=\frac{1}{\hbar}c+2ic\phi_3(t)+2ib\phi_1(t)-4\hbar b\phi_1(t)\phi_3(t)\\ \psi'(t)=-2\hbar b\phi_1(t)\psi(t)\end{cases}\tag{31}$$

with the initial condition $\phi_i(0)=0$ and $\psi(0)=1$.

Proposition 9. *There exists a unique analytic solution of* (31) *given by the following form:*

$$\begin{cases}\phi_1(t)=\dfrac{a}{2\hbar}\dfrac{\sin(2\sqrt{D}\,t)}{\sqrt{D}\cos(2\sqrt{D}\,t)-ic\sin(2\sqrt{D}\,t)},\\ \phi_2(t)=\dfrac{b}{2\hbar}\dfrac{\sin(2\sqrt{D}\,t)}{\sqrt{D}\cos(2\hbar\sqrt{D}\,t)-ic\sin(2\sqrt{D}\,t)},\\ \phi_3(t)=\dfrac{i}{2\hbar}\Big(1-\dfrac{\sqrt{D}}{\sqrt{D}\cos(2\sqrt{D}\,t)-ic\sin(2\sqrt{D}\,t)}\Big),\\ \psi(t)=e^{-cit}\Big(\dfrac{\sqrt{D}}{\sqrt{D}\cos(2\sqrt{D}\,t)-ic\sin(2\sqrt{D}\,t)}\Big)^{1/2}\end{cases}\tag{32}$$

where $D = c^2 - ab$. *This works also for the case* $D = 0$ *by setting* $\frac{1}{\sqrt{D}}\sin(2\sqrt{D}\,t) = 2t$ *via Taylor expansion. The* $\pm$ *ambiguity of* $\sqrt{D}$ *makes no difference for the result, but the* $\pm$ *ambiguity of* $(\ )^{1/2}$ *remains in the expression of* $\psi(t)$.

Remark that taking the *complex conjugate* of (32), we obtain the formula of $*$-exponential function for the anti-normal ordering. By this observation, we have the following:

Lemma 10. *In every ordering, the* $*$*-exponential function* $e_*^{\frac{1}{\hbar}au^2+bv^2+2cuv}$ *has singularities in* $(a,b,c)\in\mathbb{C}^3$. *However, there is no common singularity of the normal ordering and of the Weyl ordering.*

Remarking $2uv = u{*}v + v{*}u = 2u{*}v + i\hbar = 2u{\circ}v + i\hbar$, we can use Lemma 9 to obtain the formula of $e_*^{\frac{t}{\hbar}(au^2+bv^2+c(u*v+v*u))}$. Remark that $e_*^{\frac{t}{\hbar}(au^2+bv^2+2cuv)}$ is a curve contained in N^3, that is, $\sqrt{1+2i\hbar\phi_3(t)} = e^{cit}\psi(t)$ should hold by (25). This is checked by direct calculation. For the special case $ab=0$, we have

$$e_*^{\frac{t}{\hbar}(au^2+bv^2+2cu\circ v)} = e_\circ^{\frac{1}{4ci\hbar}(e^{4cit}-1)(au^2+bv^2)+\frac{1}{2i\hbar}(e^{2cit}-1)2u\circ v}, \quad ab=0. \tag{33}$$

This is because by setting $\sqrt{c^2}=c$, (33) gives the real analytic solution of (31) with initial data 1. Remark (33) has no singularity in $t\in\mathbb{C}$. Using (33), we have

$$e_*^{\frac{t}{\hbar}(au^2+bv^2+c(u*v+v*u))} = e^{cit}e_\circ^{\frac{1}{4ci\hbar}(e^{4cit}-1)(au^2+bv^2)+\frac{1}{2i\hbar}(e^{2cit}-1)2u\circ v}, \quad ab=0. \tag{34}$$

By remarking $u{*}v + v{*}u = 2u{*}v + i\hbar$ again, Proposition 9 gives the formula of $e_*^{\frac{\pi}{2\hbar}(au^2+bv^2+c(u*v+v*u))}$ for $c^2-ab=1$, and we have a very strange formula:

Lemma 11. *In the normal ordering w.r.t.* (u,v), $e_*^{\frac{\pi}{2\hbar}(au^2+bv^2+c(u*v+v*u))}$ *with* $c^2-ab=1$ *is given identically as* $\sqrt{-1}e_\circ^{\frac{2i}{\hbar}u\circ v}$.

4.2. Polar element.

Here a new question arises whether the $\pm$ ambiguity of $\sqrt{-1}$ of Lemma 11 can be eliminated for all $a,b,c\in\mathbb{C}$. Our conclusion in this section is that the ambiguity can *not* be eliminated.

Remark first that $e_*^{\frac{\pi}{2\hbar}(au^2+bv^2+c(u*v+v*u))}$ diverges in the Weyl ordering for $c^2-ab=1$.

By Lemma 11, $e_*^{\frac{\pi}{2\hbar}(au^2+bv^2+c(u*v+v*u))}$ is independent of a,b,c whenever $c^2-ab=1$. Thus, it must be viewed as a single element. We call it the *polar element* and denote by ε_{00}. We have in particular that

$$\varepsilon_{00}*\varepsilon_{00} = -1, \quad \mathrm{Ad}(\varepsilon_{00}) = -I \tag{35}$$

but ε_{00} has several strange features.

4.3. It looks like a contradiction.

It is clear that $\pi(\varepsilon_{00}) = (0,0,i)$. Moreover $\varepsilon_{00}{*}\varepsilon_{00} = -1$ by the exponential law. But this does not imply that $\varepsilon_{00} = \sqrt{-1}$, because the following holds by the bumping Lemma 8:

Proposition 12. $u*\varepsilon_{00}+\varepsilon_{00}*u=0$, $v*\varepsilon_{00}+\varepsilon_{00}*v=0$. *In particular,* ε_{00} *commutes with every* $\frac{t}{\hbar}(au^2+bv^2+2cuv)$, *and hence with* $e_*^{\frac{t}{\hbar}(au^2+bv^2+2cuv)}$.

Moreover, since $(a,b,c)=(0,0,1)$ and $(0,0,-1)$ are arcwise connected in the set $c^2-ab=1$, Lemma 11 gives

$$e_*^{\frac{\pi}{2\hbar}(u*v+v*u)}=\sqrt{-1}e_\circ^{\frac{2i}{\hbar}u\circ v}=e_*^{-\frac{\pi}{2\hbar}(u*v+v*u)}.$$

Considering the exponential law of the $*$-exponential function $e_*^{\frac{t}{2\hbar}(u*v+v*u)}$ for $t\in\mathbb{C}-\{\text{singular set}\}$, we must set by (34)

$$e_*^{\frac{\pi}{2\hbar}(u*v+v*u)}=ie_\circ^{\frac{2i}{\hbar}u\circ v},\quad e_*^{-\frac{\pi}{2\hbar}(u*v+v*u)}=-ie_\circ^{\frac{2i}{\hbar}u\circ v}.$$

If one wants to fix $\pm$ ambiguity, the exponential law gives

$$-1=e_*^{\frac{\pi}{2\hbar}(u*v+v*u)}*e_*^{\frac{\pi}{2\hbar}(u*v+v*u)}=e_*^{\frac{\pi}{2\hbar}(u*v+v*u)}*e_*^{-\frac{\pi}{2\hbar}(u*v+v*u)}=1.$$

Remark that $(-v,u)$ is a canonical conjugate pair and the set of all canonical conjugate pairs is arcwise connected. Then, it seems rigorous to set $\varepsilon_{00}=ie_\circ^{\frac{2i}{\hbar}u\circ v}=ie_\circ^{\frac{2i}{\hbar}(-v)\circ u}$, but it causes the same trouble. Therefore, we must conclude that the $\pm$ ambiguity in Proposition 9 cannot be eliminated. One has to set $\varepsilon_{00}=\sqrt{-1}e_\circ^{\frac{2i}{\hbar}u\circ v}$ with $\pm$ ambiguity. It is better to understood ε_{00} as a “two-valued” element. But since such notion does not exist in the set theory, it seems to be impossible to define ε_{00} as a point of a point set. Because of this anomalous character of ε_{00} we had spent a lot of time to check our calculation, and remark a conclusion that *the polar element* ε_{00} *should be understood as a “two-valued” element.* Remark that if one considers m-tensor power of our system, we have an element $\prod_{i=1}^m\varepsilon_{00}^{(i)}$ which should be treated as a 2^m-valued element. In §5, we claim this anomalous element is still useful in the calculation of $*$-product.

Olver [Ol] gave some examples of local Lie groups which does not form a group, because the associativity breaks down globally. Though these are examples within point sets, the situation seems similar and helpful to understand the appearance of the ambiguity. We have to study such phenomena more closely, in order to understand the difficulties we must manage in treating exact deformation quantizations. (Cf. also [R].)

To understand ε_{00} rigorously within the set theory, we give several notions in §7.

5. Product formulas, restriction to real forms

We now study the “group” generated by $e_*^{aH+bX+cY}$ by using the Weyl ordering and the polar element ε_{00}. We see that the $*$-product

$$e_*^{aH+bX+cY}*e_*^{a'H+b'X+c'Y}$$

is defined in general with an ambiguity of $\pm$-sign of $\sqrt{\cdot}$, and the ambiguity can only be eliminated locally.

5.1. Product formula with $\pm$ ambiguity and singularity.

We first want to establish the product formula with $\pm$ ambiguity. If we use the Weyl ordering, Proposition 2 gives that the general product formula for quadratic exponential functions can be obtained from the two cases as follows:

$$e^{tu^2} * e^{au^2+bv^2+2cuv}, \quad e^{\tau uv} * e^{au^2+bv^2+2cuv}.$$

Solving (27) with the general initial condition

$$(a(0), b(0), c(0); s(0)) = (a, b, c; 1), \tag{36}$$

we see that the first one is written as

$$e_*^{\frac{t}{\hbar}u^2} * e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)} = \frac{1}{\sqrt{1+bt}} e^{\frac{1}{\hbar(1+bt)}\{(a+(ab-c^2-2ci+1)t)u^2+bv^2+2(c-ibt)uv\}} \tag{37}$$

where the ambiguity of $\pm\sqrt{1+bt}$ can not be eliminated for all t, b. Remark also that the discriminant of $(a+(ab-c^2-2ci+1)t)u^2+bv^2+2(c-ibt)uv$ is $(c^2-ab+1)(1+bt)-(1+bt)^2$. Thus, $e^{\frac{t}{\hbar}u^2} * \sqrt{c^2-ab+1}\, e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$ is contained in M^3. (Compute the discriminant $+1$ of the phase function.)

Remarking $e_*^{\frac{t}{\hbar}u^2} = e^{\frac{t}{\hbar}u^2}$, we have the following:

Lemma 13. *For $e^{\frac{t}{\hbar}u^2}$ and $Q \in M^3$ such that $\pi(Q) = e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$ and $bt \neq -1$, Q is written in the form $\sqrt{c^2-ab+1}\, e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$, and the product $e^{\frac{t}{\hbar}u^2} * Q$ is an element of M^3.*

Similar to (37), we have in the Weyl ordering that

$$e_*^{\frac{t}{\hbar}v^2} * e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)} = \frac{1}{\sqrt{1+at}}\, e^{\frac{1}{\hbar(1+at)}\{au^2+(b+(ab-c^2+2ci+1)t)v^2+2(c+iat)uv\}}, \tag{38}$$

and hence we have the similar result as Lemma 13.

Remark $e_*^{\frac{t}{\hbar}2uv} = \sqrt{1+s^2}\, e^{\frac{s}{\hbar}2uv}$, where $s = \tan t$. Solving (27) with the initial condition (36), we have in the Weyl ordering that

$$e^{\frac{s}{\hbar}2uv} * e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)} = \\ \frac{1}{\sqrt{1-2cs+(c^2-ab)s^2}} e^{\frac{1}{\hbar(1-2cs+(c^2-ab)s^2)}(a(1+is)^2u^2+b(1-is)^2v^2+(c-(c^2-ab-1)s-cs^2)2uv)} \tag{39}$$

where the ambiguity of $\pm\sqrt{1-2cs+(c^2-ab)s^2}$ can not be eliminated.

Note the following identity for the computation of discriminant of the phase function of (39):

$$\begin{aligned}\big(1-2cs+(c^2-ab)s^2\big)^2+\big(c-(c^2-ab-1)s-cs^2\big)^2-ab(1+is)^2(1-is)^2 \\ =(c^2-ab+1)(1+s^2)\big((c^2-ab)s^2-2cs+1\big).\end{aligned} \tag{40}$$

Using (39) and (40) for the computation of 1+discriminant, we see :

Lemma 14. *If $Q_1, Q_2 \in M^3$ such that $\pi(Q_1)=e^{\frac{s}{\hbar}2uv}$, $\pi(Q_2)=e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$, then*

$$Q_1=\pm\sqrt{1+s^2}\,e^{\frac{s}{\hbar}2uv},\qquad Q_2=\pm\sqrt{c^2-ab+1}\,e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$$

with $1+s^2\neq 0$, $c^2-ab+1\neq 0$. Furthermore, if $1-2cs+(c^2-ab)s^2\neq 0$, then the $*$-product $Q_1 * Q_2$ is defined as an element of M^3.

Though every product formula in Weyl ordering has some singularity, this does not mean that the $*$-product can not be defined at such points. Recall that every quadratic form $Q(u,v)$ is written in the form $(\alpha u+\beta v)^2$ if $ab-c^2=0$, or the form $\lambda(\alpha u+\beta v)(\gamma u+\delta v)$ with $\alpha\delta-\beta\gamma=1$ otherwise. Solving the evolution equation by using the ΨDO-product formula (1) under the standard procedure of Lie theory, we have the following product formula in the normal ordering:

$$\begin{aligned} e_\circ^{\frac{t}{\hbar}u^2} * e_\circ^{\frac{1}{\hbar}(au^2+bv^2+2cuv)} &= e_\circ^{\frac{1}{\hbar}((a+t)u^2+bv^2+2cuv)},\\ e_\circ^{\frac{t}{\hbar}uv} * e_\circ^{\frac{1}{\hbar}(au^2+bv^2+2cuv)} &= e_\circ^{\frac{1}{\hbar}(a(1+it)^2u^2+bv^2+(c+(ci+\frac{1}{2})t)2uv)}. \end{aligned} \tag{41}$$

This shows that every product can be computed without ambiguity by a canonical conjugate pair which is suitably chosen depending by elements to be calculated. However, since we have to use various canonical conjugate pair to write elements in the above standard form, this does not necessarily imply that $*$-products are defined without ambiguity.

Here, we remark on the associativity as follows: Though all formulas that we have given are written by using elements written in the form $e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$, but we can replace (a,b,c) by $(\hbar a,\hbar b,\hbar c)$. After this replacement, whole formulas (except the formula involving such an element as $e^{\frac{2i}{\hbar}uv}$, where one can not eliminate $\hbar$ in the expression) are changed into those which is real analytic in $\hbar$ and these are meaningful at $\hbar=0$.

Now if $\hbar$ is viewed as a formal parameter and everything is treated in formal power series in $\hbar$, we see that the associativity holds well, and $\pm$ ambiguities disappear in the product formula. Using the Taylor expansion in $\hbar$, we have:

Proposition 15. *The associativity*

$$\begin{aligned}\big(e^{au^2+bv^2+2cuv} * e^{a'u^2+b'v^2+2c'uv}\big) * e^{a''u^2+b''v^2+2c''uv}&\\ =e^{au^2+bv^2+2cuv} * \big(e^{a'u^2+b'v^2+2c'uv} * e^{a''u^2+b''v^2+2c''uv}\big)&\end{aligned}$$

holds if both sides are defined.

5.2. Product formulas involving polar element.

Even though ε_{00} is viewed as a two-valued element, we make product formulas. We first remark the following lemma, which shows the mapping $a\to\varepsilon_{00} * a$ is better to be understood as a 2-to-2-mapping:

Lemma 16. *If $D=c^2-ab\neq 0$, then*

$$\varepsilon_{00} * e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)} = \frac{1}{\sqrt{c^2-ab}}e^{-\frac{1}{\hbar(c^2-ab)}(au^2+bv^2+2cuv)}.$$

Proof. Set $s=\tan t$. Then we have $e_*^{\pm\frac{t}{\hbar}2uv}=\sqrt{1+s^2}\,e^{\frac{s}{\hbar}2uv}$, and we remark that $s\to\pm\infty$ as $t\to\pm\frac{\pi}{2}$. Multiplying $\sqrt{1+s^2}$ to (39) and taking the limit $s^2\to\infty$, we have Lemma. □

Since the ambiguity of $\sqrt{c^2-ab}$ depends on the choice of paths for $s^2\to\infty$, it is better to understand the equality of Lemma 16 as

$$\varepsilon_{00}*\pm e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}=\frac{\pm 1}{\sqrt{c^2-ab}}e^{-\frac{1}{\hbar(c^2-ab)}(au^2+bv^2+2cuv)}.$$

Remark that ε_{00} commutes with every $e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$ and ${\varepsilon_{00}}_*^2=-1$. Then, $\varepsilon_{00}*$ gives a 2-to-2-diffeomorphism of $M^3-M^3_0$ onto itself, where

$$M^3_0=\{e_*^{\frac{1}{\hbar}(au^2+bv^2+2cuv)};c^2-ab=0\}.$$

For a point P of M^3_0, the computation of $\varepsilon_{00}*P$ is represented by the case $P=e^{\frac{1}{\hbar}au^2}$. Since we see for $t\neq\pm\frac{\pi}{2}$ that

$$e_*^{\frac{t}{\hbar}2uv}*e^{\frac{1}{\hbar}au^2}=\sqrt{1+s^2}\,e^{\frac{1}{\hbar}(a(1+is)^2u^2+2suv)},\quad \tan t=s, \tag{42}$$

this is written in the form of $*$-exponential function and is a member of M^3. As $t\to\pm\frac{\pi}{2}$, then the r.h.s of (42) diverges. Hence, we see that $e_*^{\pm\frac{\pi}{\hbar}uv}*e^{\frac{1}{\hbar}au^2}$ can not be a member of M^3. Thus, we see $M^3\cap\varepsilon_{00}*M^3_0=\emptyset$.

In the normal ordering w.r.t. (u,v), we see by (41)

$$\varepsilon_{00}*e_\circ^{\frac{1}{\hbar}au^2}=\sqrt{-1}e_\circ^{\frac{1}{\hbar}au^2},\quad \varepsilon_{00}*e_\circ^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}=\sqrt{-1}e_\circ^{\frac{1}{\hbar}(au^2+bv^2+(i-c)2uv)}.$$

Since $\varepsilon^2_{00}=-1$, we see that $N^3_*=\varepsilon_{00}*N^3_*$ for every canonical conjugate pair.

In other words, we see that

$$N^3_*\subset M^3_*\cup(\varepsilon_{00}*M^3_*)\quad\text{for every canonical conjugate pair.}\tag{43}$$

Note that $e_*^{\frac{1}{\hbar}t(\alpha u+\beta v)^2}\in N^3_*$ by the normal ordering w.r.t. a certain canonical conjugate pair $u'=\alpha u+\beta v, v'=\gamma u+\delta v$, and $\varepsilon_{00}*N^3_*=N^3_*$. Then, we have that every element of $\varepsilon_{00}*M^3_0$ is contained in N^3_* w.r.t. a certain canonical conjugate pair. Combined these arguments with (43), we see that

$$M^3_*\cup(\bigcup_{(u,v)}N^3_*)=M^3_*\cup(\varepsilon_{00}*M^3_*)\tag{44}$$

where the union takes for all canonical conjugate pair mutually related linearly.

By (44), we have only to consider the "set" obtained by "gluing" M^3_* and $\varepsilon_{00}*M^3_*$ by the "2-to-2-diffeomorphism" $\varepsilon_{00}*$ given by Lemma 16.

5.3. General product formula.

By the argument in § 5.1, we have two cases that Q_1*Q_2 are not defined in the Weyl ordering:

$$e^{\frac{t}{\hbar}u^2}*\sqrt{c^2-ab+1}\,e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)},\quad\text{for}\quad 1+bt=0,$$

and

$$\sqrt{1+s^2}\,e^{\frac{s}{\hbar}2uv}*\sqrt{c^2-ab+1}\,e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)},\ \text{for}\ 1-2cs+(c^2-ab)s^2=0.$$

By Lemma16 and (37) we obtain the following:

Corollary 17. *If $1+bt=0$, then*

$$e^{\frac{t}{\hbar}u^2}*(\varepsilon_{00}*\sqrt{c^2-ab+1}\,e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}) = e^{\frac{1}{\hbar(c^2-ab+1)}((a+(ab+(ci-1)^2)t)u^2+bv^2+2(c-ibt)uv)}$$

and the r.h.s. is written in the form $e^{\frac{1}{\hbar}(\alpha u+\beta v)^2}\in M_0^3$.

If $1-2cs+(c^2-ab)s^2=0$, then remarking (40) again, we have

$$(\varepsilon_{00}*\sqrt{1+s^2}\,e^{\frac{s}{\hbar}2uv})*\sqrt{c^2-ab+1}\,e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$$
$$=e^{\frac{1}{\hbar(c^2-ab+1)(1+s^2)}(a(1+is)^2u^2+b(1-is)^2v^2+(c-(c^2-ab-1)s-cs^2)2uv)}.$$

This is written in the form $e^{\frac{1}{\hbar}(\alpha u+\beta v)^2}$, since the discriminant of the r.h.s. vanishes.

As a consequence, we have the following:

Theorem 18. *$M^3_*\cup(\varepsilon_{00}*M^3_*)$ is closed under the $*$-product. Moreover the set $\{e_*^{aH+bX+cY}; a,b,c\in\mathbb{C}\}$ generate $M^3_*\cup(\varepsilon_{00}*M^3_*)$.*

For the proof, it is enough to remark the following: For $Q_1,Q_2\in M^3$, if Q_1*Q_2 is not defined in the Weyl ordering, then $Q_1*(\varepsilon_{00}*Q_2)$ or $(Q_1*\varepsilon_{00})*Q_2$ is defined in the Weyl ordering as an element of M^3 and

$$Q_1*(\varepsilon_{00}*Q_2)=(Q_1*\varepsilon_{00})*Q_2$$

holds, whenever both sides are defined. If Q_1*Q_2, $Q_1*(\varepsilon_{00}*Q_2)$, and $(\varepsilon_{00}*Q_1)*Q_2$ are defined, then we have $Q_1*(\varepsilon_{00}*Q_2)=(\varepsilon_{00}*Q_1)*Q_2=\varepsilon_{00}*(Q_1*Q_2)$. Moreover, $(\varepsilon_{00}*Q_1)*(\varepsilon_{00}*Q_2)$ is defined as $-Q_1*Q_2$.

$M^3_*\cup(\varepsilon_{00}*M^3_*)$ is "locally" isomorphic to $SL_{\mathbb{C}}(2)$. As is remarked at (30), $M^3_*\cup\varepsilon_{00}*M^3_*$ may be viewed as a non-trivial double cover of $SL_{\mathbb{C}}(2)$, although such it can not be treated as a point set.

Looking product formulas (37), (38), (39) and Lemma16 more carefully, we see the following:

Theorem 19. *Restrict the coefficients a,b,c in the real number and consider $(ia,ib,ic)\in\mathbb{C}^3$, or $(a,b,ic)\in\mathbb{C}^3$, then all product formula are closed in these spaces respectively. That is, $\{e_*^{\frac{i}{\hbar}(au^2+bv^2+2cuv)}\}$ and $\{e_*^{\frac{1}{\hbar}(au^2+bv^2+2ciuv)}\}$ generate Lie subgroup-like objects under the $*$-product whose Lie algebra is $sl_{\mathbb{R}}(2), su(1,1)$ respectively.*

Since the first homotopy group π_1 of $SL_{\mathbb{R}}(2), SU(1,1)$ are both $\mathbb{Z}$, we see that the $\pm$ ambiguity can be treated as genuine double coverings in such subgroup-like objects. Hence, we see that $M^3_*\cup(\varepsilon_{00}*M^3_*)$ contains the double covering group of $SL_{\mathbb{R}}(2)=Sp(2;\mathbb{R})$, which can be regarded as the metaplectic group $Mp(2;\mathbb{R})$. Thus, $M^3_*\cup(\varepsilon_{00}*M^3_*)$ may be viewed as the complexification of $Mp(2;\mathbb{R})$. It is obvious that there is no such Lie group in the genuine group theory. Similarly, $M^3_*\cup(\varepsilon_{00}*M^3_*)$ contains the double covering group of $SU(1,1)$.

6. Intertwiners and extensions

Remark that ε_{00} is defined by using several orderings. To understand the anomalous element ε_{00}, and anomalous phenomena related to $*$-exponential functions of quadratic forms, we fix the notion of intertwiners.

We mentioned that, as in §4, ε_{00} is viewed as a two-valued element, by using a canonical conjugate pair (u,v). If we take another canonical conjugate pair $u'=au+bv, v'=cu+dv$ with $ad-bc=1$, then $e_*^{\frac{\pi}{2\hbar}(u'*v'+v'*u')}$ must equal $\sqrt{-1}e_\diamond^{\frac{2i}{\hbar}u'\diamond v'}$ where $\diamond$ indicate this element is expressed in the normal ordering w.r.t. (u',v'). We view ε_{00} as the collection of expressions by the various normal orderings. Thus, it is not clear whether one may identify $\sqrt{-1}e_\circ^{\frac{2i}{\hbar}u\circ v}$ with $\sqrt{-1}e_\diamond^{\frac{2i}{\hbar}u'\diamond v'}$. (See Proposition 23 for our conclusion.)

To consider this problem, we have to fix the notion of intertwiners between several orderings. In this section, we consider intertwiners between these ordering expressions. In particular, we consider the intertwiner between normal ordering and the Weyl ordering w.r.t (u,v).

6.1. Intertwiners or coordinate transformations.

The principle of making intertwiners is based on that the $*$-exponential function $e_*^{\alpha u+\beta v}$ is given as follows by solving the evolution equation (4):

$$
\begin{aligned}
e_{*_M}^{\alpha u+\beta v} &= e_{\cdot}^{\alpha u+\beta v}, && \text{(in Weyl ordering)}\\
e_{*_N}^{\alpha u+\beta v} &= e^{\frac{i\hbar}{2}\alpha\beta}e_\circ^{\alpha u+\beta v}, && \text{(in normal ordering)}\\
e_{*_{\bar N}}^{\alpha u+\beta v} &= e^{-\frac{i\hbar}{2}\alpha\beta}e_\bullet^{\alpha u+\beta v} && \text{(in antinormal ordering)}
\end{aligned}
$$

We define the intertwiner from the Weyl ordering to the normal ordering w.r.t. (u,v) by a densely defined linear operator

$$I_{\cdot}^{\circ} = e^{\frac{i\hbar}{2}\partial_u\partial_v}. \tag{45}$$

In particular, we remark that $I_{\cdot}^{\circ}(f*_M g)=(I_{\cdot}^{\circ}f)*_N(I_{\cdot}^{\circ}g)$ holds if both sides are defined, where $*_M$, $*_N$ denote the Moyal-product and ΨDO-product respectively. This is because that both sides are the expressions of $f*g$ by different ordering.

Suppose (u,v) and (u',v') are related by

$$u'=au+bv,\quad v'=cu+dv,\quad ad-bc=1,\quad u=du'-bv',\quad v=-cu'+av'.$$

The $*$-exponential function is given by $e_*^{\alpha'u'+\beta'v'}=e^{\frac{i\hbar}{2}\alpha'\beta'}e_\diamond^{\alpha'u'+\beta'v'}$ w.r.t. a canonical conjugate pair (u',v'), where $\diamond$ indicates the normal ordering expression w.r.t. (u',v'). Thus, we must identify $e^{\frac{i\hbar}{2}\alpha\beta}e_\circ^{\alpha u+\beta v}$ with $e^{\frac{i\hbar}{2}\alpha'\beta'}e_\diamond^{\alpha'u'+\beta'v'}$. Hence, we have to define the intertwiner $I_\circ^\diamond$ from the normal ordering w.r.t. (u,v) to that w.r.t. (u',v') as follows:

$$I_\circ^\diamond f = e^{\frac{i\hbar}{2}\partial_{u'}\partial_{v'}-\frac{i\hbar}{2}\partial_u\partial_v}f. \tag{46}$$

Precisely, we must consider the exponential of the operator

$$\partial_{u'}\partial_{v'}-\partial_u\partial_v = -bd\partial_u^2+(ad+bc-1)\partial_u\partial_v-ac\partial_v^2$$

If

$$g_\diamond(u,v)=e^{\frac{i\hbar}{2}(-bd\partial_u^2+(ad+bc-1)\partial_u\partial_v-ac\partial_v^2)}f_\circ(u,v),$$

then $g_\diamond(u,v)$ is regarded as the normal ordering w.r.t. (u',v'). By the following decomposition

$$e^{-bd\partial_u^2+(ad+bc-1)\partial_u\partial_v-ac\partial_v^2}=e^{-bd\partial_u^2}e^{(ad+bc-1)\partial_u\partial_v}e^{-ac\partial_v^2},$$

we may treat these intertwiners separately. Remark here that these intertwiners are welldefined if $\hbar$ is a formal parameter. For real parameter $\hbar$, intertwiners are first defined on the space $\mathbb{C}[u,v]$, and extended as follows:

Theorem 20. *For every $0<p\leq 2$, $I_\circ^\diamond$ extends to continuous linear isomorphisms of $\mathcal{E}_p(\mathbb{C}^2)$ onto itself, and $I_\circ^\diamond$ is also an algebra isomorphism of $(\mathcal{E}_p(\mathbb{C}^2)_\circ;*)$ onto $(\mathcal{E}_p(\mathbb{C}^2)_\diamond;*)$.*

Proof. We first remark that by [OMMY1] Proposition 6.1, the system of seminorms (5) can be replaced by the following system of norms: Set $\tau=\frac{1}{p}$, and for $f=\sum a_{m,n}u^mv^n$ we define

$$\|f\|_{\tau,s}=\sum_{m,n}|a_{m,n}|(m+n)^{\tau(m+n)}s^{\tau(m+n)},\quad s>0.$$

This system defines the same Fréchet space as $\mathcal{E}_p(\mathbb{C}^2)$ for every $p>0$.

We show $e^{\alpha\partial_u\partial_v}$, $e^{\beta\partial_u^2}$ and $e^{\gamma\partial_v^2}$ extend to continuous linear isomorphisms of $\mathcal{E}_p(\mathbb{C}^2)$ onto itself for every $0<p\leq 2$. For every $f=\sum a_{m,n}u^mv^n$, we see that

$$e^{\alpha\partial_u\partial_v}f=\sum_{m,n,k}\frac{\alpha^k}{k!}\frac{(m+k)!}{k!}\frac{(n+k)!}{k!}a_{m+k,n+k}u^mv^n.$$

Hence,

$$\|e^{\alpha\partial_u\partial_v}f\|_{\tau,s}=\sum_{m,n}\sum_{0\leq k\leq m\wedge n}\frac{\alpha^k}{k!}\frac{m!}{(m-k)!}\frac{n!}{(n-k)!}|a_{m,n}|(m+n-2k)^{\tau(m+n-2k)}s^{\tau(m+n-2k)}$$

$$<\sum_{m,n}\sum_{0\leq k\leq m\wedge n}\frac{\alpha^k}{k!}(mn)^{(1-\tau)k}s^{\tau(m+n)}|a_{m,n}|(m+n)^{\tau(m+n)}.$$

If $\frac{1}{2}\leq\tau$, then we have $(mn)^{(1-\tau)}\leq K(m+n)$. Thus, we have

$$\|e^{\alpha\partial_u\partial_v}f\|_{\tau,s}\leq|a_{m,n}|(m+n)^{\tau(m+n)}(e^{K/\tau}\alpha s)^{\tau(m+n)}.$$

Since $e^{-\alpha\partial_u\partial_v}$ gives the inverse of $e^{\alpha\partial_u\partial_v}$ we have that for every $p\leq 2$, $e^{\alpha\partial_u\partial_v}$ is a linear isomorphism of $\mathcal{E}_p(\mathbb{C}^2)$ onto itself. By the similar proof, we obtain the same result for $e^{\beta\partial_u^2}$ and $e^{\gamma\partial_v^2}$.

We now choose α,β,γ appropriately so that $e^{\alpha\partial_u\partial_v}e^{\beta\partial_u^2}e^{\gamma\partial_v^2}$ defines an intertwiner $I_\circ^\diamond$. Since it is clear that $I_\circ^\diamond$ is an algebra isomorphism of $\mathbb{C}[u,v]$ onto itself, the continuity of $I_\circ^\diamond$ gives the second half of the theorem. □

It is remarkable that the composition of intertwiners gives another intertwiner, symbolically as $I_\diamond^{\diamond'}I_\circ^\diamond(f)=I_\circ^{\diamond'}(f)$ for $f\in\mathcal{E}_2(\mathbb{C}^2)$, but this holds also for $f=e^{au^2+bv^2+2cuv}$ if both sides are defined. This is because these are real analytic w.r.t. $\hbar$ and the formula holds on the formal level w.r.t. $\hbar$.

6.2. Strange characters of extended intertwiners.

However, the intertwiner does not extend to the space $\mathcal{E}_{2+}(\mathbb{C}^2)$. In such spaces, intertwiners are calculated only for exponential functions of quadratic forms $f=e^{au^2+bv^2+2cuv}$. It is not clear to what extent intertwiners are defined.

Normal orderings have less symmetries than Weyl ordering. Thus, it seems to be natural to define as follows:

Definition 21. Let A and B be elements of $\mathcal{E}_p$, defined by normal ordering expressions w.r.t. some canonical conjugate pairs. We denote by $A_\circ$, $B_\diamond$ normal ordering expressions of A, B w.r.t. $(u,v;\circ)$, $(u',v';\diamond)$ respectively. We define *the notion of equal $A=B$, if and only if $I_\circ^\diamond(A_\circ)=B_\diamond$* through the intertwiner between normal ordering expressions w.r.t. these canonical conjugate pairs.

However, *it occurs that intertwiners are defined only as 2-to-2 mappings* because of the ambiguity of $\sqrt{\ }$. This shows that Definition 21 does not work in the genuine sense. We have to understand intertwiners under 2-to-2 mappings in the following. Consider the differential equation

$$\frac{\partial}{\partial t}f=i\hbar\partial_u\partial_v f, \tag{47}$$

A real analytic solution in t is unique, if it exists. The solution with the initial function e^{au+bv} is given by $e^{i\hbar abt}e^{au+bv}$.

To obtain the solution with initial function $e^{\frac{1}{\hbar}(\alpha u^2+\beta v^2+2\gamma uv)}$, we set as follows by following the standard techniques in the classical Lie theory:

$$f=s(t)e^{\frac{1}{\hbar}(\phi_1(t)u^2+\phi_2(t)v^2+\phi_3(t)2uv)}.$$

Then, (47) is rewritten as a system of ordinary differential equations:

$$\begin{aligned}(\phi_1'(t),&\phi_2'(t),\phi_3'(t);s'(t))\\ =&\big(4i\phi_1(t)\phi_3(t),\,4i\phi_2(t)\phi_3(t),\,2i\big(\phi_1(t)\phi_2(t)+\phi_3(t)^2\big);\,2is(t)\phi_3(t)\big).\end{aligned} \tag{48}$$

First we see that

$$\phi_1(t)=\alpha e^{4i\int_0^t\phi_3(\tau)d\tau},\quad \phi_2(t)=\beta e^{4i\int_0^t\phi_3(\tau)d\tau}$$

Setting $x(t)=\int_0^t\phi_3(\tau)d\tau$, we have

$$x''(t)=2i\alpha\beta e^{8ix(t)}+2ix'(t)^2,\quad x(0)=0,\ x'(0)=\gamma.$$

We regard x as an independent variable and set $\phi_3(t)=p(x(t))$. Then since $\phi_3=x'$, we have $x''(t)=p\frac{dp}{dx}$. It follows

$$\frac{1}{2}\frac{dp^2}{dx}-2ip^2=2i\alpha\beta e^{8ix},\quad p(0)=\gamma,$$

and we have $p^2(x)=(\gamma^2-\alpha\beta)e^{4ix}+\alpha\beta e^{8ix}$. Thus, we obtain

$$\begin{aligned}&e^{\hbar ti\partial_u\partial_v}e^{\frac{1}{\hbar}(\alpha u^2+\beta v^2+2\gamma uv)}\\ &=\frac{1}{\sqrt{1-4i\gamma t-4(\gamma^2-\alpha\beta)t^2}}\,e^{\frac{1}{1-4i\gamma t-4(\gamma^2-\alpha\beta)t^2}\frac{1}{\hbar}(\alpha u^2+\beta v^2+(\gamma-2i(\gamma^2-\alpha\beta)t)2uv)},\end{aligned} \tag{49}$$

where the ambiguity of the sign of $\sqrt{1-4i\gamma t+4(\alpha\beta-\gamma^2)t^2}$ will be discussed in §6.3.

Set $t=\frac{1}{2}$. Then, we have the intertwiner $I_{\cdot}^{\circ}$ from the Weyl ordering to the normal ordering:

$$(a',b',c';s')=I_{\cdot}^{\circ}(a,b,c;s)=\frac{1}{1-2ic-D}(a,b,c-iD;s\sqrt{1-2ic-D}), \qquad (50)$$

where $D=c^2-ab$.

Proposition 22. $I_{\cdot}^{\circ}(e_{\cdot}^{\frac{1}{\hbar}(au^2+bv^2+2cu\cdot v)})$ *is singular if and only if* $(1-ci)^2+ab=0$.

It is easy to see that $D'=(c')^2-a'b'=\frac{D}{1-2ic-D}$, and the inverse mapping $I_{\circ}^{\cdot}=(I_{\cdot}^{\circ})^{-1}$ is given by setting $t=-\frac{1}{2}$. Indeed, we have

$$I_{\circ}^{\cdot}(a',b',c';s)=\frac{1}{1+2ic'-D'}(a',b',c'+iD';s'\sqrt{1+2ic'-D'}).$$

To confirm the result, we check that applying the intertwiner $I_{\cdot}^{\circ}$ given by (50) through (49) to the l.h.s. of (28) gives the normal ordering expression given in Proposition 9.

6.3. The case $e^{-i\hbar\partial_u^2}$.

Remark first that the intertwiner from the normal ordering w.r.t. (u,v) to that w.r.t. $(u',v')=(u+v,v)$ is given by $e^{-\frac{i\hbar}{2}\partial_u^2}$.

Consider now the operator $\frac{d}{d\tau}f=i\hbar\partial_u^2 f$, and set

$$f=s(\tau)e^{\frac{1}{\hbar}(\phi_1(\tau)u^2+\phi_2(\tau)v^2+2\phi_3(\tau)uv)}. \qquad (51)$$

Thus, we have

$$(\phi_1'(\tau),\phi_2'(\tau),\phi_3'(\tau);s'(\tau))=(4i\phi_1(\tau)^2,4i\phi_3(\tau)^2,4i\phi_1(\tau)\phi_3(\tau);2i\phi_1(\tau)s(\tau)).$$

The solution (51) with the initial data $(\phi_1(0),\phi_2(0),\phi_3(0);s(0))=(a,b,c;s)$ is obtained as follows:

$$(\phi_1(t),\phi_2(t),\phi_3(t);s(t))=(\frac{a}{1-4iat},\frac{b+4iDt}{1-4iat},\frac{c}{1-4iat};\frac{s}{\sqrt{1-4iat}}) \qquad (52)$$

where $D=c^2-ab$. Setting $t=-\frac{1}{2}$, we have the intertwiner

$$(a'',b'',c'';s'')=I_{\circ}^{\diamond}(a,b,c,s)=\frac{1}{1+2ia}(a,\ b-2iD,c;\ s\sqrt{1+2ia}). \qquad (53)$$

It is easy to see $D''=(c'')^2-a''b''=\frac{D}{1+2ia}$. The reversed relation of (53) is given by setting $t=\frac{1}{2}$:

$$(a,b,c;s)=(\frac{a''}{1-2ia''},\frac{b''+2iD''}{1-2ia''},\frac{c''}{1-2ia''};\frac{s''}{\sqrt{1-2ia''}}).$$

By a similar calculation, we have $e^{ti\hbar\partial_v^2}$: Consider the operator $\frac{d}{d\tau}f=i\hbar\partial_v^2 f$, and set f as in (51). The solution with the initial data $(\phi_1(0),\phi_2(0),\phi_3(0);s(0))=(a,b,c;s)$ is given by:

$$(\phi_1(t),\phi_2(t),\phi_3(t);s(t))=(\frac{a+4iDt}{1-4ibt},\frac{b}{1-4ibt},\frac{c}{1-4ibt};\frac{s}{\sqrt{1-4ibt}}) \qquad (54)$$

Then, we have the intertwiner from the normal ordering w.r.t. (u,v) to that w.r.t. $(u,u+v)$ is given by

$$I_{\circ}^{\bullet}(a,b,c;s) = \frac{1}{1-2ib}(a+2iD, b, c; s\sqrt{1-2ib}).$$

We now combine the above results. The general intertwiner is obtained by composing (50), (52), (54). For instance, the intertwiner $I_{\cdot}^{\diamond}$ from the Weyl ordering to the normal ordering w.r.t. $\frac{1}{\sqrt{2}}(u-v, u+v)$ is given by

$$I_{\cdot}^{\diamond}(a,b,c;s) = \frac{1}{1+i(b-a)-D}(a-iD, b+iD, c; \sqrt{1+i(b-a)-D}).$$

It is remarkable that intertwiners between exponential functions of quadratic forms contain always $\pm$ ambiguity on the amplitude.

The following shows that polar element is defined globally only as a two-valued element:

Proposition 23. *For a canonical conjugate pair $u' = au+bv, v' = cu+dv$ with $ad-bc=1$, then*

$$e^{\frac{i\hbar}{2}(-bd\partial_u^2+(ad+bc-1)\partial_u\partial_v-ac\partial_v^2)}\sqrt{-1}e_{\circ}^{\frac{2i}{\hbar}u\circ v} = \sqrt{-1}e_{\diamond}^{\frac{2i}{\hbar}u'\diamond v'}$$

where $\diamond$ means the normal ordering w.r.t. (u',v').

7. Gluing via intertwiners

In this section we first want to glue $\mathbb{C}^3\times\mathbb{C}_*$ and $\mathbb{C}^3\times\mathbb{C}_*$ together by the intertwiner $I_{\cdot}^{\circ}$, where $I_{\cdot}^{\circ}$ is the intertwiner from the Weyl ordering to the normal ordering w.r.t. (u,v). Let $I_{\circ}^{\cdot}$ be the inverse of $I_{\cdot}^{\circ}$. Let $P_{\cdot}^{\circ}$, $P_{\circ}^{\cdot}$ be the phase part of the intertwiners $I_{\cdot}^{\circ}$, $I_{\circ}^{\cdot}$;

$$P_{\cdot}^{\circ}(a,b,c) = \frac{1}{1-2ic-D}(a,b,c-iD), \quad P_{\circ}^{\cdot}(a',b',c') = \frac{1}{1+2ic'-D'}(a',b',c'+iD').$$

Recall that

$$I_{\cdot}^{\circ}(a,b,c;s) = \Big(P_{\cdot}^{\circ}(a,b,c); \frac{s}{\sqrt{1-2ic-D}}\Big).$$

By Proposition 22, it is not hard to see that $P_{\cdot}^{\circ}((V_\mu - \{c=-i\}) \subset V_\nu$ and $P_{\circ}^{\cdot}(V_\nu - \{4a'b'=-1\}) \subset V_\mu$. Then, the space of vacuums is preserved by the intertwiner.

To understand such gluing, we define $\Sigma = \{(x,y,z)\in\mathbb{C}^3 \,|\, z^2-xy=0\}$. Make a copy $\mathbb{C}^3-\Sigma'$ of $\mathbb{C}^3-\Sigma$, and consider a holomorphic diffeomorphism

$$T_0 : \mathbb{C}^3-\Sigma \to \mathbb{C}^3-\Sigma', \quad T_0(x,y,z) = -\frac{1}{z^2-xy}(x,y,z),$$

Gluing two $\mathbb{C}^3$ by T_0, we have a complex 3-dimensional manifold B^3. On the other hand, consider

$$\Delta : \mathbb{C}^3 \to \mathbb{C}, \quad \Delta(x,y,z) = z^2-xy. \tag{55}$$

Reminding the identity $\Delta T_0(x,y,z) = \frac{1}{\Delta(x,y,z)}$, we extends Δ naturally to the mapping of B^3 onto the riemann sphere $S^2 = \mathbb{C}\cup\{\infty\}$. We denote this mapping by the same notation $\Delta : B^3 \to S^2$.

We now consider the functions given by intertwiners:

$$f_\mu(a,b,c) = 1-2ic-D, \quad f_\nu(a',b',c') = 1+2ic'-D', \tag{56}$$

where $D = c^2-ab$, and $D' = (c')^2-a'b'$. Take new coordinate functions as follows:

$$(x,y,z) = (a,b,-(c+i)), \quad (x',y',z') = (a',b',c'-i).$$

We see easily that $\Delta(x,y,z) = -f_\mu(a,b,c)$, $\Delta(x',y',z') = -f_\nu(a',b',c')$. Hence, the manifold B^3 glued by T_0 is obtained also via the gluing diffeomorphism $T_{\mu\nu} : \mathbb{C}^3-\{f_\mu=0\} \to \mathbb{C}^3-\{f_\nu=0\}$:

$$(a',b',c') = T_{\mu\nu}(a,b,c) = \frac{1}{f_\mu(a,b,c)}(a,b,c-iD). \tag{57}$$

Indeed (57) is equivalent to

$$\big(a',b',c'-i\big) = -\frac{1}{(c{+}i)^2-ab}\big(a,b,-(c+i)\big).$$

Considering the path replacing i by $\tau i; \tau\in[0,1]$, we see the mapping $f_\mu\cup f_\nu : B^3 \to S^2$ is homotopic to Δ. Therefore, we must consider the gluing of $\mathbb{C}^3{\times}\mathbb{C}_*$ and $\mathbb{C}^3{\times}\mathbb{C}_*$ by $\tilde{T}_{\mu\nu}$, where

$$\tilde{T}_{\mu\nu}(a,b,c;s) = \big(\frac{1}{f_\mu}a, \frac{1}{f_\mu}b, \frac{1}{f_\mu}(c-iD); \frac{1}{\sqrt{f_\mu}}s\big). \tag{58}$$

As we look for the group-like object generated by exponential functions of quadratic forms, we want to glue $\mathbb{C}^3{-}V_\mu$ and $\mathbb{C}^3{-}V_\nu$ by $P^\circ_\cdot(a,b,c)$. We denote the glued manifold by $\tilde{B}^3 = B^3{-}\{\text{vacuums}\}$, where $\tilde{B}^3$ is simply connected. Through the adjoint mapping $\mathrm{Ad}(g)$, $\tilde{B}^3$ is diffeomorphic to $SL_{\mathbb{C}}(2)$. Hence, we must glue $(\mathbb{C}^3-V_\mu){\times}\mathbb{C}_*$ and $(\mathbb{C}^3-V_\nu){\times}\mathbb{C}_*$ by $\tilde{T}_{\mu\nu}$ given in (58). However, the gluing is impossible as a manifold. Otherwise, we must obtain a nontrivial double over of $SL_{\mathbb{C}}(2)$.

Thus, we need a little wider notion, which seems similar to *an object of a gerbe* of Giraud, or it may be better to say a flat unitary Diximier-Douady sheaf of groupoids (cf. [Br]). Since these mathematical lingo does not fit directly to our situation, we prefer to use other languages. These will be given in the next subsection.

7.1. Blurred $\mathbb{C}_*$-bundles.

We introduce a notion of *blurred* $\mathbb{C}_*$*-bundles* on S^2 as follows: For a simple open covering $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ of S^2, we give a system of holomorphic transition functions $t_{\alpha\beta} : U_\alpha\cap U_\beta \to \mathbb{C}_*$ such that $t_{\alpha\alpha}=1$, $t_{\alpha\beta} = t^{-1}_{\beta\alpha}$, but $t_{\alpha\beta}t_{\beta\gamma}t_{\gamma\alpha} \in \{e^{\frac{2\pi ik}{m}}; k\in\mathbb{Z}\}$ on $U_\alpha\cap U_\beta\cap U_\gamma \neq \emptyset$. $t_{\alpha\beta}$ is viewed as the gluing diffeomorphism

$$T_{\alpha\beta} : U_\beta{\times}\mathbb{C}_* \to U_\alpha{\times}\mathbb{C}_*, \quad T_{\alpha\beta}(p,z) = (p, t_{\alpha\beta}(p)z).$$

Set $t_{\alpha\beta}(p)=e^{2\pi i\lambda_{\alpha\beta}(p)}$, $\lambda_{\alpha\beta}(p)\in\mathbb{C}$, where $\lambda_{\alpha\beta}=-\lambda_{\beta\alpha}$, $\lambda_{\alpha\alpha}=0$.

For $U_\alpha\cap U_\beta\cap U_\gamma\neq\emptyset$, we set $\ell_{\alpha\beta\gamma}=(\delta\lambda)_{\alpha\beta\gamma}$ so that $t_{\alpha\beta}t_{\beta\gamma}t_{\gamma\alpha}=e^{2\pi i\ell_{\alpha\beta\gamma}}$. Then, $\ell=\{\ell_{\alpha\beta\gamma}\}$ is a Čech 2-cocycle over $\frac{1}{m}\mathbb{Z}$. Two such systems $\{\mathcal{U},\{t_{\alpha\beta}\}\}$, $\{\mathcal{U},\{\tilde{t}_{\alpha\beta}\}\}$ are said to be *equivalent*, if $\{\ell_{\alpha\beta\gamma}\}$ and $\{\tilde{\ell}_{\alpha\beta\gamma}\}$ defines the same cohomology class in $H^2(S^2,\frac{1}{m}\mathbb{Z})$. We call this equivalence class a *blurred* $\mathbb{C}_*$*-bundle* over S^2 and denote this by $\mathcal{M}_{S^2}^{(\frac{1}{m})}$. This notion seems a simple example of *gerbe* of Giraud (cf. [Br] §4 and §7).

Proposition 24. *If* $\{\mathcal{U},\{t_{\alpha\beta}\}\}$, $\{\mathcal{U},\{\tilde{t}_{\alpha\beta}\}\}$ *are equivalent, then* $\{t_{\alpha\beta}^m\}$ *and* $\{\tilde{t}_{\alpha\beta}^m\}$ *define the same* $\mathbb{C}_*$*-bundle.*

Proof. Suppose there is a 1-cochain $\{\xi_{\alpha\beta}\}\subset\frac{1}{m}\mathbb{Z}$ such that

$$\tilde{\ell}_{\alpha\beta\gamma}-\ell_{\alpha\beta\gamma}=\xi_{\alpha\beta}+\xi_{\beta\gamma}+\xi_{\gamma\alpha}.$$

Setting $\lambda_{\alpha\beta}-\tilde{\lambda}_{\alpha\beta}-\xi_{\alpha\beta}=M_{\alpha\beta}$ so that $\tilde{t}_{\alpha\beta}^{-1}t_{\alpha\beta}e^{-2\pi i\xi_{\alpha\beta}}=e^{2\pi iM_{\alpha\beta}}$, we see that $\{M_{\alpha\beta}\}$ is a Čech 1-cocycle over holomorphic functions $\mathcal{O}$. Note that $H^1(S^2,\mathcal{O})=\{0\}$. Thus, we set $M_{\alpha\beta}=\eta_\alpha-\eta_\beta$, where η_α,η_β are holomorphic functions on U_α,U_β respectively. The replacing $\tilde{t}_{\alpha\beta}$ by $e^{2\pi i\eta_\alpha}\tilde{t}_{\alpha\beta}e^{-2\pi i\eta_\beta}$ is a gauge transformation. Since

$$\lambda_{\alpha\beta}-\xi_{\alpha\beta}=\eta_\alpha+\tilde{\lambda}_{\alpha\beta}-\eta_\beta,$$

and $e^{2\pi m\xi_{\alpha\beta}}=1$, $\{\tilde{t}_{\alpha\beta}^m\}$ and $\{t_{\alpha\beta}^m\}$ define the same $\mathbb{C}_*$-bundle over S^2. □

Thus, if $H^2(N;\frac{1}{m}\mathbb{Z})=\{0\}$, then the restriction $\mathcal{M}_N^{(\frac{1}{m})}$ on a subset N of S^2 gives a genuine $\mathbb{C}_*$-bundle over N. We see that restrictions $\mathcal{M}_{S^2}^{(\frac{1}{m})}\big|_{\mathbb{C}}$ and $\mathcal{M}_{S^2}^{(\frac{1}{m})}\big|_{\mathbb{C}'}$ are trivial $\mathbb{C}_*$-bundles. We denote also by $\mathcal{M}_{S^2}^{(1)}$ the $\mathbb{C}_*$-bundle defined by using $\{t_{\alpha\beta}^m\}$ as transition functions. The projections $\pi:\mathcal{M}_{S^2}^{(\frac{1}{m})}\to S^2$, $\tilde{\pi}:\mathcal{M}_{S^2}^{(1)}\to S^2$ are well defined. $\mathcal{M}_{S^2}^{(\frac{1}{m})}$ is naturally viewed as a m-covering space of $\mathcal{M}_{S^2}^{(1)}$. We call $\mathcal{M}_{S^2}^{(\frac{1}{m})}$ the *blurred* m*-covering space* of $\mathcal{M}_{S^2}^{(1)}$.

Now, we define the blurred $\mathbb{C}_*$-bundle $\mathcal{M}_{S^2}^{(\frac{1}{2})}$ such that $\mathcal{M}_{S^2}^{(1)}$ is the tautological $\mathbb{C}_*$-bundle of $\mathbb{C}^2-\{0\}$, and consider the desired glued object as the *pull back* of $\mathcal{M}_{S^2}^{(\frac{1}{2})}$. Consider the pull-back $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{2})}$ of $\mathcal{M}_{S^2}^{(\frac{1}{2})}$ via the map $\Delta:\tilde{B}^3\to S^2$ given by (55). We remark that blurred bundles over B^3 are always considered as pull-back bundles. That is, we use only coverings of B^3 obtained by the pull back $\Delta'^{-1}\mathcal{U}$ by Δ' which is homotopically equivalent with Δ. Hence, this will be denoted by $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{2})}$.

7.2. Involutive distributions.

Let $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{2})}$ be a blurred $\mathbb{C}_*$-bundle over a manifold B^3. Though this forms neither a manifold nor has an underlying point set, several notions defined on manifolds work under the condition that these are invariant under the 2-to-2 local coordinate transformations.

The projection $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{2})}\to B^3$ is welldefined. The notion of distributions is also welldefined on $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{2})}$, and a notion of involutive distribution can be given. An involutive distribution is understood as a horizontal distribution of a flat connection on an object of gerbe.

If an involutive distribution is restricted on an open subset N where $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{m})}\big|_N$ is a genuine $\mathbb{C}_*$-bundle, then we can take an integral submanifold as a point set. By viewing S^2 as the riemann sphere $\mathbb{C}\cup\mathbb{C}'$ glued by $z\leftrightarrow z'=\frac{1}{z}$, restricted bundles $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{m})}\big|_{\Delta^{-1}\mathbb{C}}$, $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{m})}\big|_{\Delta^{-1}\mathbb{C}'}$, form genuine $\mathbb{C}^*$-bundles respectively.

Distributions $\mathcal{D}_\mu$ and $\mathcal{D}_\nu$ are glued together by $\tilde{T}_{\mu\nu}$, and gives a distribution on $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{2})}$. If one understands this distribution as a horizontal distribution of a connection, then the curvature of this connection is identically 0. In spite of this, integral submanifolds M^3 and N^3 given in §3.1, §3.2 can not be glued together as a manifold. How the union $M^3_*\cup N^3_*$ should be considered ? Apparently we have no way to explain such object directly in the set theoretical term. The only possible way is to give the whole collection of usage, or axioms of total applications. The notion of gerbes is the one of this direction.

¿From this point of view, we prefer the following explanation, because this is simple and intuitive: $M^3_*\cup N^3_*$ is the maximal "blurred integral submanifold" of $\mathcal{D}_\mu\cup\mathcal{D}_\nu$, glued together by a 2-to-2 local diffeomorphism. This looks like a non-trivial double cover of $SL_{\mathbb{C}}(2)$.

7.3. $*$-exponential mapping.

In §4, we showed in that the $*$-exponential mapping $\exp_*$ is a holomorphic mapping of $\mathbb{C}^3-\Pi_\mu$ into $M^3\subset(\mathbb{C}^3-V_\mu)\times\mathbb{C}_*$. Let Π_ν be the subset where $e_*^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$ is singular in the normal ordering w.r.t. (u,v). Then, $\exp_*:\mathbb{C}^3-\Pi_\nu\to(\mathbb{C}^3-V_\nu)\times\mathbb{C}_*$ is a holomorphic mapping, and $\exp_*(\mathbb{C}^3-\Pi_\nu)\subset N^3_*$. Since $\Pi_\mu\cap\Pi_\nu=\emptyset$, the $*$-exponential mapping is defined from $\mathbb{C}^3$ into the "space" $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{2})}$.

8. Concluding remarks

We saw that $\mathcal{D}_\mu\cup\mathcal{D}_\nu$ is viewed as a horizontal distribution of a connection defined on $\Delta^*\mathcal{M}_{S^2}^{(\frac{1}{2})}$. Since this is involutive, the curvature of this connection vanishes identically on $\tilde{B}^3$. However, it is natural for physicists to say that the curvature tensor is supported only on equilibrium states (cf. Corollary 7).

The maximal integral submanifold $M^3_*\cup(\varepsilon_{00}*M^3_*)$ contains the double covering group of $SL_{\mathbb{R}}(2)=Sp(2;\mathbb{R})$, which may be regarded as the metaplectic group $Mp(2;\mathbb{R})$. Thus, $M^3_*\cup(\varepsilon_{00}*M^3_*)$ may be viewed as a complexification of $Mp(2;\mathbb{R})$. It is obvious that there is no such Lie group in the genuine group theory. Moreover, $M^3_*\cup(\varepsilon_{00}*M^3_*)$ contains the double covering group of $SU(1,1)$ as another real form than $Mp(2;\mathbb{R})$.

As a matter of cause, $M^3\cup(\varepsilon_{00}*M^3)$ can not be recognized as a genuine object of mathematics based on the point set theory, since it is *not* a point set. In spite of this, we want to claim that such objects should be involved appropriately in the rigorous mathematics after relaxing the definition of manifolds.

Strange elements such as ε_{00} are not recognized as members of the set theory. However, for physicists such elements are easily acceptable, because they are computable. Physicists may have already used such elements heuristically in the calculus of Feynmann diagram. We might think that the connection between mathematics and physics is *not* so straight within the set theoretical mathematics.

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