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An Explicit Formula for the Projectively Invariant Quantization on Degree Three on a Manifold

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Abstract

We give an explicit formula for the projectively invariant quantization map between the space of symbols of degree three and the space of third-order linear differential operators, both viewed as modules over the group of diffeomorphisms and the Lie algebra of vector fields on a manifold.

Une formule explicite pour la quantification projectivement invariante en degré trois sur une variété différentiable

Résumé

Nous donnerons une formule explicite pour la quantification projectivement invariante entre l'espace des symboles de degrés trois et l'espace des opérateurs différentiels linéaires d'ordres trois, vus comme modules sur le groupe des difféomorphismes et l'algèbre de Lie des champs de vecteurs sur une variété différentiable.

1 Introduction

Let M be a manifold of dimension n. Fix an affine connection ∇ on M. Denote by $\mathcal{F}_\lambda(M)$ the space of $\lambda-$densities on M (i.e. sections of the bundle $(\wedge^n T^*M)^{\otimes\lambda}$). This space admits naturally a structure of module over the group of diffeomorphisms Diff(M) and the Lie algebra of vector fields Vect(M). Consider $\mathcal{D}_{\lambda,\mu}(M)$ the space of linear differential operators acting from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$. This space is a module over Diff(M) and Vect(M) (see [1, 3, 6, 7]). The action is given as follows: take $f\in$ Diff(M) and $A\in\mathcal{D}_{\lambda,\mu}(M)$ then

$$f^*A = f^*_\mu\circ A\circ f^{*-1}_\lambda, \tag{1.1}$$

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KSTS/RR-01/003
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where f_λ^* is the standard action of a diffeomorphism on $\mathcal{F}_\lambda(M)$.

Differentiating the action of the flow of a vector field, one gets the corresponding action of $\mathrm{Vect}(M)$.

Denote by $\mathcal{D}^3_{\lambda,\mu}(M)$ the space of third-order linear differential operators endowed with the structure of module (1.1). The module $\mathcal{D}^3_{\lambda,\mu}(M)$ is viewed as a submodule of $\mathcal{D}_{\lambda,\mu}(M)$.

Consider now $\mathrm{Pol}(T^*M)$ the space of functions on the cotangent bundle T^*M, polynomials on the fibers. This space is naturally isomorphic to the space of symmetric contravariant tensor fields on M. One can define a one-parameter family of $\mathrm{Diff}(M)-$modules by taking $\mathrm{Pol}_\delta(T^*M) := \mathrm{Pol}(T^*M) \otimes \mathcal{F}_\delta(M)$. Let us give explicitly this action: take $f \in \mathrm{Diff}(M)$ and $P \in \mathrm{Pol}_\delta(T^*M)$ then

$$f_\delta^* P = f^* P \cdot (J_f)^{-\delta}, \tag{1.2}$$

where f^* is the natural action of a diffeomorphism on contravariant tensor fields, and J_f is the Jacobian of f.

Differentiating the action of the flow of a vector field, one gets the corresponding action of $\mathrm{Vect}(M)$.

Denote by $\mathrm{Pol}^3_\delta(T^*M)$ the space of symbols of degree three endowed with the module structure given by (1.2).

Suppose $M := \mathbb{R}^n$ is endowed with a flat projective structure (coordinates change are projective transformations). In this case, Lecomte and Ovsienko in [6] construct a quantization map between the space $\mathrm{Pol}_\delta(T^*\mathbb{R}^n)$ and the space $\mathcal{D}_{\lambda,\mu}(\mathbb{R}^n)$, equivariant with respect to the action of the Lie algebra $\mathrm{sl}_{n+1}(\mathbb{R}) \subset \mathrm{Vect}(\mathbb{R}^n)$. Consider now any manifold M and fix an affine connection on it. It is interesting to ask if there exists a canonical quantization map associated to the given connection. On degree two, the author construct in [1] a quantization map depending only on the projective class of the connection (see also [3] for the conformal case). This approach generalizes Lecomte and Ovsienko's approach for the flat case. On higher order, the problem of existence of the projectively invariant quantization map is open.

2 Main result

The main result of this note is

Theorem 2.1 *For $n > 1$, and for $\delta \neq \frac{n+3}{n+1}, \frac{n+4}{n+1}, \frac{n+5}{n+1}$, there exits a quantization map $Q : \mathrm{Pol}^3_\delta(T^*M) \to \mathcal{D}^3_{\lambda,\mu}(M)$ given by*

$$\begin{aligned} P^{ijk} \mapsto\ & P^{ijk}\nabla_i\nabla_j\nabla_k + \alpha\nabla_k P^{ijk}\,\nabla_i\nabla_j + \left(\beta_1\nabla_i\nabla_j P^{ijk} + \beta_2 P^{ijk}R_{ij}\right)\nabla_k \\ & + (\eta_1\nabla_i\nabla_j\nabla_k P^{ijk} + \eta_2 R_{ij}\nabla_k P^{ijk} + \eta_3\nabla_i R_{jk}\,P^{ijk}) \end{aligned} \tag{2.1}$$

where R_{ij} are the components of the Ricci tensor of the connection ∇, the constants $\alpha, \beta_1, \beta_2, \eta_1, \eta_2, \eta_3$, are given by

$$\alpha = \frac{6 + 3\lambda(1+n)}{4 + (1-\delta)(1+n)}, \qquad \beta_1 = \frac{1 + \lambda(n+1)}{3 + (1-\delta)(1+n)}\alpha,$$

$$\beta_2 = \frac{2 + 3\lambda(1+n) - (4 + (1-\delta)(1+n))\beta_1}{n-1}, \qquad \eta_1 = \frac{\lambda(1+n)}{(6 + 3(1-\delta)(1+n))}\beta_1,$$

$$\eta_3 = \frac{\lambda(1+n) - \eta_1(4 + (1-\delta)(1+n))}{n-1}, \qquad \eta_2 = \frac{\lambda(1+n)\alpha - (10 + 3(1-\delta)(1+n))\eta_1}{n-1},$$

and have the following properties
(i) It depends only on the projective class of the connection ∇ *(see [5]).*
(ii) If $M = \mathbb{R}^n$ *is endowed with a flat projective structure the map (2.1) is the unique map that preserves the principal symbols, equivariant with respect to the action of the Lie algebra* $\mathrm{sl}_{n+1}(\mathbb{R}) \subset \mathrm{Vect}(\mathbb{R}^n)$.

The proof is straightforward.
For the particular values of δ :

Proposition 2.2 *If* $\delta = \frac{n+3}{n+1}, \frac{n+4}{n+1}, \frac{n+5}{n+1}$, *there exists a quantization map given by (2.1) with particular values of* λ, μ, *given in the following table*

δ	λ	μ	α	β_1	β_2	η_1	η_2	η_3
$\frac{n+5}{n+1}$	$\frac{-2}{n+1}$	$\frac{n+3}{n+1}$	t	t	$\frac{4}{1-n}$	$\frac{1}{3}t$	$\frac{4}{3}\frac{t}{(1-n)}$	$\frac{2}{1-n}$
$\frac{n+4}{n+1}$	$\frac{-2}{n+1}$	$\frac{n+2}{n+1}$	0	t	$\frac{4+t}{(1-n)}$	$\frac{2}{3}t$	$\frac{2}{3}\frac{t}{(1-n)}$	$\frac{6+2t}{(3-3n)}$
$\frac{n+4}{n+1}$	$\frac{-1}{n+1}$	$\frac{n+3}{n+1}$	3	t	$\frac{1+t}{1-n}$	$\frac{1}{3}t$	$\frac{9+t}{3-3n}$	$\frac{3+t}{3-3n}$
$\frac{n+3}{n+1}$	$\frac{-2}{n+1}$	$\frac{n+1}{n+1}$	0	0	$\frac{4}{1-n}$	t	$\frac{4}{1-n}t$	$2\frac{1+t}{1-n}$
$\frac{n+3}{n+1}$	$\frac{-1}{n+1}$	$\frac{n+2}{n+1}$	$\frac{3}{2}$	0	$\frac{1}{1-n}$	t	$\frac{1}{2}\frac{(8t+3)}{(1-n)}$	$\frac{1+2t}{1-n}$
$\frac{n+3}{n+1}$	0	$\frac{n+3}{n+1}$	3	3	$\frac{4}{1-n}$	t	$\frac{4}{1-n}t$	$\frac{2}{1-n}t$

Here t is a parameter.

Remark 2.3 (i) For the particular values of δ, the quantization map (2.1) is not unique (it is given by the parameter t).

(ii) In the one dimensional case, the quantization map was given in [2, 4].

(iii) Another approach to the quantization map equivariant with respect to the action of the conformal group in a Riemannian manifold was given in [3, 7].

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