Research Report

KSTS/RR-01/004
Jul. 26, 2001

On the Equivalent Martingale Measures for Poisson Jump Type Model

by

Takuji Arai

Takuji Arai
Department of Mathematics
Keio University

Department of Mathematics
Faculty of Science and Technology
Keio University

3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522 Japan

KSTS/RR-01/004
July 26, 2001

On the Equivalent Martingale Measures for Poisson Jump Type Model

Takuji Arai

Department of Mathematics, Keio University

Abstract

We consider an incomplete market model whose stock price fluctuation is given by a solution of a one-dimensional stochastic differential equation with Poisson jumps. For this model, we calculate the density process of the minimal martingale measure and minimal entropy martingale measure comprehensively by using the exponential martingale and martingale representation theorem. Also, we state relations to locally risk minimizing strategy or the Esscher transforms.

1 Introduction

We consider an incomplete market whose stock price fluctuation is given by a solution of a one-dimensional stochastic differential equation with Poisson jumps. Throughout this paper, we call such a model Poisson jump type model. Our aim is to calculate the density processes of various *equivalent martingale measures* (abbreviated as EMM) for Poisson jump type model. There are two similar research papers. One is Chan (1999), another is Fujiwara and Miyahara (2001). However, they treated only special cases. Thus, we purpose to lead to various equivalent martingale measures comprehensively and give a simplified proof. Moreover, we mention relations between various equivalent martingale measures and locally risk minimizing strategy or the Esscher transforms, comprehensively.

In Section 3, we study the density process of the *minimal martingale measure* (abbreviated as MMM) which was introduced in Föllmer and Schweizer (1991). As for jump type models, Chan (1999) considered a special case of Poisson jump type model. Hereafter, we call this special model Chan's model. In Chan's model, the coefficients of stochastic differential equation describing the stock price process are deterministic. He obtained the density process of the MMM for his model. We calculate the density process of the MMM for general Poisson jump type model. The density process of the MMM is given by the exponential martingale form. However, since the stock price process of Poisson jump type model is a jump process, the density process of the MMM is not necessarily positive. Hence we need to consider a necessary and sufficient condition which assures the positivity of the density process of the MMM. We call, throughout this paper, this necessary and sufficient condition the positivity condition for the MMM. Moreover, the MMM relate closely to the locally risk minimizing strategy defined by Schweizer (1991). In Section 4, we investigate some properties of the MMM. Also, we lead to locally risk minimizing strategy for Poisson jump type model.

Moreover, we calculate the general form of density processes under some conditions in Section 5. Ansel and Stricker (1992) obtained the general form of density processes for a continuous semimartingale model. Furthermore, Schweizer (1995) extended it to the general RCLL case. According to the results of these papers, the general form of density processes is given by the exponential martingale form. By using Tang-Li's martingale representation theorem, we show that the general form of density processes for Poisson jump type model also is given by the exponential martingale form under the positivity condition.

In Section 6, we study the *minimal entropy martingale measure* (abbreviated as MEMM) by using the results in Section 5. We obtain the density process of the MEMM for Poisson jump type model under the positivity condition. The MEMM was introduced in Miyahara (1996a) and Frittelli (2000). They considered only continuous case. As for jump type models, Miyahara (1999) obtained the density process of the MEMM for a geometric Lévy process model being a special case of Poisson jump type model. In particular, Miyahara (1999) treated a model which stochastic integrals of Brownian motion is not contained. Note that the geometric Lévy process model is called the log-Lévy process model in Miyahara (1999). Also, Chan (1999) considered the density process of the MEMM for Chan's model. On the other hand, Fujiwara and Miyahara (2001) treated the general geometric Lévy process model and stated the condition for the existence of the MEMM for geometric Lévy process model. Moreover, they obtained the density process of the MEMM.

Chan (1999) and Fujiwara and Miyahara (2001) mentioned that the MEMM has closely connected to the Esscher transform for Chan's model and geometric Lévy process model. Recently, many people research relations to the option pricing theory and the insurance mathematics. Above all, the Esscher transform is well-known as an important tool. In Section 7, we treat the Esscher transforms for more general Poisson jump type model than Chan's model and geometric L'evy process model. Moreover, we consider relations between the Esscher transform and the MEMM.

Furthermore, we consider three examples in Section 8. We treat Chan's model, the geometric Lévy process model and Chan's model with random coefficients. Under the positivity condition, by using the result in Section 5, we lead to the density processes of the MEMM for these three models. On the other hand, Miyahara (1996b) and Arai (2001) proved that, if the stock price process is given by a solution of a general continuous stochastic differential equation, then the MMM coincides with the MEMM. Now, we have a natural question, "Does this fact hold for the Poisson jump type model?" The answer is "No". However, in Section 9, we mention an interesting relationship between the MMM and the MEMM. In addition to this, in Section 9, we also remark some important facts.

2 Preliminaries.

We consider a Poisson jump type model whose maturity is $T > 0$. Suppose that there exist one riskless asset and only one risky asset in our market. Without loss of generality, we assume that the price of the riskless asset is 1.

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a right continuous complete filtration $\mathbf{F} = \{\mathcal{F}_t\}_{0 \le t \le T}$ satisfying that $\mathcal{F}_0$ is trivial and contains all null sets of $\mathcal{F}$, and $\mathcal{F}_T = \mathcal{F}$.

Let a process $S = \{S_t\}_{0 \le t \le T}$ be a semimartingale and suppose that the fluctuation of the risky asset is given by S. The process S is said to be the stock price process. An $\mathbf{R}$-valued process Z is called the density process for S if Z is a P-local martingale with $Z_0 = 1$ and the product SZ is a P-local martingale. If a signed measure P^*, which is equivalent to P, satisfies (1) $P^*(\Omega) = 1$, (2) $P^* = P$ on $\mathcal{F}_0$, (3) $P^* \ll P$ on $\mathcal{F}$, and (4) the stock price process S is a P^*-local martingale in the sense that the process $\left\{ \frac{dP^*}{dP}\Big|_{\mathcal{F}_t} \right\}_{0 \le t \le T}$ is a density process, then P^* is said to be a signed local martingale measure. In particular, if a signed local martingale measure P^* is a probability measure, then we call P^* an *equivalent local martingale measure* (abbreviated as ELMM). We denote by $\mathbf{M}$ the set of all ELMM's. Moreover, for every $P^* \in \mathbf{M}$, the $\mathbf{R}$-valued martingale G^* defined by

$$G_t^* := E\left[\frac{dP^*}{dP} | \mathcal{F}_t\right]$$

is called the density process of P^*.

Now, let a process $W = \{W_t\}_{0 \le t \le T}$ be a one-dimensional standard Brownian motion. Next, we denote by $N(dt, dx)$ a counting measure of a stationary Poisson point process on $[0, T] \times \mathbf{R}_0$, where $\mathbf{R}_0 = \mathbf{R} \backslash \{0\}$. Suppose that $N(dt, dx)$ is independent of W and the compensator of $N(dt, dx)$ is $\nu(dx)dt$, where the measure $\nu(\cdot)$ is a Lévy measure, that is, $\nu(\cdot)$ satisfies $\nu(\{0\}) = 0$ and

$$\int_{\mathbf{R}_0} (1 \wedge |x|^2) \nu(dx) < \infty.$$

Moreover, we set $\widetilde{N}(dt, dx) := N(dt, dx) - \nu(dx)dt$. Let the filtration $\mathbf{F}$ be given by

$$\mathcal{F}_t := \sigma\left[\int_0^{s+} \int_A N(du, dx); s \le t, A \in \mathcal{B}(\mathbf{R}_0)\right] \vee \sigma\left[W_s; s \le t\right] \vee \mathcal{N},$$

where $\mathcal{N}$ is all null sets in $\mathcal{F}$. Suppose that $\mathcal{F}_T = \mathcal{F}$. Remark that, by Lemma A.1 of Tang and Li (1994), the filtration $\mathbf{F}$ is right continuous and, by Lemma A.2 of Tang and Li (1994), $\mathbf{F}$ has the quasi-left continuous property, that is, for any increasing sequence $\{\tau_n\}$ of $\mathbf{F}$-stopping times, we have

$$\bigvee_n \mathcal{F}_{\tau_n} = \mathcal{F}_\tau,$$

where $\tau = \lim_{n \to \infty} \tau_n$.

We define three spaces of **F**-predictable processes:

$$\mathbf{F}_p^1 := \Big\{ f(t,x,\omega); f \text{ is an } \mathbf{F}\text{-predictable process and, for all } t \in [0,T], \\ E\left[\int_0^t \int_{\mathbf{R}_0} |f(s,x,\cdot)|\nu(dx)ds\right] < \infty \Big\},$$

$$\mathbf{F}_p^2 := \Big\{ f(t,x,\omega); f \text{ is an } \mathbf{F}\text{-predictable process and, for all } t \in [0,T], \\ E\left[\int_0^t \int_{\mathbf{R}_0} |f(s,x,\cdot)|^2\nu(dx)ds\right] < \infty \Big\},$$

and

$$\mathbf{F}_p := \mathbf{F}_p^1 \cap \mathbf{F}_p^2.$$

Throughout this paper, we fix one **F**-predictable process $f \in \mathbf{F}_p$. This fixed **F**-predictable process f is a function of t, x and ω, while we assume that we can regarded f as a function of t, x and S. Hereafter, we treat this fixed **F**-predictable process f as a function of t, x and S. Let us define the stock price process by using this fixed $f \in \mathbf{F}_p$. The stock price process $S = \{S_t\}_{0 \le t \le T}$ is given by a solution of the following stochastic differential equation (abbreviated as SDE) with Poisson jumps:

$$S_t = S_0 + \int_0^t \mu(s, S_{s-})ds + \int_0^t \sigma(s, S_{s-})dW_s + \int_0^{t+} \int_{\mathbf{R}_0} f(s, x, S_{s-})\widetilde{N}(ds, dx), \quad (2.1)$$

where S_0 is a positive constant. Throughout this paper, we denote $\mu_t := \mu(t, S_{t-})$, $\sigma_t := \sigma(t, S_{t-})$ and $f(t,x) := f(t,x,S_{t-})$, for notational simplicity. Moreover, we assume that the SDE (2.1) has a unique strong solution. In other words, we assume the following:

Assumption A 1. In order to ensure the existence and uniqueness of the strong solution of SDE (2.1), we assume that SDE (2.1) satisfies the conditions of Theorems 13 or 14 or 17 of Situ (1985).

2. In addition to this, we assume the following:

$$|\mu_t| > 0 \quad \text{a.e. in } (t,\omega), \quad (2.2)$$

$$|\sigma_t| > 0 \quad \text{a.e. in } (t,\omega), \quad (2.3)$$

$$|f(t,x)| > 0 \quad \text{a.e. in } (t,x,\omega),$$

$$\sigma_t \in L^2(P) \quad \text{for all } t \in [0,T],$$

$$\int_0^t |\mu_s|ds < \infty \quad \text{a.e. in } (t,\omega).$$

Since the process S is a semimartingale, we can decompose of S into a local martingale $M = \{M_t\}_{0 \le t \le T}$ and a finite variation process $A = \{A_t\}_{0 \le t \le T}$, where M and A have the following expressions:

$$M_t = \int_0^t \sigma_s dW_s + \int_0^{t+} \int_{\mathbf{R}_0} f(s,x) \widetilde{N}(ds, dx),$$

$$A_t = \int_0^t \mu_s ds.$$

Remark that the local martingale part M is a square integrable P-martingale with $M_0 = 0$, and the stock price process S is a special semimartingale. Moreover, for any $P^* \in \mathbf{M}$, the process S is not only a local martingale, but a martingale under P^*. Hereafter, we call any $P^* \in \mathbf{M}$ an *equivalent martingale measure* (EMM) replacing with an ELMM.

Generally, for a process X, we denote by $[X]$ the quadratic variation process for X and by $\langle X \rangle$ the compensator of the quadratic process $[X]$. Then, we have

$$\langle M \rangle_t = \int_0^t \left\{ \sigma_s^2 + \int_{\mathbf{R}_0} f^2(s,x) \nu(dx) \right\} ds.$$

Therefore, by Assumption A, we have $A \ll \langle M \rangle$ and are able to write as follows:

$$A_t = \int_0^t \frac{\mu_s}{\sigma_s^2 + \int_{\mathbf{R}_0} f^2(s,x)\nu(dx)} d\langle M \rangle_s. \tag{2.4}$$

Hence, by Corollary 3 of Schweizer (1995), the stock price process S has a positive density process. For simplicity, we write

$$v_t^f := \int_{\mathbf{R}_0} f^2(t,x) \nu(dx).$$

Let us denote by $\alpha = \{\alpha_t\}_{0 \le t \le T}$ the integrand of (2.4), namely, define a predictable process α by

$$\alpha_t = \frac{\mu_t}{\sigma_t^2 + v_t^f}.$$

Finally, we define one more predictable process. An $\mathbf{F}$-predictable process $K = \{K_t\}_{0 \le t \le T}$ is called the *mean-variance trade-off process* if

$$K_t = \int_0^t \frac{\mu_s^2}{\sigma_s^2 + v_s^f} ds.$$

which is assumed to be uniformly bounded throughout this paper. Consequently, we assume that the process S satisfies the structure condition (SC).

3 The minimal martingale measure.

In this section, we calculate the density process of the MMM. We define a local martingale $G^M = \{G_t^M\}_{0\le t\le T}$ by

$$G_t^M = \mathcal{E}\left(-\int_0^{\cdot} \alpha_s dM_s\right)_t,$$

where, for a semimartingale X, $\mathcal{E}(X) = \{\mathcal{E}(X)_t\}_{0\le t\le T}$ is a solution of SDE

$$\mathcal{E}(X)_t = 1 + \int_0^t \mathcal{E}(X)_{s-} dX_s,$$

and we call $\mathcal{E}(X)$ the exponential martingale for X. If a signed martingale measure P^M satisfies

$$G_t^M = E\left[\frac{dP^M}{dP}\Big|\mathcal{F}_t\right],$$

then P^M is called the signed *minimal martingale measure* (MMM). Moreover, if the singed MMM P^M is a probability measure, then we call P^M the MMM. Remark that, since the process M is not continuous, generally, G^M is not necessarily positive. Namely, even if the signed MMM P^M exists, then it is not necessarily the MMM.

From now on, we assume that the existence of the signed MMM. Firstly, we give a necessary and sufficient condition for the positivity of G^M, namely, for the existence of P^M as a probability measure. We call this necessary and sufficient condition the positivity condition for the MMM. The following proposition was proved by Schweizer (1995). However, we give here an alternative proof by using the exponential martingale.

Proposition 3.1 (The positivity condition) *The density process G^M is positive if and only if we have*

$$\alpha_t f(t,x) < 1 \quad \text{a.e. } in\ (t,x,\omega). \tag{3.1}$$

Proof. By Theorem 36 of Chap. II of Protter (1990), if we set

$$X_t := -\int_0^t \alpha_s dM_s,$$

then the exponential martingale for X is represented as

$$\mathcal{E}(X)_t = \exp\left(X_t - \frac{1}{2}[X]_t^c\right) \prod_{0<s\le t} (1+\Delta X_s)\exp(-\Delta X_s), \tag{3.2}$$

where $[X]^c$ means the continuous part of $[X]$. Also, we have

$$\prod_{0<s\le t} \Delta X_s = -\int_0^{t+}\int_{\mathbf{R}_0} \alpha_s f(s,x) N(ds,dx).$$

Hence, we see that the right hand side of (3.2) is positive if and only if we have

$$\alpha_t f(t,x) < 1 \quad \text{a.e. in } (t,x,\omega).$$

This completes the proof. □

We need some assumptions in order to calculate the density process of the MMM.

Assumption B In the rest of this section, we assume that the fixed f satisfies (3.1), that is, the MMM $P^M \in \mathbf{M}$ exists.

We calculate the density process of the MMM.

Theorem 3.2 *Under Assumption B, the density process G^M of the MMM P^M is given by*

$$\begin{aligned} G_t^M &= \exp\Big\{ -\int_0^t \alpha_s\sigma_s dW_s - \frac{1}{2}\int_0^t (\alpha_s\sigma_s)^2 ds \\ &\quad + \int_0^{t+}\int_{\mathbf{R}_0} g^f(s,x)N(ds,dx) + \int_0^t\int_{\mathbf{R}_0} \alpha_s f(s,x)\nu(dx)ds \Big\}, \end{aligned} \tag{3.3}$$

where $g^f(t,x,S_{t-}) = g^f(t,x) := \log(1-\alpha_t f(t,x))$.

To show this theorem, we need the following lemma.

Lemma 3.3 *Suppose* $\mathbf{F}$*-predictable process $g(t,x,\omega) = g(t,x) \in \mathbf{F}_p^1$ satisfies that $1 - e^{g(t,x)} \in \mathbf{F}_p^1$. Then, we have*

$$\mathcal{E}\left(\int_0^\cdot \int_{\mathbf{R}_0} \left(e^{g(s,x)} - 1\right) \widetilde{N}(ds,dx)\right)_t = \exp Y_t,$$

where the process Y is defined by

$$Y_t := \int_0^{t+}\int_{\mathbf{R}_0} g(s,x)N(ds,dx) + \int_0^t\int_{\mathbf{R}_0}\left(1 - e^{g(s,x)}\right)\nu(dx)ds.$$

Proof of Lemma 3.3 By Ito's formula, we have

$$\begin{aligned} d(\exp Y_t) &= \exp Y_t \int_{\mathbf{R}_0}\left(1 - e^{g(t,x)}\right)\nu(dx)dt \\ &\quad + \int_{\mathbf{R}_0}\left[\exp(Y_t + g(t,x)) - \exp Y_t\right] N(dt,dx) \\ &= \exp Y_t \int_{\mathbf{R}_0}\left(e^{g(t,x)} - 1\right)\left(N(dt,dx) - \nu(dx)dt\right). \end{aligned}$$

This completes the proof of Lemma 3.3. □

Proof of Theorem 3.2 Since $g^f(t,x)=\log(1-\alpha_t f(t,x))\in\mathbf{F}_p^1$, we can apply Lemma 3.3, and have

$$\begin{aligned}
G_t^M &= \mathcal{E}\left(-\int_0^{\cdot}\alpha_s dM_s\right)_t \\
&= \mathcal{E}\left(-\int_0^{\cdot}\alpha_s\left(\sigma_s dW_s+\int_{\mathbf{R}_0}f(s,x)\tilde{N}(ds,dx)\right)\right)_t \\
&= \exp\Big\{-\int_0^t\alpha_s\sigma_s dW_s-\frac{1}{2}\int_0^t(\alpha_s\sigma_s)^2ds \\
&\quad +\int_0^{t+}\int_{\mathbf{R}_0}g^f(s,x)N(ds,dx)+\int_0^t\int_{\mathbf{R}_0}\alpha_s f(s,x)\nu(dx)ds\Big\}.
\end{aligned}$$

This completes the proof. □

Let us consider the following two examples.

Example 3.4 (Chan's model) Let the stock price process S be given by

$$S_t=S_0+\int_0^t b_sS_{s-}ds+c\int_0^t\sigma_sS_{s-}dW_s+\int_0^{t+}\int_{\mathbf{R}_0}\sigma_sS_{s-}x\tilde{N}(ds,dx), \tag{3.4}$$

where σ_t and b_t are deterministic, c is a constant and S_0 is a positive constant. This model is corresponding to the case of $f(t,x,S_{t-})=\sigma_tS_{t-}x$. We assume that $\sigma_tS_{t-}x\in\mathbf{F}_p$. Moreover, we assume that the support of ν is bounded, that is, there exist two constants $0\le c_1,c_2<\infty$ such that

$$\operatorname{supp}(\nu)=[-c_1,c_2].$$

The solution of (3.4) is represented as

$$\begin{aligned}
S_t &= S_0\exp\left\{\int_0^t c\sigma_s dW_s+\int_0^t\left(b_s-\frac{c^2\sigma_s^2}{2}\right)ds+\int_0^{t+}\int_{\mathbf{R}_0}\sigma_s\tilde{N}(ds,dx)\right\} \\
&\quad\times\prod_{0<s\le t}(1+\sigma_s\Delta X_s)\exp(-\sigma_s\Delta X_s),
\end{aligned}$$

where $X_t:=\int_0^{t+}\int_{\mathbf{R}_0}x\tilde{N}(ds,dx)$. In order to ensure the positivity of S, we need to assume that

$$-\frac{1}{c_2}\le\sigma_t\le\frac{1}{c_1}\quad\text{for all } t\in[0,T].$$

Under Assumption B, let us calculate the density process G^M of the MMM. We denote

$$v_2:=\int_{\mathbf{R}_0}x^2\nu(dx).$$

Then, we have

$$\alpha_t=\frac{b_t}{\sigma_t^2(c^2+v_2)}\frac{1}{S_{t-}}.$$

Assumption B means that, for $-c_1 < x < c_2$,

$$\alpha_t \sigma_t S_{t-} x = \frac{b_t x}{\sigma_t(c^2 + v_2)} < 1.$$

By (3.3), we have

$$\begin{aligned} G_t^M &= \exp\Big\{ \int_0^t c\gamma_s dW_s - \frac{1}{2}\int_0^t c^2\gamma_s^2 ds + \int_0^{t+}\int_{\mathbf{R}_0} \log(1+\gamma_s x)\, N(ds,dx) \\ &\quad - \int_0^t \int_{\mathbf{R}_0} \gamma_s x \nu(dx) ds \Big\}, \end{aligned} \tag{3.5}$$

where we denote

$$\gamma_t := -\frac{b_t}{\sigma_t(c^2+v_2)}.$$

Example 3.5 In Example 3.4, replace x by $e^x - 1$. Namely, the stock price process is given by

$$S_t = S_0 + \int_0^t b_s S_{s-} ds + c\int_0^t \sigma_s S_{s-} dW_s + \int_0^{t+}\int_{\mathbf{R}_0} \sigma_s S_{s-}(e^x - 1)\widetilde{N}(ds,dx).$$

We assume that $\sigma_t S_{t-}(e^x - 1) \in \mathbf{F}_p$. Denote

$$v_2 := \int_{\mathbf{R}_0} (e^x - 1)^2 \nu(dx).$$

In order to assure the positivity of S, we assume that

$$\frac{1}{1-e^{c_2}} \le \sigma_t \le \frac{1}{1-e^{-c_1}}.$$

Then, under Assumption B, the density process G^M of the MMM is given by the same form as in (3.5), that is, if we define γ_t as above, then we have

$$\begin{aligned} G_t^M &= \exp\Big\{ \int_0^t c\gamma_s dW_s - \frac{1}{2}\int_0^t c^2\gamma_s^2 ds + \int_0^{t+}\int_{\mathbf{R}_0} \log(1+\gamma_s(e^x - 1))\, N(ds,dx) \\ &\quad - \int_0^t \int_{\mathbf{R}_0} \gamma_s (e^x - 1) \nu(dx) ds \Big\}. \end{aligned} \tag{3.6}$$

4 Locally risk minimizing strategy.

In this section, we lead to locally risk minimizing strategy for Poisson jump type model. Firstly, we investigate some properties of the MMM obtained by Theorem 3.2. We obtain the following proposition.

Proposition 4.1 *1. The density process G^M of the MMM is a square integrable martingale.*

2. *Let $L = \{L_t\}_{0\leq t\leq T}$ be a square integrable martingale starting at zero, which is strongly orthogonal to M under P. Then, L remains a martingale under the MMM P^M.*

Proof. 1. We set

$$X_t := -\int_0^t \alpha_s dM_s.$$

Namely, we have $G_t^M = \mathcal{E}(X)_t$. Since $E[[X]_T] = E[K_T] < \infty$, by Lemma of Theorem 28 of Chapter IV of Protter (1990), X is a square integrable martingale. Hence, by Théoréme II.2 of Lepingle and Mémin (1978), G^M is a square integrable martingale.

2. It is enough to show that the product $G^M L$ is a martingale under P. Remark that G^M is a solution of

$$dG_t^M = -\alpha_t G_t^M dM_t.$$

By assumption, $[G^M, L]$ is a martingale. On the other hand, we have

$$d(G^M L)_t = G_t^M dL_t + L_t dG_t^M + [G^M, L]_t.$$

This completes the proof. □

Now, we consider locally risk minimizing strategy. Risk minimizing approach was undertaken by Föllmer and Sondermann (1986). However, they studied only model whose stock price process is a local martingale. Thus, Schweizer (1991) undertaken locally risk minimizing approach extending risk minimizing. We owe it to this extension that we can treat semimartingale case. We do not mention the definition of locally risk minimizing in this paper. For instance, see Schweizer (1991) and Schweizer (2000). We compute locally risk minimizing strategy for Poisson jump type model. Firstly, we need to prepare some definitions.

Definition 4.2 1. For a martingale M, a predictable process θ belongs to $L^2(M)$ if the process $\int \theta^2 d\langle M\rangle$ is integrable.

2. For a finite variation process A, a predictable process θ belongs to $L^2(A)$ if the process $\int |\theta dA|$ is square integrable.
3. $\Theta := L^2(M) \cap L^2(A)$.
4. For an $\mathbf{F}$-adapted process V such that $V_T \in L^2(P)$ and $\theta \in \Theta$, we call the pair $\varphi = (V, \theta)$ a portfolio.
5. The cost process of a portfolio $\varphi = (V, \theta)$ is defined by

$$C_t(\varphi) = V_t - \int_0^t \theta_u dS_u.$$

6. A portfolio is called self-financing if its cost process is constant P-a.s.
7. A portfolio is called mean-self-financing if its cost process is a martingale under P.

Let $H \in L^2(\mathcal{F}_T, P)$ be a contingent claim. Hereafter, we call H an L^2-contingent claim. Monat and Stricker (1995) proved that, if the mean-variance trade-off process K is uniformly bounded, then every L^2-contingent claim admits a unique Föllemr-Schweizer decomposition. In other words, we can decompose H into

$$H = H_0 + \int_0^T \theta_t^H dS_t + L_T^H, \tag{4.1}$$

where $H_0 \in \mathbf{R}$, $\theta^H \in \Theta$, $L^H = \{L_t^H\}_{0\le t\le T}$ is a square integrable martingale strongly orthogonal to M. We have the following theorem.

Theorem 4.3 *For L^2-contingent claim H, there exists a unique locally risk minimizing strategy $\varphi^* = (V^*, \theta^*)$ given by:*

$$V_t^* = E^M[H|\mathcal{F}_t], \tag{4.2}$$
$$\theta^* = \theta^H,$$

where E^M means expectation under P^M and θ^H is the integrand in the Föllmer-Schweizer decomposition (4.1).

Proof. Since $H \in L^2(P)$ and $G^M \in L^2(P)$, we have $H \in L^1(P^M)$. Hence, V^* of (4.2) is well-defined. Remark that $\int \theta^H dS$ is a P^M-martingale. By Proposition 4.1, L^H in (4.1) is a P^M-martingale. Therefore, by (4.1), we have

$$V_t^* = H_0 + \int_0^t \theta_u^H dS_u + L_t^H.$$

The cost process $C(\varphi^*)$ of the strategy $\varphi^* = (V^*, \theta^*)$ is represented as

$$\begin{aligned} C_t(\varphi^*) &= V_t^* - \int_0^t \theta_s^* dS_s \\ &= H_0 + L_t^H. \end{aligned}$$

Thus, $C(\varphi^*)$ is strongly orthogonal to M and φ^* is mean-self-financing. By Assumption A and continuity of A, we can apply Theorem 3.3 of Schweizer (2000), from which φ^* is locally risk minimizing. □

5 The general form of density processes.

The aim of this section is to lead to the general form of density processes for a Poisson jump type model. We can apply Theorem 1 of Schweizer (1995), because M is a square integrable P-martingale with $M_0 = 0$, S is a special semimartingale and satisfies the structure condition. By this theorem, we can say that, if the density process of an EMM P^*, denoted by $G^* = \{G_t^*\}_{0\le t\le T}$, is square integrable, then we can write

$$G_t^* = \mathcal{E}\left(-\int_0^\cdot \alpha_s dM_s + L^*\right)_t, \tag{5.1}$$

where a process L^* is a square integrable P-local martingale being strongly orthogonal to M with $L^*_0 = 0$. We call L^* the reference local martingale of P^*. In order to lead to the general form of density processes, we need to assume the following.

Assumption C Throughout this section, we only consider a square integrable martingale density process whose reference local martingale is a square integrable P-martingale.

Hereafter, let $\mathbf{M}$ be the set of EMM's whose density processes satisfy Assumption C. Throughout this section, let us fix one $P^* \in \mathbf{M}$. Now, we need the following theorem which is said to be Tang-Li's representation theorem.

Theorem 5.1 (Lemma 2.3 of Tang and Li (1994)) *Let $\{m_t\}_{0\le t\le T}$ be an $\mathbf{R}$-valued $\mathbf{F}$-adapted square integrable martingale. Then, there exist an $\mathbf{F}$-predictable process $r \in \mathbf{F}^2_p$ and an $\mathbf{R}$-valued square integrable $\mathbf{F}$-predictable process q such that*

$$m_t = m_0 + \int_0^t q_s dW_s + \int_0^{t+}\int_{\mathbf{R}_0} r(s,x)\widetilde{N}(ds,dx).$$

By this theorem, for the reference local martingale L^* of P^*, there exist an $f^*(t,x,\omega) = f^*(t,x) \in \mathbf{F}^2_p$ and a square integrable $\mathbf{F}$-predictable process H^* such that

$$L^*_t = \int_0^t H^*_s dW_s + \int_0^{t+}\int_{\mathbf{R}_0} f^*(s,x)\widetilde{N}(ds,dx),$$

since the process L^* is a square integrable P-martingale. On the other hand, since the process L^* is strongly orthogonal to the process M, we have, for all $t \in [0,T]$,

$$\sigma_t H^*_t + \int_{\mathbf{R}_0} f(t,x)f^*(t,x)\nu(dx) = 0.$$

Therefore, by Assumption A, we can write

$$H^*_t = -\frac{1}{\sigma_t}\int_{\mathbf{R}_0} f(t,x)f^*(t,x)\nu(dx).$$

Consequently, the $\mathbf{F}$-predictable process H^* depends only on f^*. Hence, the function f^* is one-to-one corresponding to the reference local martingale L^*. We call f^* the reference function of P^*.

We now start to calculate the density process G^*. By (5.1), we have

$$\begin{aligned} G^*_t &= \mathcal{E}\left(-\int_0^\cdot \alpha_s dM_s + L^*\right)_t \\ &= \mathcal{E}\Bigg(-\int_0^\cdot \frac{\mu_s[\sigma_s dW_s + \int_{\mathbf{R}_0} f(s,x)\widetilde{N}(ds,dx)]}{\sigma_s^2 + v_s^f} \\ &\quad - \int_0^\cdot \frac{\int_{\mathbf{R}_0} f(s,x)f^*(s,x)\nu(dx)}{\sigma_s} dW_s + \int_0^\cdot\int_{\mathbf{R}_0} f^*(s,x)\widetilde{N}(ds,dx)\Bigg)_t. \quad (5.2) \end{aligned}$$

Moreover, we denote

$$\widehat{\sigma}_t^* := -\alpha_t\sigma_t + H_t^*, \tag{5.3}$$

$$\widehat{f}^*(t,x,\omega) = \widehat{f}^*(t,x) := -\alpha_t f(t,x) + f^*(t,x). \tag{5.4}$$

In addition, we assume the following.

Assumption D (The positivity condition) $\widehat{f}^* \in \mathbf{F}_p$ and

$$\widehat{f}^*(t,x) > -1 \quad \text{a.e. in } (t,x,\omega). \tag{5.5}$$

By (5.2)–(5.4), we have

$$G_t^* = \mathcal{E}\left(\int_0^{\cdot} \widehat{\sigma}_s^* dW_s + \int_0^{\cdot}\int_{\mathbf{R}_0} \widehat{f}^*(s,x)\widetilde{N}(ds,dx)\right)_t. \tag{5.6}$$

Moreover, by Assumption D and Lemma 3.3, the density process G^* is a positive process and is represented as

$$\begin{aligned} G_t^* &= \exp\left\{\int_0^t \widehat{\sigma}_s^* dW_s - \frac{1}{2}\int_0^t (\widehat{\sigma}_s^*)^2 ds \right. \\ &\quad \left. + \int_0^{t+}\int_{\mathbf{R}_0} \widehat{g}^*(s,x)N(ds,dx) - \int_0^t\int_{\mathbf{R}_0} \widehat{f}^*(s,x)\nu(dx)ds\right\}, \end{aligned} \tag{5.7}$$

where $\widehat{g}^*(t,x,\omega) = \widehat{g}^*(t,x) := \log\{1+\widehat{f}^*(t,x)\}$. Remark that $\widehat{g}^*(t,x) = \log\{1+\widehat{f}^*(t,x)\} \in \mathbf{F}_p^1$. So the conclusion is:

Theorem 5.2 *Let P^* be an EMM whose density process has square integrability. Under Assumption D, the density process of P^*, denoted G^*, is given by*

$$\begin{aligned} G_t^* &= \exp\left\{\int_0^t \widehat{\sigma}_s^* dW_s - \frac{1}{2}\int_0^t (\widehat{\sigma}_s^*)^2 ds \right. \\ &\quad \left. + \int_0^t\int_{\mathbf{R}_0} \widehat{g}^*(s,x)N(ds,dx) - \int_0^t\int_{\mathbf{R}_0} \widehat{f}^*(s,x)\nu(dx)ds\right\}. \end{aligned}$$

Remark 1 Let us denote by $\widehat{M}$ the contents of $\mathcal{E}$ of (5.1). If we assume $\Delta\widehat{M} > -1$, then we have $\mathcal{E}(\widehat{M}) > 0$. By Proposition I.5 of Lepingle and Mémin (1978), the quadratic variation process $[\widehat{M}]$ is integrable. Since $\widehat{M}$ is a local martingale, $\widehat{M}$ is a square integrable martingale, that is, the reference local martingale L^* is a square integrable martingale. Namely, we can omit the description with respect to the reference local martingale in Assumption C. Therefore, we can replace Assumption C by the following:

Assumption C′ We only consider a square integrable martingale as a density process.

6 The minimal entropy martingale measure.

In this section, we focus on calculating the density process of the minimal entropy martingale measure (MEMM) under the positivity condition. Firstly, we define the relative entropy. For two probability measures P and Q on a measurable space $(\Omega, \mathcal{F})$, the relative entropy $H(Q|P)$ of Q with respect to P is defined by

$$H(Q|P) := \begin{cases} \int \ln \frac{dQ}{dP} dQ, & \text{if } Q \ll P, \\ +\infty, & \text{otherwise.} \end{cases}$$

For a set of measures $\mathcal{P}$ on $(\Omega, \mathcal{F})$, we define

$$H(\mathcal{P}|P) := \inf_{Q \in \mathcal{P}} H(Q|P).$$

Now, let us define the MEMM. A probability measure $P^E \in \mathbf{M}$ is called the *minimal entropy martingale measure* (MEMM) if

$$H(P^E|P) = H(\mathbf{M}|P).$$

Similarly to the previous section, we assume the following.

Assumption E Throughout this section, we assume that

1. there exists the MEMM uniquely,
2. we only consider EMM's whose relative entropy with respect to P are finite and whose density process is square integrable.

Hereafter, let $\mathbf{M}$ be the set of EMM's satisfying 2 of Assumption E. In addition to this, we assume the following.

Assumption F (The positivity condition) 1. For a fixed f, there exists a unique $\mathbf{F}$-predictable process $f^E(t,x,\omega) = f^E(t,x) \in \mathbf{F}_p$ satisfying the following:

$$f^E(t,x) = \alpha_t f(t,x) + \exp\left\{ -\alpha_t f(t,x) - \frac{\int_{\mathbf{R}_0} f(t,y) f^E(t,y) \nu(dy)}{\sigma_t^2} f(t,x) \right\} - 1. \tag{6.1}$$

2. There exists an EMM $P^E \in \mathbf{M}$ whose reference function is f^E.

We prepare some notation.

Definition 6.1 1.

$$\begin{aligned} \widehat{f}^E(t,x,\omega) = \widehat{f}^E(t,x) &:= -\alpha_t f(t,x) + f^E(t,x) \\ &= \exp\left\{ \frac{\widehat{\sigma}_t^E f(t,x)}{\sigma_t} \right\} - 1 \\ &> -1. \end{aligned}$$

2. $\widehat{g}^E(t,x,\omega) = \widehat{g}^E(t,x) := \dfrac{\widehat{\sigma}_t^E f(t,x)}{\sigma_t}$.

Remark 2 1. We can replace 1 of Assumption F by the following:
There exists a unique **F**-predictable process $\widehat{\sigma}^E$ such that

$$\mu_t + \sigma_t \widehat{\sigma}_t^E + \int_{\mathbf{R}_0} f(t,x)\left(\exp\left\{\frac{\widehat{\sigma}_t^E f(t,x)}{\sigma_t}\right\} - 1\right)\nu(dx) = 0,$$

and an **F**-predictable process f^E is defined by

$$\widehat{\sigma}_t^E = -\alpha_t\sigma_t - \frac{\int_{\mathbf{R}_0} f(t,x) f^E(t,x)\nu(dx)}{\sigma_t}. \tag{6.2}$$

2. By their definition, we have $\widehat{f}^E \in \mathbf{F}_p$ and $\widehat{g}^E \in \mathbf{F}_p$.

3. Since we are assuming that f and $\widehat{f}^E$ belong to $\mathbf{F}_p$, the integrability of the above equation is ensured.

We denote by $G^E = \{G_t^E\}_{0\le t\le T}$ the density process of P^E. Under Assumption F, let us calculate the density process G^E. By (5.7), we have

$$\begin{aligned} G_t^E &= \exp\left\{\int_0^t \widehat{\sigma}_s^E dW_s - \frac{1}{2}\int_0^t (\widehat{\sigma}_s^E)^2 ds \right. \\ &\quad \left. + \int_0^{t+}\int_{\mathbf{R}_0} \widehat{g}^E(s,x) N(ds,dx) - \int_0^t\int_{\mathbf{R}_0} \widehat{f}^E(s,x)\nu(dx)ds\right\}. \end{aligned} \tag{6.3}$$

In the following, we prove that this P^E is the MEMM under some conditions. Let us calculate $\log G^E$. We have

$$\begin{aligned} \log G_t^E &= \int_0^t \widehat{\sigma}_s^E dW_s - \frac{1}{2}\int_0^t (\widehat{\sigma}_s^E)^2 ds + \int_0^{t+}\int_{\mathbf{R}_0} \frac{\widehat{\sigma}_s^E f(s,x)}{\sigma_s}\widetilde{N}(ds,dx) \\ &\quad - \int_0^t\int_{\mathbf{R}_0}\left[\widehat{f}^E(s,x) - \frac{\widehat{\sigma}_s^E f(s,x)}{\sigma_s}\right]\nu(dx)ds \\ &= \int_0^t \frac{\widehat{\sigma}_s^E}{\sigma_s}\sigma_s dW_s + \int_0^{t+}\int_{\mathbf{R}_0}\frac{\widehat{\sigma}_s^E f(s,x)}{\sigma_s} f(s,x)\widetilde{N}(ds,dx) + \int_0^t \frac{\widehat{\sigma}_s^E f(s,x)}{\sigma_s}\mu_s ds \\ &\quad -\frac{1}{2}\int_0^t (\widehat{\sigma}_s^E)^2 ds - \int_0^t \frac{\widehat{\sigma}_s^E \mu_s}{\sigma_s} ds \\ &\quad - \int_0^t\int_{\mathbf{R}_0}\left[\widehat{f}^E(s,x) - \frac{\widehat{\sigma}_s^E f(s,x)}{\sigma_s}\right]\nu(dx)ds \\ &=: \int_0^t \frac{\widehat{\sigma}_s^E}{\sigma_s} dS_s - J_t, \end{aligned}$$

say. Let us make sure that the process J is a negative process. We denote

$$h(t,x) := \frac{\widehat{\sigma}_t^E f(t,x)}{\sigma_t}.$$

Then, we have

$$\begin{aligned} J_t &= \frac{1}{2}\int_0^t (\widehat{\sigma}_s^E)^2 ds + \int_0^t \frac{\widehat{\sigma}_s^E \mu_s}{\sigma_s} ds \\ &\quad + \int_0^t \int_{\mathbf{R}_0} \left[\widehat{f}^E(s,x) - \frac{\widehat{\sigma}_s^E f(s,x)}{\sigma_s} \right] \nu(dx)ds \\ &= -\frac{1}{2}\int_0^t \left(\widehat{\sigma}_s^E\right)^2 ds - \int_0^t \int_{\mathbf{R}_0} [\exp\{h(s,x)\}(h(s,x)-1)+1]\,\nu(dx)ds. \quad (6.4) \end{aligned}$$

Since the function $e^x(x-1)+1$ is positive, the process J is non-positive. Now, we have the following theorem.

Theorem 6.2 *Under Assumptions E and F, if the process J is deterministic, then P^E, defined in 2 of Assumption F, is the MEMM. Hence, the density process of the MEMM is given by (6.3).*

Proof. For any $P^* \in \mathbf{M}$, the relative entropy with respect to P is represented as

$$\begin{aligned} H(P^*|P) &= H(P^*|P^E) + E^{P^*}[\log G_T^E] \\ &= H(P^*|P^E) - E^{P^*}[J_T], \end{aligned}$$

where E^{P^*} denotes expectation under P^*. Then, we have

$$H(P^*|P) + E^{P^*}[J_T] = H(P^*|P^E) \geq 0.$$

In particular, we have that $H(P^*|P^E) = 0$ if and only if $P^* = P^E$. This completes the proof of Theorem 6.2. □

Corollary 6.3 *Under Assumptions E and F, if the mean-variance trade-off process K, the process v_t^f/σ_t^2 and $f^E(t,x)$, defined in 1 of Assumption F, are deterministic, then the EMM P^E is the MEMM.*

Proof. It is enough to show that the process J is deterministic. Hence, by (6.4), the proof would be complete if we could show that $\widehat{\sigma}^E$ is deterministic. Firstly, by the condition (2.3), we can write

$$\frac{dK_t}{dt} = \frac{\mu_t^2/\sigma_t^2}{1 + v_t^f/\sigma_t^2}.$$

By the condition of this Corollary, we see that μ_t^2/σ_t^2 is deterministic. Next, we prove that $\widehat{\sigma}^E$, defined by (6.2), is deterministic. The second term of the right hand side of (6.2) is easily seen to be deterministic. On the other hand, by (2.2), we can write

$$\frac{\mu_t \sigma_t}{\sigma_t^2 + v_t^f} = \frac{\mu_t^2}{\sigma_t^2 + v_t^f}\frac{\sigma_t}{\mu_t}.$$

Consequently, $\widehat{\sigma}^E$ is deterministic. □

7 The Esscher transforms.

In this section, we investigate the Esscher transforms for Poisson jump type model. Also, we state relations between the Esscher transforms and the MEMM. Throughout this section, we assume that the stock price process is given by the solution of the following SDE:

$$S_t = S_0 + \int_0^t \mu_s S_{s-} ds + \int_0^t \sigma_s S_{s-} dW_s + \int_0^{t+} \int_{\mathbf{R}_0} f(s,x) S_{s-} \widetilde{N}(ds,dx).$$

We define $dR_t = dS_t/S_t$ and call the process $R = \{R_t\}_{0 \leq t \leq T}$ the return process of S. For a deterministic function $\zeta = \{\zeta_t\}_{0 \leq t \leq T}$, we define

$$U_t := \frac{e^{\zeta_t R_t}}{E\left[e^{\zeta_t R_t}\right]}.$$

Then, we call the process $U = \{U_t\}_{0 \leq t \leq T}$ the Esscher transform of the return process R with parameter function ζ. Remark that U is a martingale starting at 1. We assume that three predictables μ, σ and $f(t,x)$ are deterministic. Then, let us make sure that the Esscher transform of the return process coincides with the MEMM. Fujiwara and Miyahara (2001) obtained a part of this relation for only case of $f(t,x) = e^x - 1$. Chan (1999) also stated a part of this relation for only case of $f(t,x) = x$, but he did not represent this result explicitly. Indeed, we can extend these results to general deterministic function $f(t,x)$.

If we denote $C_t = e^{\zeta_t R_t}$, we have

$$\begin{aligned} dC_t &= \zeta_t \mu_t C_t dt + \zeta_t \sigma_t C_t dW_t + \frac{1}{2} {\zeta_t}^2 \sigma_t^2 C_t dt \\ &\quad + C_t \int_{\mathbf{R}_0} \left(e^{\zeta_t f(t,x)} - 1 \right) \widetilde{N}(dt,dx) \\ &\quad + C_t \int_{\mathbf{R}_0} \left(e^{\zeta_t f(t,x)} - 1 - \zeta_t f(t,x) \right) \nu(dx) dt. \end{aligned}$$

Moreover, if we denote $C'_t = E\left[e^{\zeta_t R_t}\right]$, we have

$$\begin{aligned} dC'_t &= \zeta_t \mu_t C'_t dt + \frac{1}{2} {\zeta_t}^2 \sigma_t^2 C'_t dt \\ &\quad + C'_t \int_{\mathbf{R}_0} \left(e^{\zeta_t f(t,x)} - 1 - \zeta_t f(t,x) \right) \nu(dx) dt. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} dU_t &= d\left(\frac{C_t}{C'_t}\right) \\ &= \frac{dC_t}{C'_t} + C_t d\left(\frac{1}{C'_t}\right) \\ &= U_t \zeta_t \sigma_t dW_t + U_t \int_{\mathbf{R}_0} \left(e^{\zeta_t f(t,x)} - 1 \right) \widetilde{N}(dt,dx). \end{aligned}$$

This can be restated that

$$U_t = \mathcal{E}\left(\int_0^\cdot \zeta_s \sigma_s dW_s + \int_0^\cdot \int_{\mathbf{R}_0} \left(e^{\zeta_s f(s,x)} - 1\right) \widetilde{N}(ds, dx)\right)_t.$$

This implies that, if we take $\zeta = \widehat{\sigma}^E/\sigma$, then all conditions of Corollary 6.3 is satisfied and we have $U = G^E$, that is, the Esscher transform of the return process coincides the MEMM.

Next, we consider the Esscher transform of $\log S$. We define a process $X = \{X_t\}_{0\leq t\leq T}$ by

$$S_t = S_0 e^{X_t},$$

that is, we have $d\log S = dX$. Now, we define the Esscher transform of X with parameter ζ by

$$V_t := \frac{e^{\zeta X_t}}{E\left[e^{\zeta X_t}\right]}.$$

We assume that $\zeta \neq 1$ and three predictable processes μ, σ and $f(t,x)$ are deterministic. Let us show that the Esscher transform of X does not coincides with the MEMM. Notice that a part of this relation was obtained by Chan (1999) for only cases $f(t,x) = e^{rx} - 1$ and $= rx$, where $r \in \mathbf{R}$. In this paper, we make sure that his result hold for general deterministic function $f(t,x)$, too.

Firstly, we have

$$\begin{aligned} dX_t &= \sigma_t dW_t + \mu_t dt - \frac{1}{2}\sigma_t^2 dt \\ &\quad + \int_{\mathbf{R}_0} \log(1 + f(t,x))\widetilde{N}(dt, dx) \\ &\quad + \int_{\mathbf{R}_0} \{\log(1 + f(t,x)) - f(s,x)\}\,\nu(dx)dt \end{aligned}$$

If we denote $D_t = e^{\zeta X_t}$, we have

$$\begin{aligned} dD_t &= d(S_t^\zeta) \\ &= \zeta\mu_t D_t dt + \zeta\sigma_t D_t dW_t + \frac{\zeta(\zeta-1)}{2}\sigma_t^2 D_t dt \\ &\quad + D_t \int_{\mathbf{R}_0} \left((1 + f(t,x))^\zeta - 1\right) \widetilde{N}(dt, dx) \\ &\quad + D_t \int_{\mathbf{R}_0} \left((1 + f(t,x))^\zeta - 1 - \zeta f(t,x)\right) \nu(dx)dt. \end{aligned}$$

Moreover, if we denote $D'_t = E\left[e^{\zeta X_t}\right]$, we have

$$\begin{aligned} dV_t &= d\left(\frac{D_t}{D'_t}\right) \\ &= \frac{dD_t}{D'_t} + D_t d\left(\frac{1}{D'_t}\right) \\ &= \zeta V_t \sigma_t dW_t + V_t \int_{\mathbf{R}_0} \left((1 + f(t,x))^\zeta - 1\right) \widetilde{N}(dt, dx). \end{aligned}$$

Consequently, we have

$$V_t = \mathcal{E}\left(\int_0^{\cdot} \zeta\sigma_s dW_s + \int_0^{\cdot}\int_{\mathbf{R}_0} \left((1+f(s,x))^{\zeta} - 1\right)\widetilde{N}(ds,dx)\right)_t.$$

From this, we can conclude that the Esscher transform of X does not coincides with the MEMM.

8 Examples.

In this section, we calculate the density processes of the MEMM's for three Poisson jump type models. The first one is Chan's model in Example 3.1. The second is the geometric Lévy process model. The third one is an extension case of Chan' model such that coefficients have randomness. We call it Chan's model with random coefficients.

Example 8.1 (Chan's model) Let the stock price process be given by (3.4). We assume all assumptions in Example 3.4. In addition, we assume the following:

Assumption There exists a unique deterministic function β_t satisfying

$$b_t + \sigma_t\beta_t c^2 + \sigma_t \int_{\mathbf{R}_0} x\left(e^{\beta_t x} - 1\right)\nu(dx) = 0.$$

We denote $v_2 := \int_{\mathbf{R}_0} x^2\nu(dx)$. Remark that we have $v_2 < \infty$, $e^{\beta_t x} - 1 \in \mathbf{F}_p^1$ and

$$\int_{\mathbf{R}_0} x\left(e^{\beta_t x} - 1\right)v(dx) < \infty \quad \text{for all } t \in [0,T],$$

because $\sigma_t S_{t-}x \in \mathbf{F}_p$ and ν is bounded. Now, we compute the mean-variance trade-off process K and $\widehat{\sigma}^E$ defined by (6.2) are given by

$$K_t = \int_0^t \frac{\mu_s^2}{\sigma_s^2(c^2+v_2)}ds,$$

$$\widehat{\sigma}_t^E = -\frac{b_t c}{\sigma_t(c^2+v_2)} - \frac{\int_{\mathbf{R}_0} x f^E(t,x)\nu(dx)}{c}.$$

We denote

$$f^E(t,x) := \left(e^{\beta_t x} - 1\right) + \frac{b_t x}{\sigma_t(c^2+v_2)}. \tag{8.1}$$

Then, we have $f^E \in \mathbf{F}_p^2$. Hence, $f^E \in \mathbf{F}_p$. Moreover, we assume that there exists an EMM, denoted P^E, whose reference function is f^E.

If the deterministic function f^E, defined by (8.1), satisfies Assumption F, then Chan's model satisfies all conditions of Corollary 6.3. We calculate the right hand side of (6.1).

$$\begin{aligned}
\text{R.H.S. of (6.1)} &= \frac{b_t x}{\sigma_t(c^2+v_2)} - 1 \\
&\quad + \exp\left\{ -\frac{b_t x}{\sigma_t(c^2+v_2)} - \frac{x}{c^2}\int_{\mathbf{R}_0} y f^E(t,y)\nu(dy) \right\} \\
&= \frac{b_t x}{\sigma_t(c^2+v_2)} - 1 + \exp\left\{ -\frac{b_t x}{\sigma_t(c^2+v_2)} \right. \\
&\quad \left. -\frac{x}{c^2}\int_{\mathbf{R}_0} \left(y e^{\beta_t y} - y + \frac{b_t y^2}{\sigma_t(c^2+v_2)} \right) \nu(dy) \right\} \\
&= \frac{b_t x}{\sigma_t(c^2+v_2)} + e^{\beta_t x} - 1 \\
&= f^E(t,x).
\end{aligned}$$

Hence, (6.1) holds. In consequence of this, Chan's model satisfies Assumption F. Namely, we can apply Corollary 6.3. Hence, P^E, defined by 2 of Assumption F, is the MEMM for Chan's model.

We calculate the density process G^E of the MEMM P^E. Firstly, we have

$$\begin{aligned}
\widehat{\sigma}_t^E &= -\frac{b_t c}{\sigma_t(c^2+v_2)} - \frac{1}{c}\int_{\mathbf{R}_0} x(e^{\beta_t x}-1)\nu(dx) - \frac{1}{c}\int_{\mathbf{R}_0} \frac{b_t x^2}{\sigma_t(c^2+v_2)}\nu(dx) \\
&= -\frac{b_t}{\sigma_t c} - \frac{1}{c}\int_{\mathbf{R}_0} x(e^{\beta_t x}-1)\nu(dx) \\
&= c\beta_t.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\widehat{f}^E(t,x) &= -\frac{b_t x}{\sigma_t(c^2+v_2)} + \frac{b_t x}{\sigma_t(c^2+v_2)} - 1 + e^{\beta_t x} \\
&= e^{\beta_t x} - 1
\end{aligned}$$

and $\widehat{g}^E(t,x) = \beta_t x$. We end up with

$$\begin{aligned}
G_t^E &= \exp\left\{ \int_0^t c\beta_s dW_s - \frac{1}{2}\int_0^t c^2\beta_s^2 ds \right. \\
&\quad \left. + \int_0^{t+}\int_{\mathbf{R}_0} \beta_s x N(ds,dx) - \int_0^t\int_{\mathbf{R}_0} \left(e^{\beta_s x}-1\right)\nu(dx)ds \right\}. \qquad (8.2)
\end{aligned}$$

Example 8.2 (The geometric Lévy process model) Let the stock price process S be the solution of the following SDE:

$$S_t = S_0 + \mu\int_0^t S_{s-}ds + \sigma\int_0^t S_{s-}dW_s + \int_0^{t+}\int_{\mathbf{R}_0} S_{s-}(e^x-1)N(ds,dx), \qquad (8.3)$$

where S_0 is a positive constant, and μ and σ are constants. The solution of (8.3) is represented as

$$S_t = S_0 \exp\left\{\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right)t + \int_0^{t+}\int_{\mathbf{R}_0} xN(ds,dx)\right\}.$$

We call the process S a geometric Lévy process. Remark that the process S is positive. We denote $\mu' := \mu + \int_{\mathbf{R}_0}(e^x - 1)\nu(dx)$. In addition, we assume the following:

Assumption 1. $S_{t-}(e^x - 1) \in \mathbf{F}_p$.

2. $\displaystyle\int_{\{x \geq \log 2\}} e^{2\beta(e^x - 1)}\nu(dx) < \infty$.

3. There exists a unique constant β such that

$$\mu + \beta\sigma^2 + \int_{\mathbf{R}_0}(e^x - 1)e^{\beta(e^x - 1)}\nu(dx) = 0.$$

We denote $v_2 := \displaystyle\int_{\mathbf{R}_0}(e^x - 1)^2\nu(dx)$. Remark that we have $v_2 < \infty$, $e^{\beta(e^x - 1)} - 1 \in \mathbf{F}_p^1$ and

$$\int_{\mathbf{R}_0}(e^x - 1)e^{\beta(e^x - 1)}\nu(dx) < \infty.$$

Moreover, the mean-variance trade-off process for this model is deterministic, and $\widehat{\sigma}^E$ is represented as

$$\widehat{\sigma}_t^E = -\frac{\mu'\sigma}{\sigma^2 + v_2} - \frac{\int_{\mathbf{R}_0}(e^x - 1)f^E(t,x)\nu(dx)}{\sigma}.$$

Now, we denote

$$f^E(t,x) = f^E(x) := e^{\beta(e^x - 1)} - 1 + \frac{\mu'(e^x - 1)}{\sigma^2 + v_2}.$$

We assume that there exists an EMM, denoted by P^E, whose reference function is f^E. We make sure that this $f^E(x)$ satisfies Assumption F. Firstly, we calculate the right hand side of (6.1) as follows.

$$\begin{aligned}
\text{R.H.S. of (6.1)} &= \frac{\mu'(e^x - 1)}{\sigma^2 + v_2} + \exp\left\{-\frac{\mu'(e^x - 1)}{\sigma^2 + v_2}\right. \\
&\quad \left. -\frac{e^x - 1}{\sigma^2}\int_{\mathbf{R}_0}(e^y - 1)f^E(y)\nu(dy)\right\} - 1 \\
&= \frac{\mu'(e^x - 1)}{\sigma^2 + v_2} + \exp\{\beta(e^x - 1)\} - 1 \\
&= f^E(x).
\end{aligned}$$

Therefore, (6.1) holds. Next, we show that $f^E \in \mathbf{F}_p^2$. By Assumption above, we have only to show that

$$\int_{\{x \le \log 2\}} \left| e^{\beta(e^x-1)} - 1 \right|^2 \nu(dx) < \infty,$$

for $\beta > 0$. For $0 < \theta < 1$ and a constant $\phi > 0$, we have

$$\theta \left(e^{\phi} - 1 \right) > e^{\phi\theta} - 1,$$

and hence

$$\int_0^{\log 2} |e^{\beta(e^x-1)} - 1|^2 \nu(dx) \le \left(e^{\beta} - 1 \right)^2 \int_0^{\log 2} |e^x - 1|^2 \nu(dx) < \infty.$$

On the other hand, for $-1 < \theta < 0$ and a constant $\phi > 0$, we have

$$-\theta(1 - e^{-\phi}) > 1 - e^{\phi\theta},$$

and thus

$$\int_{-\infty}^0 |e^{\beta(e^x-1)} - 1|^2 \nu(dx) \le \left(1 - e^{-\beta} \right)^2 \int_{-\infty}^0 |e^x - 1|^2 \nu(dx) < \infty.$$

Hence, we have that $f^E \in \mathbf{F}_p^2$, namely, $f^E \in \mathbf{F}_p$. Consequently, the present model satisfies all conditions of Corollary 6.3. Namely, P^E being as in 2 of Assumption F is the MEMM. We have

$$\begin{aligned} \widehat{\sigma}_t^E &= \frac{\mu'\sigma}{\sigma^2 + v_2} - \frac{\int_{\mathbf{R}_0} (e^x - 1) f^E(t,x) \nu(dx)}{\sigma} \\ &= -\frac{1}{\sigma} \left(\mu + \int_{\mathbf{R}_0} (e^x - 1) e^{\beta(e^x-1)} \nu(dx) \right) \\ &= \beta\sigma. \end{aligned}$$

Furthermore, we have $\widehat{f}^E(t,x) = \widehat{f}^E(x) = \exp\{\beta(e^x - 1)\} - 1$ and $\widehat{g}^E(t,x) = \widehat{g}^E(x) = \beta(e^x - 1)$. Finally,

$$\begin{aligned} G_t^E &= \exp\left\{ \beta\sigma W_t - \frac{1}{2}(\beta\sigma)^2 t \right. \\ &\quad \left. + \int_0^{t+} \int_{\mathbf{R}_0} \beta(e^x - 1) N(ds, dx) - \int_0^t \int_{\mathbf{R}_0} \left(e^{\beta(e^x-1)} - 1 \right) \nu(dx) ds \right\}. \end{aligned}$$

Example 8.3 (Chan's model with random coefficients) Let the stock price process S be given by

$$S_t = S_0 + \int_0^t \mu_s S_{s-} ds + \int_0^t \sigma_s S_{s-} dW_s + \int_0^{t+} \int_{\mathbf{R}_0} \lambda_s S_{s-} x \widetilde{N}(ds, dx),$$

where μ_t, σ_t and λ_t are continuous bounded $\mathbf{F}$-predictable processes and S_0 is a positive constant. This model is corresponding to the case $f(t, x, S_{t-}) = \lambda_t S_{t-} x$. We assume the following:

Assumption 1. $\lambda_t S_{t-} x \in \mathbf{F}_p$.

2. The support of ν is bounded, that is, there exist two constants $0 \leq c_1, c_2 < \infty$ such that
$$\mathrm{supp}(\nu) = [-c_1, c_2].$$

3. The predictable processes μ_t, σ_t and λ_t are mutually absolutely continuous. Furthermore, the quotients μ_t/σ_t and μ_t/λ_t are deterministic.

4. There exists a unique $\mathbf{F}$-predictable process β_t satisfying
$$\mu_t + \beta_t\sigma_t^2 + \lambda_t \int_{\mathbf{R}_0} x \left(e^{\beta_t\lambda_t x} - 1\right) \nu(dx) = 0$$
and the product $\beta_t\lambda_t$ is deterministic.

We denote $v_2 := \displaystyle\int_{\mathbf{R}_0} x^2 \nu(dx)$ and
$$f^E(t,x) := \left(e^{\beta_t\lambda_t x} - 1\right) + \frac{\mu_t\lambda_t x}{\sigma_t^2 + \lambda_t^2 v_2}.$$

Then, f^E is deterministic and belongs to $\mathbf{F}_p$. We assume that there exists an EMM, denoted P^E, whose reference function is f^E. It is easily to show that the present model satisfies all conditions of Corollary 6.3. Hence, we can calculate the density process of MEMM P^E, denoted G^E. We obtain
$$\begin{aligned} G_t^E &= \exp\left\{ \int_0^t \beta_s\sigma_s dW_s - \frac{1}{2}\int_0^t \beta_s^2\sigma_s^2 ds \right. \\ &\quad \left. + \int_0^{t+}\int_{\mathbf{R}_0} \lambda_s\beta_s x N(ds,dx) - \int_0^t\int_{\mathbf{R}_0} \left(e^{\lambda_s\beta_s x} - 1\right)\nu(dx)ds \right\}. \end{aligned}$$

9 Concluding remarks.

1. Even if we replace W by a d-dimensional Brownian motion ($d \geq 2$), Tang-Li's representation theorem (Theorem 5.1) remains true. Hence, for a d-dimensional Brownian motion, we can obtain the same result as Theorem 5.2.

Moreover, we can extend our results to the general RCLL semimartingale model holding martingale representation theorem under some adequate conditions. Namely, if let S be the stock price process which is a RCLL special semimartingale, M the local martingale part of S which is a square integrable martingale, and $\mathbf{F}$ the filtration generated by M, then any square integrable $\mathbf{F}$-adapted P-martingales are represented as a stochastic integral of M. In this case, under Assumption C or C′, the reference local martingale is represented as a stochastic integral of M. By the results of Examples 3.4 and 8.1, we may show that the MMM does not coincide with the MEMM in general RCLL semimartingale case.

2. We have defined the MEMM as an EMM which minimizes the relative entropy, that is, the MEMM is an EMM defined by mathematical motivation. However, we have not known yet the financial meaning of the MEMM. In order to consider this problem, it is important to investigate the relationship to the MMM having financial meaning. In the continuous case, Arai (2001) studied the relationship between the MMM and the MEMM by using the concept of the base of filtration. In particular, if the stock price process is given by the solution of a SDE, then the MMM coincides with the MEMM.

Contrary to this, the results of Examples 3.4 and 8.1 mean that the MMM does not coincide with the MEMM in Chan's model. Namely, we can say that the MMM does not coincide with the MEMM in general jump type model. The reason is as follows. For two continuous semimartingales X and Y, if X is strongly orthogonal to Y, then we have

$$\mathcal{E}(X+Y) = \mathcal{E}(X)\mathcal{E}(Y). \tag{9.1}$$

For general RCLL semimartingales, (9.1) does not hold, since $[X,Y] \neq \langle X,Y \rangle$. However, we can see an interesting connection between the MMM and the MEMM. In Examples 3.5 and 8.1, let coefficient functions b_t and σ_t be constants. Now, let us set $\gamma_t = 1$ in Example 3.5 and $\beta_t = 1$ in Example 8.1. Then, the density process G^M being as in (3.6) is equivalent to the density process G^E being as in (8.2). Very roughly speaking, the MMM with respect to $e^x - 1$ is corresponding to the MEMM with respect to x. Whereas, we can regard Chan's model as the Black-Scholes model with Poisson jumps. Hence, the MMM for the geometric Lévy process model is corresponding to the MEMM for the Black-Scholes model with Poisson jumps.

References

[1] ANSEL, J. P. AND STRICKER, C. (1992). Lois de martingale, densités et décomposition de Föllmer Schweizer. *Ann. Inst. H. Poincaré Probab. Statist.* **28**, 375–392.

[2] ARAI, T. (2001). The relations between minimal martingale measure and minimal entropy martingale measure. To appear in *Asia Pacific Financial Markets.*

[3] CHAN, T. (1999). Pricing contingent claims on stocks driven by Lévy processes. *Ann. Appl. Prob.* **9**, 504–528.

[4] FÖLLMER, H. AND SCHWEIZER, M. (1991). Hedging of contingent claims under incomplete information. In: *Applied Stochastic Analysis*, ed. Davis, M. H. A. and Elliott, R. J. Stochastic Monographs, Vol. 5. Gordon and Breach, New York, pp. 389–414.

[5] FÖLLMER, H. AND SONDERMANN, D. (1986). Hedging of non-redundant contingent claims. In: *Contributions to Mathematical Economics*, ed. Hildenbrand, W. and Mas-Colell, A. pp. 205–223.

[6] FRITTELLI, M. (2000). The minimal entropy martingale measures and the valuation problem in incomplete markets. *Math. Fin.* **10,** 39–52.

[7] FUJIWARA, T. AND MIYAHARA, Y. (2001). The minimal entropy martingale measure for geometric Lévy processes. Preprint.

[8] LEPINGLE, D. AND MÉMIN, J. (1978). Sur l'intégrabilité uniforme des martingales exponentielles. *Z. Wahrscheinlichkeitsth.* **42,** 175–203.

[9] MIYAHARA, Y. (1996a). Canonical martingale measures of incomplete assets markets. In: *Probability Theory and Mathematical Statistics: Proceedings of the Seventh Japan-Russia Symposium, Tokyo 1995.* World Sci. pp. 343–352.

[10] MIYAHARA, Y. (1996b). Canonical martingale measures and minimal martingale measures of incomplete assets markets. *The Australian National University Research Report.* No.FMRR **007-96,** 95–100.

[11] MIYAHARA, Y. (1999). Minimal entropy martingale measures of jump type price processes in incomplete assets markets. *Asia Pacific Fin. Markets.* **6,** 97–113.

[12] MONAT, P. AND STRICKER, C. (1995). Föllmer-Schweizer decomposition and mean-variance hedging of general claims. *Ann. Prob.* **23,** 605–628.

[13] PROTTER, P. (1990). *Stochastic Integration and Differential Equations. A new approach*, Springer-Verlag, Berlin.

[14] SCHWEIZER, M. (1991). Option hedging for semimartingales. *Stoch. Proc. Appl.* **37,** 339–363.

[15] SCHWEIZER, M. (1995). On the minimal martingale measure and the Föllmer-Schweizer decomposition. *Stoc. Anal. Appl.* **13,** 573–599.

[16] SCHWEIZER, M. (1999). A guided tour through quadratic hedging approaches. Preprint.

[17] SITU, R. (1985). On strong solution, uniqueness, stability and comparison theorems for a stochastic system with Poisson jumps. In: *Distributed Parameter System*, Lecture Notes in Control & Infor. Sci.,Vol. 75, pp. 352–381.

[18] TANG, S. AND LI, X. (1994). Necessary conditions for optimal control of stochastic systems with random jumps. *SIAM J. Control Optim.* **32,** 1447–1475.