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by

Sofiane Bouarroudj

Sofiane Bouarroudj
Department of Mathematics
Keio University

Department of Mathematics
Faculty of Science and Technology
Keio University

3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522 Japan

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Sofiane BOUARROUDJ*
Department of Mathematics, Keio University, Faculty of Science & Technology
3-14-1, Hiyoshi, Kohoku-ku, Yokohama, 223-8522, Japan.
e-mail:sofbou@math.keio.ac.jp

Abstract

Let (M, g) be a pseudo-Riemannian manifold. We propose a new approach for the conformal Schwarzian derivatives. These derivatives are 1-cocycles on the group of diffeomorphisms of M related to the modules of linear differential operators. As operators, these derivatives depend only on the conformal class of the metric g. In particular if the manifold (M, g) is conformally flat, these derivatives vanish on the conformal group $\mathrm{O}(p+1, q+1)$, where $\dim(M) = p+q$. This work is a continuation of [1, 3] where the Schwarzian derivative was defined on a manifold endowed with a projective connection.

1 Introduction

Let S^1 be the circle identified with the projective line $\mathbb{RP}^1$. For any diffeomorphism f of S^1, the following expression

$$S(f) := \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2, \tag{1.1}$$

where x is an affine parameter on S^1, is called Schwarzian derivative (see [5]).
The Schwarzian derivative has the following properties

(i) It defines a 1-cocycle on the group of diffeomorphisms $\mathrm{Diff}(S^1)$ with value in differential quadratics (cf. [12, 19]).

(ii) Its kernel is the group of projective transformations $\mathrm{PSL}_2(\mathbb{R})$.

The aim of this paper is to propose a new approach for the multi-dimensional conformal Schwarzian derivative. This approach has recently been used in [1, 3] to introduce the multi-dimensional "projective" Schwarzian derivative. The standpoint of our approach is the relation between the Schwarzian derivative (1.1) and the space of Sturm-Liouville operators (see e.g. [20]). The space of Sturm-Liouville operators is not isomorphic as $\mathrm{Diff}(S^1)$−module to the space of differential quadratics. More precisely, the space of Sturm-Liouville operators is a non-trivial deformation of the space of differential quadratics in the sense of Neijenhuis & Richardson's theory of deformation (see [17]), generating by the

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1-cocycle (1.1) (see [11] for more details). From this point of view, we believe that the multi-dimensional Schwarzian derivative is closely related to the modules of linear differential operators. To explicate our approach, let us introduce some notations.

Let M be a smooth manifold. We consider the space of linear differential operators with arguments are $\lambda-$densities on M and values are $\mu-$densities on M. We have, therefore, a two parameter family of $\mathrm{Diff}(M)-$modules noted $\mathcal{D}_{\lambda,\mu}(M)$. The corresponding space of symbols is the space of fiberwise polynomials on T^*M with values in $\delta-$densities, where $\delta = \mu - \lambda$. Denote this space by $\mathcal{S}_\delta(M)$. The space $\mathcal{D}_{\lambda,\mu}(M)$ is not isomorphic as $\mathrm{Diff}(M)-$module to the space $\mathcal{S}_\delta(M)$ (cf. [9, 15]). One can distinguishes two cases:

(i) If $M := \mathbb{R}^n$ is endowed with a flat projective structure (i.e. local action of the group $\mathrm{SL}_{n+1}(\mathbb{R})$ by linear fractional transformations) there exits an isomorphism between $\mathcal{D}_{\lambda,\mu}(\mathbb{R}^n)$ and $\mathcal{S}_\delta(\mathbb{R}^n)$, for δ generic, intertwining the action of $\mathrm{SL}_{n+1}(\mathbb{R})$ (cf. [15]). The multi-dimensional "projective" Schwarzian derivative was defined in [1, 3] as an obstruction to extend this isomorphism to the full group $\mathrm{Diff}(\mathbb{R}^n)$.

(ii) If $M := \mathbb{R}^n$ is endowed with a flat conformal structure (i.e. local action of the conformal group $\mathrm{O}(p+1, q+1)$, where $p + q = n$), there exists an isomorphism between $\mathcal{D}_{\lambda,\mu}(\mathbb{R}^n)$ and $\mathcal{S}_\delta(\mathbb{R}^n)$, for δ generic, intertwining the action of $\mathrm{O}(p+1, q+1)$ (cf. [8, 9]). In this paper we introduce the multi-dimensional "conformal" Schwarzian derivative in this context. Recall that in the one dimensional case these two notions coincide in the sense that the conformal Lie algebra $\mathrm{o}(2, 1)$ is isomorphic to the projective Lie algebra $\mathrm{sl}_2(\mathbb{R})$.

2 Differential operators and symbols

Let (M, g) be a pseudo-Riemannian manifold of dimension n. We denote by Γ the Levi-Civita connection associated to the metric g.

2.1 Space of linear differential operators as a module

Let $\mathcal{F}_\lambda(M)$, or $\mathcal{F}_\lambda$ for simplify, be the space of tensor densities on M. This space is nothing but the space of sections of the line bundle $(\wedge^n T^*M)^{\otimes\lambda}$. One can define naturally a $\mathrm{Diff}(M)$-module structure on it as follows: let $f \in \mathrm{Diff}(M)$ and $\phi \in \mathcal{F}_\lambda$. In a local coordinates (x^i), the action is given by

$$f^*\phi = \phi \circ f^{-1} \cdot (J_{f^{-1}})^\lambda, \tag{2.1}$$

where $J_f = |Df/Dx|$ is the Jacobian of f.

Differentiate the action above one can obtain the action of the Lie algebra of vector fields $\mathrm{Vect}(M)$.

Example 2.1 $\mathcal{F}_0 = C^\infty(M)$, $\mathcal{F}_1 = \Omega^n(M)$ (space of differential $n-$forms).

Let us recall the definition of a covariant derivative on densities. If $\phi \in \mathcal{F}_\lambda$, then $\nabla\phi \in \Omega^1(M) \otimes \mathcal{F}_\lambda$ given, in a local coordinates, by

$$\nabla_i\phi = \partial_i\phi - \lambda\Gamma_i\phi,$$

with $\Gamma_i = \Gamma^t_{ti}$. (Here and bellow summation is understood over repeated indices).

Consider now $\mathcal{D}_{\lambda,\mu}(M)$ the space of linear differential operators acting on tensor densities

$$A : \mathcal{F}_\lambda \to \mathcal{F}_\mu. \tag{2.2}$$

The action of Diff(M) on $\mathcal{D}_{\lambda,\mu}(M)$ depends on two parameters λ and μ. This action is given by the equation

$$f_{\lambda,\mu}(A) = f^* \circ A \circ f^{*-1}, \tag{2.3}$$

where f^* is the action (2.1) of Diff(M) on $\mathcal{F}_\lambda$.

Differentiate the action above one can obtain the action of the Lie algebra Vect(M).

The formulæ (2.1) and (2.3) do not depend on the choice of a system of coordinates.

Denote by $\mathcal{D}^2_{\lambda,\mu}(M)$ the space of second-order linear differential operators with the Diff$(M)-$module structure given by (2.3). The space $\mathcal{D}^2_{\lambda,\mu}(M)$ is in fact a Diff(M)-submodule of $\mathcal{D}_{\lambda,\mu}(M)$.

Example 2.2 The space of Sturm-Liouville operators on S^1: $\frac{d^2}{dx^2} + u(x) : \mathcal{F}_{-1/2} \to \mathcal{F}_{3/2}$, where $u(x) \in \mathcal{F}_2$ is the potential, is a submodule of $\mathcal{D}^2_{-\frac{1}{2},\frac{3}{2}}(S^1)$ (see [20]).

2.2 The module of symbols

The space of symbols, Pol(T^*M), is the space of functions on the cotangent bundle T^*M polynomial on the fibers. This space is naturally isomorphic to the space $\mathcal{S}(M)$ of symmetric contravariant tensor fields on M. In a local coordinates (x^i, ξ_i), one can write $P \in \mathcal{S}(M)$ in the form

$$P = \sum_{l \geq 0} P^{i_1,\ldots,i_l} \xi_{i_1} \cdots \xi_{i_l},$$

with $P^{i_1,\ldots,i_l}(x) \in C^\infty(M)$.

One defines a one parameter family of Diff$(M)-$module on the space of symbols by

$$\mathcal{S}_\delta(M) := \mathcal{S}(M) \otimes \mathcal{F}_\delta.$$

Let us explicate this action.

Take $f \in$ Diff(M) and $P \in \mathcal{S}_\delta(M)$. Then, in a local coordinates (x^i), one has

$$f_\delta(P) \ = \ f^*P \cdot (J_{f^{-1}})^\delta, \tag{2.4}$$

where $J_f = |Df/Dx|$ is the Jacobian of f, and f^* is the natural action of Diff(M) on $\mathcal{S}(M)$.

We have then a filtration of Diff$(M)-$module given by

$$\mathcal{S}_\delta(M) = \bigoplus_{k=0}^{\infty} \mathcal{S}^k_\delta(M),$$

where $S^k_\delta(M)$ is the space of contravariant tensor fields of degree k endowed with the Diff$(M)-$module structure (2.4).

We are interested to study the space of contravariant tensor fields of degree less than two noted $\mathcal{S}_{\delta,2}(M)$ (i.e. $\mathcal{S}_{\delta,2}(M) := \mathcal{S}^2_\delta(M) \oplus \mathcal{S}^1_\delta(M) \oplus \mathcal{S}^0_\delta(M)$).

2.3 Conformally equivariant quantization

The quantization procedure explained in this paper was first introduced in [9, 14]. By an equivariant quantization we mean an identification between the space of linear differential operators and the corresponding space of symbols, equivariant with respect to the action of a (finite) sub-group $G \subset \mathrm{Diff}(\mathbb{R}^n)$. Recall that in the one-dimensional case the equivariant quantization process was carried out for $G = \mathrm{SL}_2(\mathbb{R})$ in [7] (see also [11]).

Following [9], there exists, for $n > 2$, a quantization map

$$Q : \mathcal{S}_{\delta,2}(M) \to \mathcal{D}^2_{\lambda,\mu}(M),$$

given: for all $\delta \neq 1, \frac{2}{n}, \frac{n+2}{2n}, \frac{n+1}{n}, \frac{n+2}{n}$, and for each $P = P^{ij}\xi_i\xi_j + P^i\xi_i + P_0 \in \mathcal{S}_{\delta,2}(M)$, one associates a linear differential operator given by

$$\begin{aligned} Q(P) &= P^{ij}\nabla_i\nabla_j \\ &\quad +(\alpha_1\nabla_i P^{ij} + \alpha_2\, \mathrm{g}^{ij}\, \mathrm{g}_{kl}\nabla_i P^{kl} + P^j)\nabla_j \qquad (2.5)\\ &\quad +\alpha_3\nabla_i\nabla_j P^{ij} + \alpha_4\, \mathrm{g}^{st}\, \mathrm{g}_{ij}\nabla_s\nabla_t P^{ij} + \alpha_5\nabla_i P^i + \alpha_6 R_{ij}P^{ij} + \alpha_7 R\, \mathrm{g}_{ij}\, P^{ij} + P_0, \end{aligned}$$

where R_{ij} (resp. R) are the Ricci tensor components (resp. the scalar curvature) of the metric g, the constants $\alpha_1, \ldots, \alpha_7$ are given by

$$\alpha_1 = \frac{2(n\lambda+1)}{2+n(1-\delta)}, \qquad \alpha_5 = \frac{\lambda}{1-\delta},$$

$$\alpha_2 = \frac{n(\lambda+\mu-1)}{(2+n(1-\delta))(2-n\delta)}, \qquad \alpha_6 = \frac{n^2\lambda(\mu-1)}{(n-2)(1+n(1-\delta))},$$

$$\alpha_3 = \frac{n\lambda(n\lambda+1)}{(1+n(1-\delta))(2+n(1-\delta))}, \qquad \alpha_7 = \frac{(n\delta-2)}{(n-1)(2+n(1-2\delta))}\alpha_6,$$

$$\alpha_4 = \frac{n\lambda(n^2\mu(2-\lambda-\mu)+2(n\lambda+1)^2-n(n+1))}{(1+n(1-\delta))(2+n(1-\delta))(2+n(1-2\delta))(2-n\delta)}.$$

The quantization map (2.5) has the following properties
(i) It depends only on the conformal class of the metric g.
(ii) If $M = \mathbb{R}^n$ is endowed with a flat conformal structure the map (2.5) is unique, equivariant with respect to the action of the group $\mathrm{O}(p+1, q+1) \subset \mathrm{Diff}(\mathbb{R}^n)$.

Before to give the formula of the conformal equivariant map in the case of surfaces, let us recall an interesting approach for the multi-dimensional Schwarzian derivative for conformal mapping (see [6, 18]). First, recall that all surfaces are conformally flat. This means that every metric can be express (locally) as

$$\mathrm{g} = F^{-1}\psi^*\mathrm{g}_0,$$

where ψ is a conformal diffeomorphism of M, and F is a non-zero positive function, g_0 is a metric of constant curvature. The Schwarzian derivative of ψ is defined in [6, 18] as the following tensors fields

$$S(\psi) = \frac{1}{2F}\nabla dF - \frac{3}{4F^2}\, dF \otimes dF + \frac{1}{8F^2}\mathrm{g}^{-1}(dF, dF)\, \mathrm{g}. \qquad (2.6)$$

Now we are in position to give the quantization map for the case of surfaces.

For $\delta \neq 1, 2, \frac{3}{2}, \frac{5}{2}$, and for each $P = P^{ij}\xi_i\xi_j + P^i\xi_i + P_0 \in \mathcal{S}_{\delta,2}(M)$ one associates a linear differential operator given by

$$\begin{aligned} Q(P) &= P^{ij}\nabla_i\nabla_j \\ &+(\alpha_1\nabla_i P^{ij} + \alpha_2 \mathrm{g}^{ij}\,\mathrm{g}_{kl}\nabla_i P^{kl} + P^j)\nabla_j \\ &+\alpha_3\nabla_i\nabla_j P^{ij} + \alpha_4\,\mathrm{g}^{st}\,\mathrm{g}_{ij}\nabla_s\nabla_t P^{ij} + \alpha_5\nabla_i P^i \\ &+\frac{4\lambda(\mu-1)}{2\delta-3}\left(S(\psi)_{ij}P^{ij} + \frac{1}{8(\delta-1)}R\,\mathrm{g}_{ij}\,P^{ij}\right) + P_0, \end{aligned} \tag{2.7}$$

where $S(\psi)$ is the tensor (2.6), R is the scalar curvature, the coefficients $\alpha_1, \ldots, \alpha_5$ are given as above.

Remark 2.3 The projectively equivariant quantization map was given in [1, 15]. The multi-dimensional projective Schwarzian derivative is defined as an obstruction to extend this isomorphism to the full group $\mathrm{Diff}(M)$. We will show in section 4 that the conformal Schwarzian derivatives defined in this paper appear as obstructions to extend the isomorphism (2.5), (2.7) to the full group $\mathrm{Diff}(M)$.

2.4 Remark on the cohomology of Vect(M)

The space $\mathcal{D}^2_{\lambda,\mu}(M)$ can be viewed as a non-trivial deformation of the module $\mathcal{S}_{2,\delta}(M)$ in the sense of Neijenhuis & Richardson's theory of deformation (cf. [9, 14]). According to the theory of deformation, the problem of "infinitesimal" deformation is related to the cohomology group

$$\mathrm{H}^1(\mathrm{Vect}(M), \mathrm{End}(\mathcal{S}_{2,\delta}(M))\cdot \tag{2.8}$$

To compute the cohomology group (2.8) we restrict coefficients to the space of linear differential operators on $\mathcal{S}_{2,\delta}(M)$, noted $\mathcal{D}(\mathcal{S}_{2,\delta}(M))$. This space is decomposed, as a $\mathrm{Vect}(M)$-module, into direct sum

$$\mathcal{D}(\mathcal{S}_{2,\delta}(M)) = \bigoplus_{k,m=0}^{2} \mathcal{D}(\mathcal{S}^k_\delta(M), \mathcal{S}^m_\delta(M)),$$

where $\mathcal{D}(\mathcal{S}^k_\delta(M), \mathcal{S}^m_\delta(M)) \subset \mathrm{Hom}(\mathcal{S}^k_\delta(M), \mathcal{S}^m_\delta(M))$.

The relation between the Schwarzian derivative (1.1) and the cohomology group above is as follows: recall that in the one dimensional case the space $\mathcal{S}^k_\delta(S^1)$ is nothing but $\mathcal{F}_{\delta-k}$. In this case, the problem of deformation with respect to the Lie algebra $\mathrm{sl}_2(\mathbb{R})$ is related to the cohomology group

$$\mathrm{H}^1(\mathrm{Vect}(S^1), \mathrm{sl}_2(\mathbb{R}); \mathcal{D}(\mathcal{F}_{\delta-k}, \mathcal{F}_{\delta-l})), \tag{2.9}$$

where $k, l = 0, 1, 2$. The cohomology group (2.9) was calculated in [4], it is one dimension for $k = 2$, $l = 0$, and zero otherwise. The (unique) non-trivial class, for $k = 2$, $l = 0$, can be integrated to the group of diffeomorphisms $\mathrm{Diff}(S^1)$; it is a zero-order operator given as a multiplication by the Schwarzian derivative (1.1) (see [4] for more details).

In the multi-dimensional case, and for $\delta = 0$, the first group of differential cohomology of $\mathrm{Vect}(M)$, with coefficients in the space $\mathcal{D}(\mathcal{S}^k(M), \mathcal{S}^m(M))$ of linear differential operators from $\mathcal{S}^k(M)$ to $\mathcal{S}^m(M)$ was calculated in [15]. For $n \geq 2$ the result is as follows

$$\mathrm{H}^1(\mathrm{Vect}(M), \mathcal{D}(\mathcal{S}^k(M), \mathcal{S}^m(M))) = \begin{cases} \mathbb{R} \oplus \mathrm{H}^1_{\mathrm{DR}}(M), & k - m = 0, \\ \mathbb{R}, & k - m = 1, m \neq 0, \\ \mathbb{R}, & k - m = 2, \\ 0, & \text{otherwise}. \end{cases} \tag{2.10}$$

We believe that the infinitesimal multi-dimensional Schwarzian derivative is a cohomology class in the cohomology group (2.10).

3 Conformal Schwarzian derivatives

Let (M, g) be a pseudo-Riemannian manifold. Denote by Γ the Levi-Civita connection associated to the metric g.

3.1 Explicit formulæ for 1-cocycles on $\mathrm{Diff}(M)$

It is well known that the difference between two connections is a well defined tensor fields of type $(2, 1)$. It follows, therefore, that the difference

$$\ell(f) := f^*\Gamma - \Gamma, \tag{3.1}$$

is a well defined $(2, 1)-$tensor fields on M.

It is easy to see that the map $f \to \ell(f^{-1})$ defines a *non-trivial* 1-cocycle on $\mathrm{Diff}(M)$ with value in tensor fields on M of type $(2, 1)$.

For the sake of brevity, let us denote $\hat{\mathrm{g}} := f^{*-1}\mathrm{g}$.

The expression

$$\begin{aligned} \mathcal{A}(f)^k_{ij} &= \left(\hat{\mathrm{g}}^{sk}\, \hat{\mathrm{g}}_{ij} - \mathrm{g}^{sk}\, \mathrm{g}_{ij}\right) \nabla_s + (2 - \delta n) \left(\ell(f)^k_{ij} - \frac{1}{n}\, Sym_{i,j}\, \delta^k_i\, \ell(f)_j\right) \\ &\quad + \hat{\mathrm{g}}^{tk} \left(Sym_{i,j}\, \hat{\mathrm{g}}_{si}\, \ell(f)^s_{jt} - \delta\, \hat{\mathrm{g}}_{ij}\, \ell(f)_t\right), \end{aligned} \tag{3.2}$$

where $\ell(f)^k_{ij}$ are the components of the tensor (3.1), is a linear differential operator from $\mathcal{S}^2_\delta(M)$ to $\mathcal{S}^1_\delta(M)$.

Theorem 3.1 *(i) For all $\delta \neq 2/n$, the map $f \to \mathcal{A}^k_{ij}(f^{-1})$ defines a non-trivial 1-cocycle on* $\mathrm{Diff}(M)$ *with value in* $\mathcal{D}(\mathcal{S}^2_\delta(M), \mathcal{S}^1_\delta(M))$.
(ii) The operator (3.2) depends only on the conformal class of the metric g. *In particular, if $M := \mathbb{R}^n$ is endowed with a flat conformal structure of signature $p - q$, this operator vanishes on the conformal group* $\mathrm{O}(p + 1, q + 1)$.

Proof. To prove (i) we have to check the 1-cocycle condition

$$\mathcal{A}(f \circ h) = h^*\mathcal{A}(f) + \mathcal{A}(h),$$

for all $f, h \in \mathrm{Diff}(M)$, and h^* is the natural action on $\mathcal{D}(\mathcal{S}_\delta^2(M), \mathcal{S}_\delta^1(M))$.
This condition can be checked using the following formula

$$\nabla_i f_\delta^* P^{kl} = f_\delta^* \nabla_i P^{kl} - Sym_{k,l}\left(\ell(f^{-1})_{it}^k f_\delta^* P^{tl}\right) + \delta \ell(f^{-1})_i f_\delta^* P^{kl}, \tag{3.3}$$

for all $f \in \mathrm{Diff}(M)$, and for all $P^{kl} \in \mathcal{S}_\delta^2(M)$.
Let us proof that, for $\delta \neq 2/n$, this 1-cocycle is not trivial. Suppose that there is a first-order differential operator $A = u_{ij}^{sk}\nabla_s + v_{ij}^k$ such that

$$\mathcal{A}(f) = f^{*-1}A - A. \tag{3.4}$$

From (3.4), it is easy to see that $f^{*-1}v_{ij}^k - v_{ij}^k = (2-\delta n)\left(\ell(f)_{ij}^k - \frac{1}{n}\,Sym_{i,j}\,\delta_i^k\,\ell(f)_j\right)$. The right side of the formula above depends on the second jet of the diffeomorphism f while the left side depends on the first jet of f which is absurd.

For $\delta = 2/n$, the 1-cocycle (3.2) turns out to be trivial. Indeed, $\mathcal{A}(f) = f^{*-1}A - A$, where $A := \mathrm{g}^{sk}\,\mathrm{g}_{ij}\nabla_s$.

To prove (ii) suppose that there is a metric $\tilde{\mathrm{g}}$ conformally equivalent to g. Denote by $\tilde{\mathcal{A}}(f)$ the operator (3.2) written with the metric $\tilde{\mathrm{g}}$. We have to prove that $\tilde{\mathcal{A}}(f) = \mathcal{A}(f)$. Since the metrics g and $\tilde{\mathrm{g}}$ are conformally equivalent, there exists a non-zero positive function F such that $\tilde{\mathrm{g}} = F \cdot \mathrm{g}$. The Levi-Civita connections associated to the previous metrics are related by

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \frac{1}{2F}\left(F_i\delta_j^k + F_j\delta_i^k - F_t\,\mathrm{g}^{tk}\mathrm{g}_{ij}\right), \tag{3.5}$$

where $F_i = \partial_i F$.
Now the proof is a simple computation. ■

The explicit formula for the 1-cocycle integrating to the group, for $k = 2, m = 0$, the cohomology class of (2.10) depends on the dimension of the manifold M. Let us start with $n > 2$.

For all $f \in \mathrm{Diff}(M)$, the following

$$\begin{aligned}
\mathcal{B}(f)_{ij} \;=\;& \left(\hat{\mathrm{g}}^{st}\,\hat{\mathrm{g}}_{ij} - \mathrm{g}^{st}\,\mathrm{g}_{ij}\right)\nabla_s\nabla_t + (2+n(1-2\delta))\left(\ell(f)_{ij}^s - \frac{1}{n}\,Sym_{i,j}\,\delta_i^s\,\ell(f)_j\right)\nabla_s \\
& -2\delta\,\hat{\mathrm{g}}_{ij}\,\hat{\mathrm{g}}^{st}\,\ell(f)_t\nabla_s - \hat{\mathrm{g}}_{ij}\,\hat{\mathrm{g}}^{st}\,\ell(f)_{st}^u\nabla_u + 2\,Sym_{i,j}\,\hat{\mathrm{g}}_{ui}\,\hat{\mathrm{g}}^{st}\ell(f)_{jt}^u\nabla_s \\
& +(2+n(1-2\delta))\left(\frac{(\delta-1)}{n}\ell(f)_i\,\ell(f)_j - \frac{n(\delta+n-\delta n)-2}{n(n-2)}\,Sym_{i,j}\nabla_i\ell_j(f)\right. \\
& \left. +\frac{2(1-\delta)(n-1)}{n-2}\ell(f)_{ij}^s\ell(f)_s + \frac{(-2+n(2-\delta))}{n-2}\nabla_s\ell(f)_{ij}^s - \frac{n(1-\delta)}{n-2}\ell(f)_{si}^u\ell(f)_{uj}^s\right) \\
& +\hat{\mathrm{g}}^{st}\,Sym_{i,j}\left(\hat{\mathrm{g}}_{ki}\left(\ell(f)_{su}^k\ell(f)_{jt}^u - \ell(f)_{st}^u\ell(f)_{uj}^k - 2\delta\ell(f)_{sj}^k\ell(f)_t\right) + \hat{\mathrm{g}}_{ui}\nabla_s\ell(f)_{tj}^u\right) \\
& +\delta\,\hat{\mathrm{g}}^{st}\,\hat{\mathrm{g}}_{ij}\left(\ell(f)_{st}^u\ell(f)_u + \delta\ell(f)_s\ell(f)_t - \nabla_s\ell(f)_t\right) + 2\hat{\mathrm{g}}^{st}\,\hat{\mathrm{g}}_{kl}\,\ell(f)_{is}^k\ell(f)_{jt}^l \\
& +\frac{n(\delta-1)(n\delta-2)}{(n-1)}\left(f^{-1^*}(R\,\mathrm{g}_{ij}) - R\,\mathrm{g}_{ij}\right),
\end{aligned} \tag{3.6}$$

where $\ell(f)$ is the tensor (3.1), R is the scalar curvature of the metric g, is a differential operator from $\mathcal{S}^2_\delta(M)$ to $\mathcal{S}^0_\delta(M)$.

Theorem 3.2 *(i) For all $\delta \neq \frac{n+2}{2n}$, the map $f \mapsto \mathcal{B}(f^{-1})$ defines a non-trivial 1-cocycle on* $\mathrm{Diff}(M)$ *with values in $\mathcal{D}(\mathcal{S}^2_\delta(M), \mathcal{S}^0_\delta(M))$.*

(ii) The operator (3.6) depends only on the conformal class of the metric g. *In the flat case, this operator vanishes on the conformal group* $\mathrm{O}(p+1, q+1)$.

Proof. To prove that the map $f \mapsto \mathcal{B}(f^{-1})$ is a 1-cocycle one has to check the 1-cocycle condition

$$\mathcal{B}(f \circ h) = h^* \mathcal{B}(f) + \mathcal{B}(h), \tag{3.7}$$

for all $f, h \in \mathrm{Diff}(M)$, and h^* is the natural action on $\mathcal{D}(\mathcal{S}^2_\delta(M), \mathcal{S}^0_\delta(M))$. To check the formula (3.7) we use the following formulæ

$$\begin{aligned}
\nabla_i \nabla_j \, f_\delta^{*-1} P^{kl} &= f_\delta^{*-1} \nabla_i \nabla_j \, P^{kl} - Sym_{l,k} \left(f_\delta^{*-1} \nabla_i \, P^{tl} \, \ell(f)^k_{tj} \right) + f_\delta^{*-1} \nabla_u \, P^{kl} \, \ell(f)^u_{ij} \\
&\quad + \delta \left(f_\delta^{*-1} \nabla_i \, P^{kl} \, \ell(f)_j \right) - Sym_{k,l} \left(\nabla_j \ell(f)^k_{it} \, f_\delta^{*-1} P^{tl} \right) \qquad (3.8) \\
&\quad + \delta \nabla_j \ell(f)_i \, f_\delta^{*-1} P^{kl} - Sym_{k,l} \left(\ell(f)^k_{it} \nabla_j \, f_\delta^{*-1} P^{tl} \right) + \delta \ell(f)_i \, \nabla_j f_\delta^{*-1} P^{kl} \\
\nabla_u h^* \ell(f)^c_{ab} &= h^* \nabla_u \ell(f)^c_{ab} - h^* \ell(f)^t_{ab} \, \ell(h^{-1})^c_{ut} + h^* \ell(f)^c_{at} \, \ell(h^{-1})^t_{bu} + h^* \ell(f)^c_{bt} \, \ell(h^{-1})^t_{au}
\end{aligned}$$

for all $f, h \in \mathrm{Diff}(M)$, and for all $P^{kl} \in \mathcal{S}^2_\delta(M)$.
Let us prove that this 1-cocycle is not trivial. Suppose that there exists an operator $B := u^{st}_{ij} \nabla_s \nabla_t + v^s_{ij} \nabla_s + t_{ij}$ such that

$$\mathcal{B}(f) = f^{*-1} B - B. \tag{3.9}$$

It is easy to see that $f^{*-1} v^s_{ij} - v^s_{ij} = (2 + n(1-2\delta)) \left(\ell(f)^s_{ij} - \frac{1}{n} Sym_{i,j} \, \delta^s_i \, \ell(f)_j \right)$. The right side of the formula above depends on the second jet of f while the the left side depends on the first jet which is absurd.

For $\delta = \frac{n+2}{2n}$, the 1-cocycle (3.6) turns out to be trivial. Indeed, $\mathcal{B}(f) = f^* B - B$, where $B := \mathrm{g}^{st} \, \mathrm{g}_{ij} \nabla_s \nabla_t - \frac{1}{4} \frac{(n-2)^2}{n-1} R \, \mathrm{g}_{ij}$.

To proof (ii) we use the formula (3.5).

■

3.2 Comparison with the projective case

Let M be a manifold of dimension n. Fix a symmetric affine connection Γ on M (here Γ is any connection not necessarily a Levi-Civita one). Let us recall the notion of projective connection (see [13]).

A projective connection is an equivalent class of symmetric affine connections giving the same unparameterized geodesics.

Following [13], the symbol of the projective connection is given by the expression

$$\Pi^k_{ij} = \Gamma^k_{ij} - \frac{1}{n+1} \left(\delta^k_i \Gamma_j + \delta^k_j \Gamma_i \right), \tag{3.10}$$

where Γ^k_{ij} are the Christoffel symbols of the connection Γ and $\Gamma_i = \Gamma^j_{ij}$.
Two affine connection Γ and $\tilde{\Gamma}$ are projectively equivalent if the corresponding symbols (3.10) coincide.
A projective connection on M is called *flat* if in a neighborhood of each point there exists a local coordinate system $(x^1, \ldots, x^n)$ such that the symbols Π^k_{ij} are identically zero (see [13] for a geometric definition). Every flat projective connection defines a projective structure on M.
Let Π and $\tilde{\Pi}$ be two projective connections on M. Then the difference $\Pi - \tilde{\Pi}$ is a well-defined $(2,1)-$tensor fields. Therefore, it is clear that a projective connection on M leads to the following 1-cocycle on $\mathrm{Diff}(M)$:

$$\mathcal{C}(f^{-1}) = \left((f^{-1})^*\Pi^k_{ij} - \Pi^k_{ij}\right) dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k} \tag{3.11}$$

This formula is independent on the choice of the coordinate system.

By definition, the tensor fields (3.11) depends only on the projective class of the connection M. In particular if $\Pi \equiv 0$, this tensor fields vanishes on the projective group $\mathrm{PSL}_{n+1}(\mathbb{R})$.

One can define a 1-cocycle on $\mathrm{Diff}(M)$ with value in $\mathcal{D}(\mathcal{S}^2_\delta(M), \mathcal{S}^1_\delta(M))$ by contracting any symmetric contravariant tensor fields with the tensor (3.11). Therefore, the operator (3.2) can be viewed as the conformal analogue of the tensor fields (3.11). In the same spirit, the operator (3.6) can be viewed as the conformal analogue of the "projective" multi-dimensional Schwarzian derivative introduced in [1, 3].

4 Relation to the modules of differential operators

The goal of this section is to explicate the relation between the 1-cocycles (3.2), (3.6) and the space of second-order linear differential operators $\mathcal{D}^2_{\lambda,\mu}(M)$. Since the space $\mathcal{D}^2_{\lambda,\mu}(M)$ is a non-trivial deformation of the space of the corresponding space of symbols $\mathcal{S}_{\delta,2}(M)$, where $\delta = \mu - \lambda$, it is interesting to give explicitly this deformation in term of the 1-cocycles (3.2), (3.6). Namely, we are looking for the operator $\bar{f}_\delta = Q^{-1} \circ f_{\lambda,\mu} \circ Q$ such that the diagram below is commutative

$$\begin{array}{ccc} \mathcal{S}_{\delta,2}(M) & \xrightarrow{\bar{f}_\delta} & \mathcal{S}_{\delta,2}(M) \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathcal{D}^2_{\lambda,\mu}(M) & \xrightarrow{f_{\lambda,\mu}} & \mathcal{D}^2_{\lambda,\mu}(M) \end{array} \tag{4.1}$$

Proposition 4.1 *If* $\dim M \geq 2$, *for all* $\delta \neq \frac{2}{n}, \frac{n+2}{2n}, 1, \frac{n+1}{n}, \frac{n+2}{n}$, *the deformation of the space of symbols* $\mathcal{S}_{\delta,2}(M)$ *by the space* $\mathcal{D}^2_{\lambda,\mu}(M)$ *as* $\mathrm{Diff}(M)-$*module is given as follows: for all* $P = P^{ij}\xi_i\xi_j + P^i\xi_i + P_0 \in \mathcal{S}_{\delta,2}(M)$, *we have*

$$\bar{f}_\delta \cdot (P^{ij}\xi_i\xi_j + P^i\xi_i + P^0) = T^{ij}\xi_i\xi_j + T^i\xi_i + T^0,$$

where

$$\begin{aligned} T^{ij} &= (f_\delta\, P)^{ij}, \\ T^i &= (f_\delta\, P)^i + \frac{n(\mu+\lambda-1)}{(2+n(1-\delta))(2-n\delta)}\, \mathcal{A}^i_{kl}(f^{-1})(f_\delta\, P)^{kl}, \\ T^0 &= (f_\delta\, P)_0 - \frac{n\lambda(\mu-1)}{(2+n(1-2\delta))(1-\delta)(1+n(1-\delta))}\, \mathcal{B}_{kl}(f^{-1})(f_\delta\, P)^{kl}, \end{aligned} \tag{4.2}$$

and f_δ *is the action (2.4).*

Proof. The proof is a simple computation using (3.3), (3.8) and the following formulæ

$$\begin{aligned} \nabla_i\, f^*\phi &= f^*\nabla_i\phi + \lambda\, \ell(f^{-1})_i\, f^*\phi \\ \nabla_j\nabla_i\, f^*\phi &= f^*\nabla_j\nabla_i\phi + \ell(f^{-1})^t_{ji}\, f^*\nabla_t\phi + \lambda Sym_{i,j}\, \ell(f^{-1})_j\, f^*\nabla_i\phi \\ &\quad + \left(\lambda\nabla_j\, \ell(f^{-1})_i + \lambda^2\ell(f^{-1})_j\, \ell(f^{-1})_i\right)\, f^*\phi \\ f^*R_{jk} - R_{jk} &= \nabla_j\, \ell(f^{-1})_k - \nabla_i\, \ell(f^{-1})^i_{jk} + \ell(f^{-1})^m_{sj}\, \ell(f^{-1})^s_{km} - \ell(f^{-1})_m\, \ell(f^{-1})^m_{jk}, \end{aligned}$$

for all $\phi \in \mathcal{F}_\lambda$, and for all $f \in \mathrm{Diff}(M)$, where R_{ij} are the Ricci tensor components.

■

5 Appendix

We will give a formula for the Schwarzian derivative for the case of surfaces. As explained in section (2.3), all surfaces are conformally flat. That means that every metric can be express (locally) as

$$\mathrm{g} = F^{-1}\psi^*\mathrm{g}_0,$$

where ψ is a conformal diffeomorphism of M, and F is a non-zero positive function, g_0 is a metric of constant curvature.

The explicit formula of the Schwarzian derivative in the case of surfaces is as follows: the following

$$\begin{aligned} \mathcal{B}'_2(f)_{ij} &= \left(\hat{\mathrm{g}}^{st}\,\hat{\mathrm{g}}_{ij} - \mathrm{g}^{st}\,\mathrm{g}_{ij}\right)\nabla_s\nabla_t + 4(1-\delta)\left(\ell(f)^s_{ij} - \frac{1}{2}Sym_{i,j}\,\delta^s_i\,\ell(f)_j\right)\nabla_s \\ &\quad + \hat{\mathrm{g}}^{st}\left(-2\delta\hat{\mathrm{g}}_{ij}\,\ell(f)_t\nabla_s - \hat{\mathrm{g}}_{ij}\,\ell(f)^u_{st}\nabla_u + 2\,Sym_{i,j}\,\hat{\mathrm{g}}_{ui}\,\ell(f)^u_{jt}\nabla_s\right) \\ &\quad + 4(1-\delta)\left(\frac{\delta-1}{2}\ell_i(f)\ell(f)_j - \frac{2-\delta}{2}\,Sym_{ij}\,\nabla_j\ell_i(f) + (1-\delta)\ell(f)^s_{ij}\ell(f)_s + \nabla_s\ell(f)^s_{ij}\right) \\ &\quad + \hat{\mathrm{g}}^{st}\,Sym_{i,j}\left(\hat{\mathrm{g}}_{ki}\left(\ell(f)^k_{su}\ell(f)^u_{jt} - \ell(f)^u_{st}\ell(f)^k_{uj} - 2\delta\ell(f)^k_{sj}\ell(f)_t\right) + \hat{\mathrm{g}}_{ui}\nabla_s\ell(f)^u_{tj}\right) \\ &\quad + \delta\,\hat{\mathrm{g}}^{st}\,\hat{\mathrm{g}}_{ij}\left(\ell(f)^u_{st}\ell(f)_u + \delta\ell(f)_s\ell(f)_t - \nabla_s\ell(f)_t\right) + 2\hat{\mathrm{g}}^{st}\,\hat{\mathrm{g}}_{kl}\,\ell(f)^k_{is}\ell(f)^l_{jt} \\ &\quad + (\delta-1)\left(f^{-1*}(S(\phi)_{ij}) - S(\phi)_{ij}\right) + \frac{1}{8}\left(f^{-1*}(R\,\mathrm{g}_{ij}) - R\,\mathrm{g}_{ij}\right), \end{aligned}$$

where $S(\psi)$ is the derivative (2.6), $\ell(f)$ is the tensor (3.1), R is the scalar curvature of the metric g, is a differential operator from from $\mathcal{S}^2_\delta(M)$ to $\mathcal{S}^0_\delta(M)$.

Theorem (3.2) remains true for this operator.

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