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Cohomology of groups of diffeomorphisms related to the modules of differential operators on a smooth manifolds

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Cohomology of groups of diffeomorphisms related to the modules of differential operators on a smooth manifold

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Abstract

Let M be a manifold and T^*M be the cotangent bundle. We introduce 1-cocycle on the group of diffeomorphisms of M with value in the space of linear differential operators acting on $C^\infty(T^*M)$. We use this 1-cocycle to exhibit 1-cocycles generating the first-cohomology group of the group of diffeomorphisms of M with coefficients in the space of linear differential operators acting on contravariant tensor fields on M.

1 Introduction

The origin of our investigation is the relationship between the space of linear differential operators acting on densities and the corresponding space of symbols, both viewed as modules over the group of diffeomorphisms and the Lie algebra of smooth vector fields. These two spaces are obviously isomorphic as vector spaces. However, these spaces are not isomorphic as modules (cf. [1, 5, 6, 8, 9]). More precisely, the module of linear differential operators is a non-trivial deformation of the module of symbols in the sense of Neijinhuis & Richardson's theory of deformation (see [10]). This theory of deformation can be summerized briefly here as follows: given a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}-$module V. Deformation of the $\mathfrak{g}-$module V means extend the action of the Lie algebra $\mathfrak{g}$ on the formal power series $V[[t]]$, where t is a parameter. The problem of deformation is related to the cohomology groups $\mathrm{H}^1(\mathfrak{g}, \mathrm{End}(V))$ and $\mathrm{H}^2(\mathfrak{g}, \mathrm{End}(V))$. The first cohomology group classifies all *infinitesimal* deformation of the module V, while the second cohomology group measures the obstruction to extend this infinitesimal deformation to a formal deformation.

In this paper we deal with the space of symbols $\mathcal{S}$; they are functions on the cotangent bundle which are polynomial on the fibers. This space is naturally a module over the Lie algebra of smooth vector fields $\mathrm{Vect}(M)$ and the group of diffeomorphisms $\mathrm{Diff}(M)$. Since the module of linear differential operators is a deformation of the space of symbols, it is interesting to exhibit the 1-cocycles generating the infinitesimal deformation. For the Lie algebra $\mathrm{Vect}(M)$, Lecomte and Ovsienko [9] have compute the cohomology group $\mathrm{H}^1(\mathrm{Vect}(M), \mathcal{D}(\mathcal{S}))$, where $\mathcal{D}(\mathcal{S})$ is the space of linear differential operators acting on $\mathcal{S}$. In this paper we shall study the counterpart cohomology of the group of diffeomorphisms

$$\mathrm{H}^1(\mathrm{Diff}(M), \mathcal{D}(\mathcal{S})). \tag{1.1}$$

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First, we introduce a non-trivial 1-cocycle on the group of diffeomorphisms $\mathrm{Diff}(M)$ with value in $\mathcal{D}(C^\infty(T^*M))$. Using this 1-cocycle, we compute and exhibit explicit formulæ for 1-cocycles generating the cohomology group (1.1).

2 Introducing a 1-cocycle on $\mathrm{Diff}(M)$

Let M be a manifold of dimension n. Fix a symmetric affine connection on it. Let us recall here the natural way to lift a connection on the cotangent bundle T^*M (see [15] for more details).

2.1 Lift of a connection

Denote by Γ^k_{ij}, for $i, j, k = 1, \dots, n$ the Christoffel symbols of the connection on M. There exists a symmetric affine connection on T^*M whose Christoffel symbols $\tilde{\Gamma}^{\mathbf{k}}_{\mathbf{ij}}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k} = 1, \dots, 2n$, given as follows: denote by i the superscript $\mathbf{i}$ if it varies from 1 to n, and by $\bar{i}$ if it varies from $n+1$ to $2n$. In a local coordinates (x^i, ξ_i) on T^*M, the Christoffel symbols of the lifted connection are given as follows

$$\begin{aligned} &\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij}, \quad \tilde{\Gamma}^k_{i\bar{j}} = 0, \quad \tilde{\Gamma}^k_{\bar{i}j} = 0, \quad \tilde{\Gamma}^k_{\bar{i}\bar{j}} = 0, \\ &\tilde{\Gamma}^{\bar{k}}_{ij} = \xi_a(\partial_k \Gamma^a_{ij} - \partial_i \Gamma^a_{jk} - \partial_j \Gamma^a_{ik} + 2\Gamma^a_{kt}\Gamma^t_{ij}), \\ &\tilde{\Gamma}^{\bar{k}}_{i\bar{j}} = -\Gamma^j_{ik}, \quad \tilde{\Gamma}^{\bar{k}}_{\bar{i}j} = -\Gamma^i_{kj}, \quad \Gamma^{\bar{k}}_{\bar{i}\bar{j}} = 0. \end{aligned} \tag{2.1}$$

Remark 2.1 *The connection $\tilde{\Gamma}$ is actually the Levi-Civita connection of some metric on T^*M (see [15]).*

2.2 Main definition

First we shall construct canonically a tensor fields on T^*M.

It is well known that the difference between two connections is a well defined tensor fields of type $(2,1)$. It follows from this fact that the following expression

$$C(F) := F^*(\tilde{\Gamma}) - \tilde{\Gamma}, \tag{2.2}$$

where F is a diffeomorphism on T^*M, $\tilde{\Gamma}$ is the connection above, is a well defined tensor fields on T^*M of type $(2,1)$. It is easy to see that the map

$$F \mapsto C(F^{-1})$$

defines a non-trivial 1-cocycle on the group of diffeomorphisms of T^*M with value in the space of tensor fields of type $(2,1)$.

Denote by $g^{\mathbf{ij}}$ the standard bivector, dual to the standard symplectic structure on T^*M.

For all $f \in \mathrm{Diff}(M)$, denote by $\tilde{f}$ its symplectic lift on T^*M.

The main of this paper is:

The following[1]

$$\begin{aligned}\mathcal{L}(f) \;:=\;& Sym_{\mathbf{j,i,k}}\left(C^{\mathbf{j}}_{\mathbf{ml}}(\tilde{f})\cdot g^{\mathbf{im}}\cdot g^{\mathbf{kl}}\right)\tilde{\nabla}_{\mathbf{i}}\tilde{\nabla}_{\mathbf{j}}\tilde{\nabla}_{\mathbf{k}}\\ &-\frac{3}{2}Sym_{\mathbf{n,m,i}}\left(C^{\mathbf{n}}_{\mathbf{lk}}(\tilde{f})\cdot g^{\mathbf{ml}}\cdot g^{\mathbf{ik}}\right)\cdot C^{\mathbf{j}}_{\mathbf{mn}}(\tilde{f})\tilde{\nabla}_{\mathbf{i}}\tilde{\nabla}_{\mathbf{j}},\end{aligned} \tag{2.3}$$

where Sym designs symmetrization, is a third-order linear differential operator on $C^\infty(T^*M)$.

Recall that the space $C^\infty(T^*M)$ is naturally a module over the group Diff(M). The action is given as follows: take $f \in$ Diff(M), and $Q \in C^\infty(T^*M)$ then

$$f\cdot Q := Q\circ\tilde{f}^{-1}. \tag{2.4}$$

Differentiating the action above one can obtain the action of the Lie algebra Vect(M).

The action (2.4) above induces an action of the group Diff(M) on $\mathcal{D}(C^\infty(T^*M))$ given as follows: take $f \in$ Diff(M), and $T \in \mathcal{D}(C^\infty(T^*M))$ then

$$f^*T := f\circ T\circ f^{-1}.$$

In this paper we shall study the cohomology arising in this context.

We have the following

Theorem 2.2 *The map*

$$f \mapsto \mathcal{L}(f^{-1}),$$

defines a non-trivial 1-cocycle on Diff(M) *with value in* $\mathcal{D}^3(C^\infty(T^*M))$.

Proof. To prove that the operator (2.3) is 1-cocycle one has to check the 1-cocycle condition

$$\mathcal{L}(f\circ h) = \tilde{h}^*\mathcal{L}(f) + \mathcal{L}(h), \tag{2.5}$$

for all $f, h \in$ Diff(M).

To verify (2.5) we use the following formulæ

$$\begin{aligned}Sym_{\mathbf{j,i,k}}\left(C^{\mathbf{j}}_{\mathbf{ml}}(\tilde{f})\cdot g^{\mathbf{im}}\cdot g^{\mathbf{kl}}\right)\tilde{\nabla}_{\mathbf{i}}\tilde{\nabla}_{\mathbf{j}}\tilde{\nabla}_{\mathbf{k}}(\tilde{h}^{-1^*}Q) \;=\;& 3\,Sym_{\mathbf{j,i,k}}\left(C^{\mathbf{j}}_{\mathbf{ml}}(\tilde{f})\cdot g^{\mathbf{im}}\cdot g^{\mathbf{kl}}\right)\cdot C^{\mathbf{t}}_{\mathbf{ij}}(\tilde{h}^{-1})\tilde{h}^{-1^*}\tilde{\nabla}_{\mathbf{t}}\tilde{\nabla}_{\mathbf{k}}Q\\ &+Sym_{\mathbf{j,i,k}}\left(C^{\mathbf{j}}_{\mathbf{ml}}(\tilde{f})\cdot g^{\mathbf{im}}\cdot g^{\mathbf{kl}}\right)\tilde{h}^{-1^*}\tilde{\nabla}_{\mathbf{i}}\tilde{\nabla}_{\mathbf{j}}\tilde{\nabla}_{\mathbf{k}}Q,\end{aligned}$$

$$Sym_{\mathbf{i,m,n}}\left(C^{\mathbf{i}}_{\mathbf{kl}}(\tilde{f})\cdot g^{\mathbf{mk}}\cdot g^{\mathbf{nl}}\right)\cdot C^{\mathbf{j}}_{\mathbf{mn}}(\tilde{f})\tilde{\nabla}_{\mathbf{i}}\tilde{\nabla}_{\mathbf{j}}(\tilde{h}^{-1^*}Q) \;=\; Sym_{\mathbf{i,m,n}}\left(C^{\mathbf{i}}_{\mathbf{kl}}(\tilde{f})\cdot g^{\mathbf{mk}}\cdot g^{\mathbf{nl}}\right)\cdot C^{\mathbf{j}}_{\mathbf{mn}}(\tilde{f})\tilde{h}^{-1^*}\tilde{\nabla}_{\mathbf{i}}\tilde{\nabla}_{\mathbf{j}}Q,$$

for all $f, h \in$ Diff(M), and for all $Q \in C^\infty(T^*M)$.

Let us prove that this 1-cocycle is not trivial. Suppose that there exits an operator A of degree three such that

$$\mathcal{L}(f) = \tilde{f}^*A - A.$$

The principal symbol of the operator A transforms under coordinates change as a contravariant tensor fields of degree three. It depends then only on the first jet of the symplectomorphism $\tilde{f}$, while the principal symbol of the operator (2.3) depends on the second jet of the symplectomorphism $\tilde{f}$. Absurd.

■

[1] Here and bellow summation is understood over repeated indices.

Remark 2.3 *The fact that the first-order coefficients in the formulæ above are trivial is due to the properties of the connection (2.1).*

2.3 Expression in local coordinates

Denote by $(x^i, \xi_i), i = 1, \ldots, n$, the local coordinates on T^*M. In these coordinates the components of the symplectic form g are

$$g_{ij} = g_{\bar{i}\bar{j}} = 0, \quad g_{i\bar{j}} = \delta_{ij}, \quad g_{i\bar{j}} = -g_{\bar{j}i},$$

where $i, j = 1, \ldots, n$.

For any diffeomorphism $f(x) = (f^1(x), \ldots, f^n(x))$ on M, its symplectic lift on T^*M is written as $\tilde{f}(x, \xi) = (f^1(x), \ldots, f^n(x), \frac{\partial x^i}{\partial f^1}\xi_i, \ldots, \frac{\partial x^i}{\partial f^n}\xi_i)$.

An easy computation show that the operator (2.3) (up to some constant) has the following form

$$\begin{aligned} &3\, C^i_{jk}(\tilde{f})\, \frac{\partial^2}{\partial \xi^j \partial \xi^k} \frac{\partial}{\partial x^i} + 3 \left(2\, C^m_{ik}(\tilde{f})\, \Gamma^k_{jm} + C^m_{ki}(\tilde{f}) C^k_{mj}(\tilde{f})\right) \frac{\partial^2}{\partial \xi^i \partial \xi^j} \\ &+ C^{\bar{i}}_{jk}(\tilde{f})\, \frac{\partial^3}{\partial \xi^i \partial \xi^j \partial \xi^k} \end{aligned} \tag{2.6}$$

where $C(\tilde{f})$ is the tensor (2.2).

In the particular case when the connection Γ is Euclidean (i.e. $\Gamma \equiv 0$) the operator above takes the following form

$$\begin{aligned} &3\, \frac{\partial^2 f^l}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial f^l}\, \frac{\partial^2}{\partial \xi^j \partial \xi^k} \frac{\partial}{\partial x^i} + 3\, \frac{\partial^2 f^k}{\partial x^q \partial x^i} \frac{\partial x^m}{\partial f^k} \frac{\partial^2 f^l}{\partial x^j \partial x^m} \frac{\partial x^q}{\partial f^l}\, \frac{\partial^2}{\partial \xi^i \partial \xi^j} \\ &+ \frac{\partial x^m}{\partial f^q} \xi_m \left(3 \frac{\partial x^l}{\partial f^p} \frac{\partial^2 f^p}{\partial x^i \partial x^j} \frac{\partial^2 f^q}{\partial x^l \partial x^k} - \frac{\partial^3 f^q}{\partial x^i \partial x^j \partial x^k}\right) \frac{\partial^3}{\partial \xi^i \partial \xi^j \partial \xi^k} \end{aligned} \tag{2.7}$$

3 Cohomology of the group of diffeomorphisms

Let M be a compact, oriented, manifold. Denote by $\mathrm{Diff}_+(M)$ the group of diffeomorphisms of M preserving the volume form on M.

3.1 Space of linear differential operators and space of symbols

Let $\mathcal{D}(\mathcal{F}_\lambda)$ be the space of linear differential operators acting on the space of tensor densities of degree λ on M. This space admits a structure of module over the group $\mathrm{Diff}(M)$ given as follows: take $f \in \mathrm{Diff}(M)$ and $A \in \mathcal{D}(\mathcal{F}_\lambda)$ then

$$f^* A := f^* \circ A \circ f^{-1^*}, \tag{3.1}$$

where f^* is the natural action of a diffeomorphism on $\lambda-$densities.

Differentiating this action one can obtain the action of $\mathrm{Vect}(M)$ on $\mathcal{D}(\mathcal{F}_\lambda)$.

Consider now $\mathcal{S}^k$ be the space of symmetric contravariant tensor fields on M of degree k. This space is naturally isomorphic as vector space to the space of functions on T^*M which are polynomials of degree k on the fibers. We shall identify these two spaces throughout this letter.

One can define a Diff(M)−module structure on $\mathcal{S}^k$ given by formula (2.4). We have then a filtration of Diff(M)−module

$$\mathcal{S} = \bigoplus_{k\geq 0} \mathcal{S}^k.$$

The module $\mathcal{S}$ and the modules $\mathcal{D}(\mathcal{F}_\lambda)$ are not isomorphic with respect to the action of Diff(M) given above (cf. [5, 6, 8, 9]).

3.2 Cohomology of Vect(M) and cohomology of Diff(M)

The problem of infinitesimal deformation of the module $\mathcal{S}$ with respect to the Lie algebra Vect(M) is related to the cohomology group $\mathrm{H}^1(\mathrm{Vect}(M), \mathrm{End}(\mathcal{S}))$. Consider now the space of differential operators $\mathcal{D}(\mathcal{S})$ as a submodule of End$(\mathcal{S})$. This module can be decomposed as Vect(M)−module into $\bigoplus_{k,l} \mathcal{D}(\mathcal{S}^k, \mathcal{S}^l)$. The following result is proven in [9].

$$\mathrm{H}^1(\mathrm{Vect}(M), \mathcal{D}(\mathcal{S}^k, \mathcal{S}^m)) = \begin{cases} \mathbb{R} \oplus \mathrm{H}^1_{\mathrm{DR}}(M), & k - m = 0, \\ \mathbb{R}, & k - m = 2, \\ \mathbb{R}, & k - m = 1, m \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Let us give explicit formulæ for 1-cocycles generating the group of cohomology (3.2).
For $\mathbf{k - m = 0}$.

Denote by Div the divergence operator associated to the volume form on M. The following map $X \mapsto a\,\mathrm{Div}(X) + i_X\omega$, where a is a constant, ω is a closed 1-form on M, defines a 1-cocycle on Vect(M) with value in $C^\infty(M)$ (cf. [7])

The 1-cocycle generating the group of cohomology (3.2), for $k - m = 0$, is given as a multiplication operator by a k−tensor fields with the cocycle above.
For $\mathbf{k - m = 1}$, and $\mathbf{m \neq 0}$.

The following map: $X \mapsto L_X\nabla$, where $L_X\nabla$ is the Lie derivative of the connection ∇ (see [9] for more details) defines a 1-cocycle on Vect(M) with values in the space of tensor fields on M of type $(2, 1)$.

The 1-cocycle generating the group of cohomology (3.2), for $k - m = 1$, is obtained by contracting a k−tensor fields with the above (2,1)-tensor fields.
For $\mathbf{k - m = 2}$.

It is well known that the cohomology group $\mathrm{H}^2(C^\infty(T^*M), C^\infty(T^*M))$ is isomorphic to $\mathbb{R} \oplus \mathrm{H}^1_{\mathrm{DR}}(M)$ (see e.g., [11] and the references therein). The components $\mathbb{R}$ is generated by the so-called Vey cocycle [14], noted usually by S^3_Γ (see [12] for an explicit construction using a connection). The Vey cocycle is important in the theory of deformation quantization; it appears in the third-order term in the Moyal product (see e.g., [11]).

Consider now the natural embedding of Vect(M) in $C^\infty(T^*M)$ given by $X \mapsto X^i\xi_i$. The map $X \mapsto S^3_{\tilde{\Gamma}}(X^i\xi_i, \cdot)$, where $\tilde{\Gamma}$ is the connection (2.1), turns out to be a 1-cocycle on Vect(M) with value in $\mathcal{D}(\mathcal{S}^k, \mathcal{S}^{k-2})$, generating the cohomology group (3.2) (see [9])

Remark 3.1 In the one dimensional case, the group of cohomology (3.2) was calculated in [4] (see [3] for the complex case).

To study the cohomology of the group of diffeomorphisms, we deal with differential cohomology. This means we consider only differential cochains (see [7]). The general case remains a difficult problem. We impose another condition that the manifold M is simply connected. Otherwise, we do not know if the De Rham class in the cohomology group (3.2) can be integrated on the group of diffeomorphisms.

The analogue result of the cohomology group (3.2) is the following

Theorem 3.2 *Let M be an oriented, compact, simply connected, manifold of dimension $n > 1$. The first-cohomology group*

$$\mathrm{H}^1(\mathrm{Diff}_+(M), \mathcal{D}(\mathcal{S}^k, \mathcal{S}^m)) = \begin{cases} \mathbb{R}, & k-m=0, \\ \mathbb{R}, & k-m=2, \\ \mathbb{R}, & k-m=1, m \neq 0, \\ 0, & \textit{otherwise}. \end{cases} \tag{3.3}$$

We start by giving explicit formulæ for 1-cocycles generating the cohomology group (3.3).

For $\mathbf{k - m = 0}$.

Any diffeomorphism $f \in \mathrm{Diff}_+(M)$ preserve the volume form on M up to some constant. The logarithm function of this constant defines a 1-cocycle on $\mathrm{Diff}(M)$, say $c(f)$, with value in $C^\infty(M)$.

The 1-cocycle generating the cohomology group (3.3) is given as a multiplication operator of any tensor field by the 1-cocycle $c(f)$.

For $\mathbf{k - m = 1, m \neq 0}$.

The difference $\ell(f) := f^*\Gamma - \Gamma$, where Γ is the connection on M, is a well defined tensor fields of type $(2,1)$. The map $f \mapsto \ell(f^{-1})$ defines a 1-cocycle on $\mathrm{Diff}(M)$ with value in the space of tensor fields of type $(2,1)$.

The 1-cocycle generating the group of cohomology (3.3) is obtained by contracting any tensor fields with the above 1-cocycle.

For $\mathbf{k - m = 2}$.

First we have the following

Proposition 3.3 *For any $f \in \mathrm{Diff}_+(M)$, the restriction $\mathcal{L}(\tilde{f})_{|_{\mathcal{S}^k}}$, where $\mathcal{L}$ is the operator (2.3), defines a map from $\mathcal{S}^k$ to $\mathcal{S}^{k-2}$.*

Proof. Straightforward from formula (2.6) and the properties of the connection (2.1).

It is easy to see that the map $f \mapsto \mathcal{L}(\tilde{f^{-1}})_{|_{\mathcal{S}^k}}$ defines a 1-cocycle on $\mathrm{Diff}_+(M)$ with value in $\mathcal{D}(\mathcal{S}^k, \mathcal{S}^{k-2})$ generating the cohomology group (3.3).

Now we are in position to proof Theorem 3.2.

Since the dimension of the cohomology group (3.3) is bounded by the dimension of the cohomology group (3.2), and by construction of the 1-cocycles above follows Theorem 3.2. ■

4 Discussion

1. In the one dimensional case $M := S^1$, the analogue of Theorem (3.3) was given in [4].

Namely, the first cohomology group

$$\mathrm{H}^1(\mathrm{Diff}_+(S^1), \mathrm{PSL}_2(\mathbb{R}); \mathcal{D}(\mathcal{F}_\lambda, \mathcal{F}_\mu)) = \begin{cases} \mathbb{R}, & \mu - \lambda = 2, 3, 4, \ (\lambda \text{ generic}) \\ \mathbb{R}, & (\lambda, \mu) = (-4, 1), (0, 5) \\ 0, & \text{otherwise.} \end{cases} \quad (4.1)$$

The 1-cocycle generating the cohomology group (4.1) are given as differential operators whose coefficients are given by the so-called Schwarzian derivative and their derivative (see [5] for explicit formulæ).
From this point of view, Ovsienko and the author [5] (see also [2]) generalize the Schwarzian derivative on $\mathbb{R}^n$ endowed with a projective structure (i.e. coordinates change are projective transformations). Namely, the multi-dimensional Schwarzian derivative is a 1-cocycle on the group $\mathrm{Diff}(\mathbb{R}^n)$ with value in $\mathcal{D}(S^2, S^0)$, vanishing on the subgroup $\mathrm{PSL}_{n+1}(\mathbb{R})$. It is interesting to ask if there is a relation between the 1-cocycle (2.7) and the multi-dimensional Schwarzian derivative. Observe that the 1-cocycle (2.7) does not vanish on the subgroup $\mathrm{PSL}_{n+1}(\mathbb{R})$.

2. It is interesting to generalize the 1-cocycle (2.3) on any symplectic manifold. In this case, the 1-cocycle (2.3) can be interpreted as a cocycle integrating on one argument the Vey 2-cocycle S^3_Γ (see section 3.2). The 1-cocycle (2.3) integrate the Vey cocycle only on the cotangent bundle T^*M.

Another 2-cocycle was construct by Tabachnikov in [13] integrating the Vey 2-cocycle. It is also interesting to compare this 2-cocycle and the operator (2.3).

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References

[1] S. Bouarroudj, *Projectively equivariant quantization map,* Lett. Math. Phy. **51:** (4), 2000, 265-274.

[2] S. Bouarroudj, *Multi-dimensional Schwarzian derivative revisited,* math. DG/0101056

[3] S. Bouarroudj & H. Gargoubi, *Projectively invariant cocycles of holomorphic vector fields on an open Riemann surface*, To appear in Tokyo J. Math.

[4] S. Bouarroudj & V. Ovsienko, *Three cocycles on* $\mathrm{Diff}(S^1)$ *generalizing the Schwarzian derivative*, **IMRN** 1998, No.1, 25–39.

[5] S. Bouarroudj & V. Ovsienko, *Schwarzian derivative related to the space of linear differential operators on a locally projective manifold*, Banach Cent. Pub., vol 51, 2000.

[6] C. Duval & V. Ovsienko, *Conformally equivariant quantization*, math. DG/9801122.

[7] D. B. Fuks, *Cohomology of infinite-dimensional Lie algebras,* Contemp. Soviet Math., Consultants Bureau, New york, 1986.

[8] P.B.A. Lecomte & V. Ovsienko, *Projectively invariant symbol calculus,* Lett. Math. Phy. **49** (3) (1999), 173-196.

[9] P. B. A. Lecomte & V. Ovsienko, *Cohomology of vector fields Lie algebras,* Compositio Mathematica. **119:** 2000.

[10] A. Neijnhuis & R. W., Jr. Richardson, *Deformation of homomorphisms of Lie groups and Lie algebras* , Bull. Amer. Math. Soc. **73** 1967, 175-179.

[11] V. Ovsienko & C. Roger, *Deformation of Poisson brackets and extensions of Lie algebras of contact vector fields,* Russian Math. Survey **47** (1992), 141-194.

[12] C. Roger, *Déformation algébriques et applications à la physique,* Gaz. Math. 1991, No. 49, 75-94.

[13] S. Tabachnikov, *Projective connections, group Vey cocycle, and deformation quantization.* Internat. Math. Res. Not. 1996, No. 14, 705–722.

[14] J. Vey, *Déformation du crochet de Poisson sur une variété symplectique,* Comment. Math. Helv. **50** (1975), no. 4, 421–454.

[15] K. Yano & S. Ishihara, *Tangent and cotangent bundles:* differential geometry. Pure and Applied Mathematics, No. 16. Marcel Dekker, Inc., New York, 1973.