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Deformation quantization of Fréchet-Poisson algebras of Heisenberg type

by

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Deformation quantization of Fréchet-Poisson algebras of Heisenberg type

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Abstract We discuss on the convergence problem of the star product for the linear Poisson algebra associated with the Heisenberg Lie algebra in a certain class of entire functions. The critical exponent of entire functions to extend the star product is obtained. We also study collapsing phenomena for convergence of the star product and give a way to extend the algebra keeping associativity via breaking symmetry.

Keywords: deformation quantization, Poisson algebra, Heisenberg Lie algebra.
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1 Introduction

The 3-dimensional Heisenberg Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ is the Lie algebra generated by x, y, z with the relation $[x, y] = z$. Viewing these generators as linear functions on the dual space $\mathfrak{g}^*$, we have a linear Poisson structure on $\mathfrak{g}^* = \mathbb{C}^3$

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with the Poisson bracket $\{x,y\}_H = z$. The Poisson algebra we discuss in this paper is as follows: On the complex 3-space $\mathbb{C}^3$, set

$$\{f,g\}_H = z(f(\overleftarrow{\partial_x}\cdot\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\cdot\overrightarrow{\partial_x})g), \tag{1.1}$$

for functions $f{=}f(z,x,y)$ and $g{=}g(z,x,y)$, where $\overleftarrow{\partial_x}\cdot\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\cdot\overrightarrow{\partial_x}$ stands for a bidifferential operator:

$$f(\overleftarrow{\partial_x}\cdot\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\cdot\overrightarrow{\partial_x})g = \partial_x f\cdot\partial_y g - \partial_y f\cdot\partial_x g. \tag{1.2}$$

We have

$$\{z,x\}_H = 0, \quad \{z,y\}_H = 0, \quad \{x,y\}_H = z, \tag{1.3}$$

which gives a linear Poisson structure on $\mathbb{C}^3$ associated with the Heisenberg Lie algebra.

The purpose of this paper is to give concrete examples of deformation quantizations of the Poisson algebra (1.1) endowed with a linear Fréchet structure.

A commutative associative Fréchet algebra $\mathcal{F}$ over $\mathbb{C}$ is called a *Fréchet-Poisson algebra* if $\mathcal{F}$ has a continuous Poisson bracket operation $\{\ ,\ \} : \mathcal{F}\times\mathcal{F}\to\mathcal{F}$, that is, skew symmetric biderivation satisfying the Jacobi identity.

Let $\mathcal{E}(\mathbb{C}^3)$ be the space of all holomorphic functions on $\mathbb{C}^3$. Then, together with the Poisson structure $\{\ ,\ \}_H$, $\mathcal{E}(\mathbb{C}^3)$ is a Fréchet-Poisson algebra under a suitable Fréchet topology.

Viewing the deformation parameter $\hbar$ as a formal parameter, we have a deformation quantization of the Fréchet-Poisson algebra $(\mathcal{E}(\mathbb{C}^3),\cdot,\{\ ,\ \}_H)$ by the Moyal product formula:

$$f *_\hbar g = \sum_{p=0}^{\infty}\frac{(i\hbar z)^p}{2^p p!}(f(\overleftarrow{\partial_x}\cdot\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\cdot\overrightarrow{\partial_x})^p g), \tag{1.4}$$

where $(\overleftarrow{\partial_x}\cdot\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\cdot\overrightarrow{\partial_x})^p$ denotes the p-th power of the bidifferential operator (1.2), and $\mathcal{E}(\mathbb{C}^3)[[\hbar]]$ is a Fréchet space under a suitable topology.

We now view $\hbar$ as a non-zero parameter, for simplicity, set $\hbar = 1$. Then, the Moyal product formula (1.4) is not defined over $\mathcal{E}(\mathbb{C}^3)$ in general. However, it satisfies the following:

(a) $f * g$ is defined if either f or g is a polynomial.

(b) The associativity $f * (g * h) = (f * g) * h$ holds, if two of f, g, h are polynomials.

Remark that the properties (**a**) and (**b**) hold even if the function space $\mathcal{E}(\mathbb{C}^3)$ is replaced by any $Hol(U)$; the space of all holomorphic functions on any open subset U of $\mathbb{C}^3$.

If $U = \mathbb{C}_* \times \mathbb{C}^2$ where $\mathbb{C}_* = \mathbb{C} - \{0\}$, then we see that z is invertible and

$$x^\bullet = \frac{1}{x}(1 - e^{-\frac{2i}{z}xy}), \quad x^\circ = \frac{1}{x}(1 - e^{\frac{2i}{z}xy}) \in Hol(U),$$

and it is easy to see by (1.4) with $\hbar = 1$ that

$$(1.5) \qquad x^\bullet * x = 1 = x * x^\circ, \quad x * x^\bullet = 1 - 2e^{-\frac{2i}{z}xy}, \quad x^\circ * x = 1 - 2e^{\frac{2i}{z}xy}.$$

This means the coordinate function x has both a left inverse and a right inverse. Since $x^\bullet \neq x^\circ$, the associativity fails:

$$(1.6) \qquad (x^\bullet * x) * x^\circ \neq x^\bullet * (x * x^\circ).$$

On the other hand, in [9], we have proposed the notion of the deformation quantization of a Fréchet-Poisson algebra, which is meant the convergence star product for the Poisson algebras in Fréchet categories. Similar notion has been studied in the C^* framework by Natsume [3], Natsume-Nest [4] and Rieffel [11]. We discuss how the associativity breaking (1.6) affects the deformation quantization of a Fréchet-Poisson algebras.

Let us recall a notion of deformation quantization of Fréchet-Poisson algebra $(\mathcal{F}, \cdot, \{,\})$ (cf. [9]-[10]):

Definition 1.1 *Let $\hbar \in \mathbb{R}$ and $(\mathcal{F}, \cdot, \{,\})$ a Fréchet-Poisson algebra. $(\mathcal{F}, *_\hbar)$ is called a deformation quantization of $(\mathcal{F}, \cdot, \{,\})$ if it satisfies the following:*

- *For each $\hbar$, $(\mathcal{F}, *_\hbar)$ is an associative Fréchet algebra.*
- *$f *_\hbar g$ is continuous in $\hbar$, and moreover $f *_\hbar g \to f \cdot g$ as $\hbar \to 0$ for every $f, g \in \mathcal{F}$.*
- *$\frac{1}{\hbar i}(f *_\hbar g - f \cdot g) \to \frac{1}{2}\{f, g\}$ as $\hbar \to 0$ for every $f, g \in \mathcal{F}$.*

Introducing a system of seminorms on $\mathcal{E}(\mathbb{C}^3)$, we define classes of Fréchet algebras $\mathcal{E}_p(\mathbb{C}^3)$ of entire functions on $\mathbb{C}^3$, where the label $p=(p_0, p_1, p_1)$ is assigned by the system of seminorms. Briefly speaking, the Fréchet algebra $\mathcal{E}_p(\mathbb{C}^3)$ is the subset of entire functions on $\mathbb{C}^3$ with the growth order less than p_0, p_1, p_1 for z, x, y-variables (see §2). We define for every $p_0 > 0, p_1 > 0$ a linear Poisson bracket $\{\ ,\ \}_H$ on $\mathcal{E}_p(\mathbb{C}^3)$ associated with the Heisenberg Lie algebra structure (see (2.4)) so that $(\mathcal{E}_p(\mathbb{C}^3), \cdot, \{\ ,\ \}_H)$ is a Fréchet-Poisson algebra. The case $p_0 = \infty$ is read that $f(z,x,y)$ is only holomorphic with respect to the variable z without any growth condition, and it is denoted by *Hol* instead of p_0.

In §4, we show that $(\mathcal{E}_p(\mathbb{C}^3), \cdot, *)$ is a deformation quantization of $(\mathcal{E}_p(\mathbb{C}^3), \cdot, \{\ ,\ \}_H)$, if and only if $0 < p_1 \leq \frac{2p_0}{p_0+1}$, or $p_0 = Hol$ and $0 < p_1 \leq 2$ (Theorem 2.1), and these are isomorphic to quotient algebras of completions of free tensor algebras $\mathcal{T}^3_{\frac{1}{p_0},\frac{1}{p_1}}$ and $\mathcal{T}^3_{Hol,\frac{1}{p_1}}$ by the closure of the ideal generated by the relation of the 3-dimensional Heisenberg Lie algebra, respectively (Theorem 4.1, Theorem 4.2, Theorem 4.3). In both cases, p_1 is restricted in the interval $0 < p_1 \leq 2$.

We find that the class of Fréchet algebras obtained above is not so large. Through computations of star exponential function (see in the proof of Theorem 5.1), we show that the star exponential functions of quadratics are not included.

The growth condition to be a deformation quantization relates to the associativity breaking (1.6). We have showed in [9] that the Moyal product made sense as a convergent product on the space of entire functions on $\mathbb{C}^2$ of order less than 2, but it failed the associative properties for entire functions of order ≥ 2. The techniques in [9] proceeds to find a similar phenomena to [9] even in this paper. Rough idea is as follows: Even in the case $p_1 > 2$, the $*$-product has the properties (**a**) and (**b**), and z commutes with any other elements. Thus, we can consider the restriction of our system to the "submanifold" $z = a \neq 0$. However, this procedure causes the associativity breaking (1.6).

Since the completion $\mathcal{T}^3_{Hol,\frac{1}{p_1}}$ of the free tensor algebra in §3 is a topological associative algebra for any $p_1 > 0$, the quotient algebra $\mathcal{A}^3_{Hol,\frac{1}{p_1}}$ of $\mathcal{T}^3_{Hol,\frac{1}{p_1}}$ by the closure of the ideal generated by the relation of the 3-dimensional Heisenberg Lie algebra is also an associative algebra. The associativity breaking (1.6) in the case $2 < p_1$ affects the degeneracy of the quotient algebra.

In §5, we study the case $p_1 > 2$. In this case, the quotient algebra $\mathcal{A}^3_{Hol,\frac{1}{p_1}}$ has the property similar to the formal deformation quantization [1]. Namely, if one substitute the central element to z a non-zero number, then the algebra collapses to the trivial one $\{0\}$ (cf. in §.5).

Through this argument, we realize that *restriction* of our system should not be taken carelessly. It may cause another difficulty of divergence. If we restricts the complex variable x to the real line, then $x - ia$, $a \in \mathbb{R}_+$, should be invertible, but $x - ia$ has a left inverse $(x - ia)^\bullet$ in a suitably extended system. Since there is no reason to set $(x - ia)^{-1} = (x - ia)^\bullet$, this makes another breakdown of the associativity. Thus, the restriction of the domain seems a crucial problem in treating non-commutative variables. Confiding a notion of *non-commutative submanifolds* seems to need a various strange phenomena.

In order to keep associativity, we have to break $SL(2, \mathbb{C})$-symmetry of our system discussed above. In §6, we impose for functions $f(x, y, z) \in Hol(\mathbb{C}^3)$ the condition that f is rapidly decreasing in $(x, y) \in \mathbb{R}^2$.

We show that a certain space, type S space, given by Gel'fand-Shilov [2] is closed under the $*$-product.

2 Fréchet-Poisson algebras

Let $\mathbb{C}^{n+1}$ be a complex $(n+1)$-space with the complex coordinates

$$(x_0, x_1, \cdots, x_n),$$

and $\mathcal{P}(\mathbb{C}^{n+1})$ the set of all polynomials on $\mathbb{C}^{n+1}$. The usual pointwise multiplication

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

for polynomial functions f, g on $\mathbb{C}^{n+1}$ gives a commutative associative structure on $\mathcal{P}(\mathbb{C}^{n+1})$.

2.1 Fréchet algebras of $\mathbb{C}^{N+1}$.

Tracing mainly the book of Gel'fand Shilov [2], we first introduce a system of seminorms on the set of polynomials to obtain Fréchet algebras by taking completions.

To define a system of seminorms, we use the following notations: $\tilde{p}$ and $\tilde{b}$, etc. denotes $(n+1)$-tuples $\tilde{p} = (p_0, p_1, \cdots, p_n)$ and $\tilde{b} = (b_0, b_1, \cdots, b_n)$ with $p_i > 0$ and $b_i > 0$ for $0 \leq i \leq n$. By forgetting p_0 and b_0, $\tilde{p}_*$ and $\tilde{b}_*$ denote $\tilde{p}_* = (p_1, \cdots, p_n)$ and $\tilde{b}_* = (b_1, \cdots, b_n)$ respectively.

Definition 2.1 *Let r_0 and N_0 be positive real numbers and non negative integers, respectively. We define seminorms $||\cdot||_{\tilde{p},\tilde{b}}$, $||\cdot||_{\tilde{p}_*,\tilde{b}_*,r_0}$ and $||\cdot||_{\tilde{p}_*,\tilde{b}_*,N_0}$ on $\mathcal{P}(\mathbb{C}^{n+1})$ as follows:*

$$(2.1) \qquad ||f||_{\tilde{p},\tilde{b}} = \sup_{(x_0,\cdots,x_n)\in\mathbb{C}^{n+1}} |f| \exp(-\sum_{i=0}^{n} b_i |x_i|^{p_i}),$$

$$(2.2)\qquad ||f||_{\tilde{p}_*,\tilde{b}_*,r_0} = \sup_{|x_0|\le r_0}\ \sup_{(x_1,\cdots,x_n)\in\mathbb{C}^n} |f|\exp(-\sum_{i=1}^{n} b_i|x_i|^{p_i}),$$

$$(2.3)\qquad ||f||_{\tilde{p}_*,\tilde{b}_*,N_0} = \sum_{k=0}^{N_0} ||f_k(x_1,\cdots,x_n)||_{\tilde{p}_*,\tilde{b}_*},$$

where we expand f *as* $f(x_0,x_1,\cdots,x_n)=\sum_{k=0}^{\infty} f_k(x_1,\cdots,x_n)x_0^k$ *as a power series of* x_0 *variable.*

We denote the completions of $\mathcal{P}(\mathbb{C}^{n+1})$ under systems of seminorms $\{||\cdot||_{\tilde{p},\tilde{b}}\}_{\tilde{b}}$, $\{||\cdot||_{\tilde{p}_*,\tilde{b}_*,r_0}\}_{\tilde{b}_*,r_0}$ and $\{||\cdot||_{\tilde{p}_*,\tilde{b}_*,N_0}\}_{\tilde{b}_*,N_0}$ respectively by

(E.1): $\mathcal{E}_{\tilde{p}}(\mathbb{C}^{n+1})$, (E.2): $\mathcal{E}_{Hol,\tilde{p}_*}(\mathbb{C}^{n+1})$, (E.3): $\mathcal{E}_{\infty,\tilde{p}_*}(\mathbb{C}^{n+1})$.

The notation $\mathcal{E}_{\Omega}(\mathbb{C}^{n+1})$ stands for one of $\mathcal{E}_{\tilde{p}}(\mathbb{C}^{n+1})$, $\mathcal{E}_{Hol,\tilde{p}_*}(\mathbb{C}^{n+1})$ and $\mathcal{E}_{\infty,\tilde{p}_*}(\mathbb{C}^{n+1})$.

Lemma 2.1 $\mathcal{E}_{\Omega}(\mathbb{C}^{n+1})$ *is a commutative associative Fréchet algebra.*

Remark that $\mathcal{E}_{\tilde{p}}(\mathbb{C}^{n+1})$ and $\mathcal{E}_{Hol,\tilde{p}_*}(\mathbb{C}^{n+1})$ are subalgebras of all entire functions $\mathcal{E}(\mathbb{C}^{n+1})$ on $\mathbb{C}^{n+1}$, and $\mathcal{E}_{\infty,\tilde{p}_*}(\mathbb{C}^{n+1})$ is the space $\mathcal{E}_{\tilde{p}_*}(\mathbb{C}^n)[[x_0]]$ of all formal power series of x_0 with coefficients in $\mathcal{E}_{\tilde{p}_*}(\mathbb{C}^n)$ with the x_0-adic direct product topology.

In this paper, we are mainly concerned with the case of $n=2$ and $p_1=p_2$. Reminding this, we denote by $p=(p_0,p_1,p_1)$ and $p_*=(p_1,p_1)$, respectively.

Lemma 2.2 *Let* $\mathcal{E}_{\Omega}(\mathbb{C}^3)$ *be one of* (E.1)-(E.3). *Then* $(\mathcal{E}_{\Omega}(\mathbb{C}^3),\cdot,\{\ ,\ \}_H)$ *is a Fréchet-Poisson algebra.*

2.2 Deformation quantization of $(\mathcal{E}_{\Omega}(\mathbb{C}^3),\cdot,\{\ ,\ \}_H)$

Let $\mathcal{E}_{\Omega}(\mathbb{C}^3)$ be a Fréchet-Poisson algebra given by Lemma 2.2. We consider a noncommutative product which gives deformation quantization of $(\mathcal{E}_{\Omega}(\mathbb{C}^3),\cdot,\{\ ,\ \}_H)$: Setting $\hbar=1$, we define a product $f*g$ for $f,g\in\mathcal{P}(\mathbb{C}^3)$ by the product formula, called also the Moyal product formula:

$$(2.4)\qquad f*g=\sum_{p=0}^{\infty}\frac{(iz)^p}{2^p p!}(f(\overleftarrow{\partial_x}\cdot\overrightarrow{\partial_y}-\overleftarrow{\partial_y}\cdot\overrightarrow{\partial_x})^p g).$$

As for $f,g\in\mathcal{P}(\mathbb{C}^3)$, the product (2.4) gives an associative product.

Moreover, if we replace z by $\hbar z$, then (1.4) gives a deformation quantization of $(C^\infty(\mathbb{C}^3), \cdot, \{,\}_H)$ (cf. [1], etc.). We focus to the question how (2.4) extends to Fréchet-Poisson algebra $(\mathcal{E}_\Omega(\mathbb{C}^3), \cdot, \{\, ,\,\}_H)$. In this section, we are concentrated with the case of the weight $p = (p_0, p_1, p_1)$, $(p_1 = p_2)$.

One of goals in this paper is the following:

Theorem 2.1 *Let $(\mathcal{E}_\Omega(\mathbb{C}^3), \cdot, \{,\}_H)$ be a Fréchet-Poisson algebra given by Lemma 2.2. Assume that p is given by $p = (p_0, p_1, p_1)$. Then the following holds:*
(1) *$(\mathcal{E}_\Omega(\mathbb{C}^3), *)$ is an associative Fréchet algebra if and only if Ω satisfies one of the following:*

(A1) *For* $\Omega = p = (p_0, p_1, p_1)$, $0 < p_1 \leq \dfrac{2p_0}{p_0+1}$,

(A2) *For* $\Omega = (Hol, p_*)$, $p_* = (p_1, p_1)$, $0 < p_1 \leq 2$,

(A3) *For* $\Omega = (\infty, p_*)$, $p_* = (p_1, p_1)$, $0 < p_1$.

(2) *For any case* (A1)–(A3), *$(\mathcal{E}_\Omega(\mathbb{C}^3), *)$ has the following properties:*

(i) $[z, \mathcal{E}_\Omega(\mathbb{C}^3)] = 0$, *i.e. z is a central element.*

(ii) $[\mathcal{E}_\Omega(\mathbb{C}^3), \mathcal{E}_\Omega(\mathbb{C}^3)] \subset z{*}\mathcal{E}_\Omega(\mathbb{C}^3)$.

(iii) $\mathcal{E}_\Omega(\mathbb{C}^3) = \mathcal{E}_{p_*}(\mathbb{C}^2) \oplus z{*}\mathcal{E}_\Omega(\mathbb{C}^3)$ *(topological direct sum).*

(iv) *$z*$, and $*z$ are continuous linear isomorphisms of $\mathcal{E}_\Omega(\mathbb{C}^3)$ onto $z{*}\mathcal{E}_\Omega(\mathbb{C}^3)$.*

(v) *There is an an involutive anti-automorphism $a \to \hat{a}$ of $(\mathcal{E}_\Omega(\mathbb{C}^3), *)$ given by setting $\hat{z} = z$, $\hat{u} = u$ and $\hat{v} = v$, but $\hat{i} = -i$.*

(vi) $\bigcap_{m\geq 0} z^m * \mathcal{E}_\Omega(\mathbb{C}^3) = \{0\}$.

An algebra with the properties (i)–(vi) is a particular regulated algebra which is given in [8]:

Definition Replace (i) by the property

(i') $[z, \mathcal{E}_\Omega(\mathbb{C}^3)] \subset z * \mathcal{E}_\Omega(\mathbb{C}^3) * z$.

An associative (Fréchet) algebra $\mathcal{A}$ with the properties (i')-(v) in Theorem 2.1 is called a *z-regulated (Fréchet) algebra.* If $\mathcal{A}$ satisfies (i), then $\mathcal{A}$ is called *z-central,* and if $\mathcal{A}$ satisfies (vi), then $\mathcal{A}$ is called *analytic.* If $\mathcal{A}$ is a formal power series of z, then $\mathcal{A}$ is called *formal.*

Replacing z by $\hbar z$ in (2.4), we have a product $*_\hbar$.

For $\mathcal{E}_\Omega(\mathbb{C}^3)$, we have the following typical deformation quantization of Fréchet-Poisson algebras:

Corollary 2.1 *Assume* $(\mathcal{E}_p(\mathbb{C}^3), \cdot, \{,\}_H)$ *satisfies (A.1) of Theorem 2.1. Then,* $(\mathcal{E}_p(\mathbb{C}^3), *_\hbar)$ *is a deformation quantization of the Fréchet-Poisson algebra* $(\mathcal{E}_p(\mathbb{C}^3), \cdot, \{,\}_H)$.

Set $z = 1$ in (1.1) and denote it by $\{\, ,\, \}_1$. Then we have a Fréchet Poisson algebra $(\mathcal{E}_{p_*}(\mathbb{C}^2), \cdot, \{,\}_1)$.

Note that $\mathcal{E}_{p_*}(\mathbb{C}^2)$ is viewed as the closed subset of $\mathcal{E}_{Hol,p_*}(\mathbb{C}^3)$ whose elements are independent of z. We consider the product defined by (1.4) by setting $z = \hbar$. We denote its product by $*_\hbar$.

Corollary 2.2 ([9]) *Assume* $p_* = (p_1, p_1)$, $0 < p_1 \leq 2$. *Then* $(\mathcal{E}_{p_*}(\mathbb{C}^2), *_\hbar)$ *is a deformation quantization of a Fréchet Poisson algebra* $(\mathcal{E}_{p_*}(\mathbb{C}^2), \cdot, \{,\}_1)$.

The case (A.3) of Theorem 2.1 implies the formal deformation quantization:

Corollary 2.3 *For every* $p_* = (p_1, p_1)$, $0 < p_1$, $(\mathcal{E}_{p_*}(\mathbb{C}^2)[[\hbar]], *_\hbar)$ *is a formal deformation quantization of* $(\mathcal{E}_{p_*}(\mathbb{C}^2), \cdot, \{,\}_1)$.

3 Free tensor algebra

Let $\mathcal{T}^{n+1}$ be the free tensor algebra of $(n+1)$-vector space V:

$$\mathcal{T}^{n+1} = \sum_{k=0}^{\infty} \oplus \mathcal{T}_k^{n+1}, \tag{3.1}$$

where $\mathcal{T}_0^{n+1} = \mathbb{C}$, $\mathcal{T}_k^{n+1} = V \otimes \cdots \otimes V$ (k times). Fix a basis $X_0, X_1, \cdots, X_n$ of V. Then the monomials $X_\alpha = X_{\alpha_1} \otimes \cdots \otimes X_{\alpha_k}$ form a basis for $\mathcal{T}_k^{n+1}$ where $\alpha = (\alpha_1, \cdots, \alpha_k)$, $0 \leq \alpha_1, \cdots, \alpha_k \leq n$.

An element $T \in \mathcal{T}^{n+1}$ is written as

$$T = \sum_{|\alpha| \geq 0} t_\alpha X_\alpha \quad \text{(finite sum)}, \quad t_\alpha \in \mathbb{C} \tag{3.2}$$

3.1 Completion of Free tensor algebra

We introduce a topology, called the *word topology* on the space $\mathcal{T}^{n+1}$ to make a completion.

We use similar notations as in §2.1: $\tilde{\tau}$ denotes $(n+1)$-tuples $\tilde{\tau} = (\tau_0, \tau_1, \cdots, \tau_n)$ and $\tilde{s} = (s_0, s_1, s_2, \cdots, s_n)$ respectively, where $\tau_i > 0$, and $s_i > 0$ for $0 \leq i \leq n$. By forgetting τ_0 and s_0, $\tilde{\tau}_*$ and $\tilde{s}_*$ denote $\tilde{\tau}_* = (\tau_1, \cdots, \tau_n)$ and $\tilde{s}_* = (s_1, s_2, \cdots, s_n)$.

Given $\alpha = (\alpha_1, \cdots, \alpha_k)$, we consider a monomial $X_\alpha = X_{\alpha_1} \otimes \cdots \otimes X_{\alpha_k}$. Let $m_i(\alpha)$ denote the number of X_i in X_α and set $m(\alpha) = (m_0(\alpha), \cdots, m_n(\alpha))$. For $\tilde{\tau} = (\tau_0, \tau_1, \cdots, \tau_n)$ and $\alpha = (\alpha_1, \cdots, \alpha_k)$, we set

$$\tilde{\tau} m(\alpha) = (\tau_0 m_0(\alpha), \tau_1 m_1(\alpha), \cdots, \tau_n m_n(\alpha)).$$

Using these notations, we set for a monomial $X_\alpha = X_{\alpha_1} \otimes \cdots \otimes X_{\alpha_k}$ as follows:

$$|X_\alpha|_{\tilde{\tau},\tilde{s}} = (\tilde{\tau} m(\alpha))^{\tilde{\tau} m(\alpha)} s_0^{\tau_0 m_0(\alpha)} \cdots s_n^{\tau_n m_n(\alpha)}, \tag{3.3}$$

$$|X_\alpha|_{\tilde{\tau}_*,\tilde{s}_*} = (\tilde{\tau}_* m_*(\alpha_*))^{\tilde{\tau}_* m_*(\alpha_*)} s_1^{\tau_1 m_1(\alpha_*)} \cdots s_n^{\tau_n m_n(\alpha_*)}, \tag{3.4}$$

where $(\tilde{\tau} m(\alpha))^{\tilde{\tau} m(\alpha)} = (\tau_0 m_0(\alpha))^{\tau_0 m_0(\alpha)} (\tau_1 m_1(\alpha))^{\tau_1 m_1(\alpha)} \cdots (\tau_n m_n(\alpha))^{\tau_n m_n(\alpha)}$ and $\tilde{\tau}_* m_*(\alpha_*) = (\tau_1 m_1(\alpha))^{\tau_1 m_1(\alpha)} \cdots (\tau_n m_n(\alpha))^{\tau_n m_n(\alpha)}$.

Definition 3.1 *Let $\tilde{\tau}, \tilde{\tau}_*, \tilde{s}, \tilde{s}_*$ be as above. Let t and N_0 be a positive real number and a nonnegative integer, respectively. For $T = \sum_\alpha t_\alpha X_\alpha \in \mathcal{T}^{n+1}$, we set $T = \sum T_j$, $T_j \in \mathcal{T}^{n+1}$ where T_j is the component of T which contains X_0 j times in the term X_α. We define seminorms as follows:*

$$||T||_{\tilde{\tau},\tilde{s}} = \sum_\alpha |t_\alpha| \cdot |X_\alpha|_{\tilde{\tau},\tilde{s}}, \tag{3.5}$$

$$||T||_{\tilde{\tau}_*,\tilde{s}_*,t} = \sum_{j=0}^{\infty} ||T_j||_{\tilde{\tau}_*,\tilde{s}_*} t^j, \tag{3.6}$$

$$||T||_{\tilde{\tau}_*,\tilde{s}_*,N_0} = \sum_{j=0}^{N_0} ||T_j||_{\tilde{\tau}_*,\tilde{s}_*}. \tag{3.7}$$

The completion of $\mathcal{T}^{n+1}$ by the system of seminorms $\{||T||_{\tilde{\tau},\tilde{s}}\}_{\tilde{s}}$, $\{||T||_{\tilde{\tau}_*,\tilde{s}_*,t}\}_{\tilde{s}_*,t}$, $\{||T||_{\tilde{\tau}_*,\tilde{s}_*,N_0}\}_{\tilde{s}_*,N_0}$ will be denoted respectively by

$$(\mathrm{T}.1)\ :\ \mathcal{T}_{\tilde{\tau}}^{n+1},\qquad (\mathrm{T}.2)\ :\ \mathcal{T}_{Hol,\tilde{\tau}_*}^{n+1},\qquad (\mathrm{T}.3)\ :\ \mathcal{T}_{\infty,\tilde{\tau}_*}^{n+1}.$$

Corresponding to §2, we denote by $\mathcal{T}_{\mho}^{n+1}$ one of the notation $\mathcal{T}_{\tilde{\tau}}^{n+1}$, $\mathcal{T}_{Hol,\tilde{\tau}_*}^{n+1}$ and $\mathcal{T}_{\infty,\tilde{\tau}_*}^{n+1}$ according to $\tilde{\mho} = \tilde{\tau}$, $(Hol, \tilde{\tau}_*)$, $(\infty, \tilde{\tau}_*)$.

The conjugation $\hat{X}_1 = X_1, \hat{X}_2 = X_2, \cdots, \hat{X}_n = X_n$ and $\hat{i} = -i$ on $\mathcal{T}^{n+1}$ is extended to $\mathcal{T}_{\Omega}^{n+1}$ and gives an involutive anti-automorphism.

Lemma 3.1 *$(\mathcal{T}_{\mho}^{n+1}, \circledast)$ is a noncommutative associative Fréchet algebra with the conjugation.*

In the proof of Lemma 3.1, we use the first inequality of the following Lemma:

Lemma 3.2 *Let $u, v > 0$. Then, we have*

$$u^u v^v \le (u+v)^{u+v} \le e^{u+v} u^u v^v. \tag{3.8}$$

The second inequality is useful in the later computation.

As in the previous section, we mainly restrict our concern to the cases of $n = 2$ and the weights $\tau = (\tau_0, \tau_1, \tau_1), \tau_0, \tau_1 > 0$ and $\tau_* = (\tau_1, \tau_1), \tau_1 > 0$.

3.2 Subspace of symmetric elements

We first introduce a symmetric product

$$F \circ G = \frac{1}{2}(F \circledast G + G \circledast F)$$

in $\mathcal{T}^{n+1}$ and set

$$(F\circ)^k \cdot H = F \circ (F \circ (\cdots (F \circ H) \cdots)), \tag{3.9}$$

$$(F\circ)^k \cdot (G\circ)^l \cdot H = F \circ (F \circ \cdots (F \circ (G\circ)^l \cdot H) \cdots). \tag{3.10}$$

(cf. [4]). Using these notations, we define a linear subspace S^{n+1} of $\mathcal{T}^{n+1}$ as

$$S^{n+1} = \{F \in \mathcal{T}^{n+1} |\ F = \sum_{\alpha=(\alpha_0,\cdots,\alpha_n)} c_\alpha (X_0\circ)^{\alpha_0} \cdots (X_n\circ)^{\alpha_n} \cdot 1\}. \tag{3.11}$$

Setting a commutative product $\odot$ for monomials:

$$((X_0\circ)^{\alpha_0} \cdots (X_n\circ)^{\alpha_n} \cdot 1) \odot ((X_0\circ)^{\beta_0} \cdots (X_n\circ)^{\beta_n} \cdot 1) = (X_0\circ)^{\alpha_0+\beta_0} \cdots (X_n\circ)^{\alpha_n+\beta_n} \cdot 1, \tag{3.12}$$

we extend $\odot$ on S^{n+1}. Thus, $(S^{n+1}, \odot)$ is a commutative associative algebra.

Denote by $S_{\mho}^{n+1}$ the closure of S^{n+1} in $\mathcal{T}_{\mho}^{n+1}$.

For $\Omega = \tilde{p}$, (Hol, p_*), (∞, p_*) respectively we consider the following correspondence $\mho = \tilde{p}^{-1}$, $(Hol, p_*{}^{-1})$, $(\infty, p_*{}^{-1})$ where $\tilde{p}^{-1} = (p_0^{-1}, p_1^{-1}, \cdots, p_n^{-1})$ and $p_*{}^{-1} = (p_1^{-1}, \cdots, p_n^{-1})$.

Proposition 3.1 *With the correspondence as above, $(S_{\mho}^{n+1}, \odot)$ is isomorphic to $\mathcal{E}_{\Omega}(\mathbb{C}^{n+1})$ as commutative topological algebras.*

Proof. Although it seems well-known facts (cf.[2],[10]), we repeat the proof. Identifications between these spaces are constructed naturally by putting the generators $X_i \mapsto x_i$, and the operation $\odot \mapsto \cdot$. In the following, we view these as the same ones, and using the same character we delete the notations of products $\odot$, $\cdot$.

We show the case $\Omega = \tilde{p}$ and for the case of 1-variable z, which yields easily the multi-variable cases. Let $\Omega = p$ and $\mho = \tau = 1/p$. Let $f = \sum a_n z^n \in S_\tau$. For every $s > 0$ there exists $C = C(s) > 0$ such that $|a_n| \leq C(\tau n)^{-\tau n} s^{-\tau n}$, $n = 1, 2, \cdots$. By the inequality $(\tau n)^{-\tau n} \leq \frac{1}{[\tau n]!}$ for $n \geq \frac{1}{\tau}$, we have

$$|\sum_{n \geq \tau^{-1}} a_n z^n| \leq C \sum_{n \geq \tau^{-1}} \frac{1}{[\tau n]!} \left(|z|^{\frac{1}{\tau}}/s\right)^{\tau n}.$$

Note that $(|z|^{\frac{1}{\tau}}/s)^{\tau n} = (|z|^{\frac{1}{\tau}}/s)^{[\tau n]}(|z|^{\frac{1}{\tau}}/s)^{\tau n - [\tau n]} \leq (|z|^{\frac{1}{\tau}}/s)^{[\tau n]} \exp(|z|^{\frac{1}{\tau}}/s)$. Thus, we have

$$|f| \leq \sum_{n < \tau^{-1}} |a_n||z|^n + C \exp\left(2|z|^{\frac{1}{\tau}}/s\right). \tag{3.13}$$

The first term of the right hand side is a polynomial and hence bounded from the above by the exponential function $\exp(2|z|^{\frac{1}{\tau}}/s)$. Then by putting $p = \frac{1}{\tau}$ we have:

$$|f| \leq CK \exp\left(2|z|^p/s\right), \tag{3.14}$$

for certain positive constant K depending only on τ. Then for every $b > 0$, we have an inequality $\|f\|_{p,b} \leq K\|f\|_{\tau,s}$ where $f = \sum a_n z^n$ and $s = 2/b$. Thus, we have $f = \sum a_n z^n \in \mathcal{E}_p(\mathbb{C})$.

Conversely, assume $f \in \mathcal{E}_p(\mathbb{C})$, i.e., $\sup_{z\in\mathbb{C}} |f(z)| \exp(-b|z|^p) < \infty$ for every $b > 0$. Put $f(z) = \sum a_n z^n$. Using the Cauchy estimate that $|a_n| \leq M(\exp br^p)/r^n$ for any $r > 0$, and taking the minimal value, we have

$$|a_n| \leq M \frac{(bpe)^{\frac{n}{p}}}{n^{\frac{n}{p}}}. \tag{3.15}$$

By choosing s such that $epbs < 1$, we have the following which gives the converse

$$(3.16)\qquad \sum |a_n|(n/p)^{n/p}s^{n/p} \leq M\sum_n (epbs)^{n/p}.$$

Thus, the case $\Omega = \tilde{p}$ is obtained by extending the above arguments to multi-variable functions.

To show the case $\Omega = (Hol, p_*)$, we remind estimate of the seminorms for X_0 in S_{τ_*} and for x_0 in $\mathcal{E}_{p_*}$is same. Thus, the above argument also yields for this case $\Omega = (Hol, p_*)$.

The case $\Omega = (\infty, p_*)$ seems rather trivial. Remark $S^{n+1}_{\infty,\tilde{\tau}_*} = S^n_{\tilde{\tau}_*}[[x_0]]$ and $\mathcal{E}_{\infty,p_*}(\mathbb{C}^{n+1}) = \mathcal{E}_{p_*}(\mathbb{C}^n)[[x_0]]$ with the direct product topology. Using the above observation for $\mathcal{E}_{p_*}(\mathbb{C}^n)$ and $S^n_{p_*^{-1}}$, we have the case $\Omega = (\infty, p_*)$. □

3.3 $*$-product on $S^3_{\mho}$

In this subsection, we work in $S^3_{\mho}$. For convenience, we write as $X_0 = Z$, $X_1 = X$, $X_2 = Y$. We introduce an (commutative) associative product, denoted by $\odot$ on $S^3_{\mho} \subset \mathcal{T}^3_{\mho}$, where $\mho$ is one of $\tilde{\tau}$, (Hol, τ_*), (∞, τ_*).

For $F = \sum a_{kmn}(Z\circ)^k(X\circ)^m(Y\circ)^n \cdot 1 \in S^3_{\mho}$, we set

$$(3.17)\qquad \begin{aligned} \partial_X F &= \sum a_{kmn} m(Z\circ)^k(X\circ)^{m-1}(Y\circ)^n \cdot 1 \\ \partial_Y F &= \sum a_{kmn} n(Z\circ)^k(X\circ)^m(Y\circ)^{n-1} \cdot 1. \end{aligned}$$

Simple estimation of (3.17) yields that, $\partial_X F, \partial_Y F \in S^3_{\mho}$. Similarly, we define higher derivatives $\partial_X^{l_1}\partial_Y^{l_2} F$ as usual, and $\partial_X^{l_1}\partial_Y^{l_2} F \in S^3_{\mho}$. For $F_1, F_2 \in S^3_{\mho}$, we set

$$(3.18)\qquad \{F_1, F_2\} = F_1\left(Z\odot\left(\overleftarrow{\partial_X}\odot\overrightarrow{\partial_Y} - \overleftarrow{\partial_Y}\odot\overrightarrow{\partial_X}\right)\right)F_2.$$

Then, by Proposition 3.1, $(S^3_{\mho}, \odot, \{\ ,\ \})$ is a Fréchet-Poisson algebra, isomorphic to $(\mathcal{E}_{\Omega}(\mathbb{C}^3), \cdot, \{\ ,\ \}_H)$.

The formula (1.4) will be read as on S^3 and on $S^3_{\mho}$, i.e. for $F_1, F_2 \in S^3_{\mho}$, we set

$$(3.19)\qquad F_1 * F_2 = \sum_{p=0}^{\infty}\frac{\sqrt{-1}^p}{2^p p!}F_1\left(Z\odot(\overleftarrow{\partial_X}\odot\overrightarrow{\partial_Y} - \overleftarrow{\partial_Y}\odot\overrightarrow{\partial_X})\right)^p F_2.$$

(3.19) is well-defined and gives an associative product on S^3.

By Proposition 3.1, in order to obtain Theorem 2.1 it suffices to show

Theorem 3.1 *Let $(S^3_{\mho}, \circledcirc, \{\ ,\ \})$ be as above. Then, $(S^3_{\mho}, *)$ is a Z-central, Z-regulated analytic Fréchet algebra, if and only if $\mho$ satisfies one of the following:*

(A'.1) $0 < \tau_0 \leq 2\tau_1 - 1$ *for* $\mho = (\tau_0, \tau_1, \tau_1)$.

(A'.2) $\frac{1}{2} \leq \tau_1$ *for* $\mho = (Hol, \tau_*), \tau_* = (\tau_1, \tau_1)$.

(A'.3) $0 < \tau_1$ *for* $\mho = (\infty, \tau_*)$, $\tau_* = (\tau_1, \tau_1)$.

4 Convergence of the product

In this section, we prove Theorem 2.1.

4.1 Case $\mathcal{T}^3_\tau$, $\tau = (\tau_0, \tau_1, \tau_1)$, $0 < \tau_0 \leq 2\tau_1 - 1$

We first prove the sufficiency part in Theorem 2.1.

Let $\mathcal{T}^3_\tau$ and S^3_τ be as in §3. To show Theorem 2.1, we consider the following product on S^3_τ:

$$(4.1) \qquad F_1 * F_2 = \sum \frac{\sqrt{-1}^p}{2^p} \sum_{i+j=p} \frac{Z^p}{i!j!} \circledcirc \partial^i_X (-\partial_Y)^j F_1 \circledcirc \partial^i_Y \partial^j_X F_2,$$

for $F_1 = \sum a_{k_1 m_1 n_1} (Z\circledcirc)^{k_1} (X\circledcirc)^{m_1} (Y\circledcirc)^{n_1}$, $F_2 = \sum b_{k_2 m_2 n_2} (Z\circledcirc)^{k_2} (X\circledcirc)^{m_2} (Y\circledcirc)^{n_2}$ $\in S^3_\tau$.

In this subsection, we show the following:

Theorem 4.1 *Assume $\tau = (\tau_0, \tau_1, \tau_1), 0 < \tau_0 \leq 2\tau_1 - 1$. Then, $(S^3_\tau, *)$ is a Z-central, Z-regulated analytic Fréchet algebra.*

Proof. Let F_1, F_2 be as in (4.1). Compute

$$\begin{aligned}(4.2) \quad F_1 * F_2 \;=\; & \sum \frac{\sqrt{-1}^p}{2^p p!} \sum_{i+j=p} a_{k_1 m_1 n_1} b_{k_2 m_2 n_2} \\ & \times \frac{p!}{i!j!} (-1)^j \frac{m_1!}{(m_1 - i)!} \frac{n_1!}{(n_1 - j)!} \frac{m_2!}{(m_2 - j)!} \frac{n_2!}{(n_2 - i)!} \\ & \times (Z\circledcirc)^{p+k_1+k_2} (X\circledcirc)^{m_1+m_2-p} (Y\circledcirc)^{n_1+n_2-p}.\end{aligned}$$

By using the definition of seminorms and $\frac{m!}{(m-i)!} \leq m^i$, we have the following estimate:

$$\begin{aligned}||F_1 * F_2||_{\tau,s} \leq \sum |a_{k_1m_1n_1}||b_{k_2m_2n_2}| \cdot \sum_{p\geq 0} \frac{1}{2^p p!} \sum_{i+j=p} \frac{p!}{i!j!} m_1^i n_1^j m_2^j n_2^i \\ \times(\tau_0(p+k_1+k_2))^{\tau_0(p+k_1+k_2)} \\ \times(\tau_1(m_1+n_1-p))^{\tau_1(m_1+n_1-p)}(\tau_1(m_2+n_2-p))^{\tau_1(m_2+n_2-p)} \\ \times s_0^{\tau_0(p+k_1+k_2)} s_1^{\tau_1(m_1+m_2+n_1+n_2-2p)}.\end{aligned}$$

Remark

$$\begin{aligned}(\tau_1(m_1+n_1-p))^{\tau_1(m_1+n_1-p)}(\tau_1(m_2+n_2-p))^{\tau_1(m_2+n_2-p)} \\ \leq (\tau_1(m_1+m_2+n_1+n_2-2p))^{\tau_1(m_1+m_2+n_1+n_2-2p)}, \\ m_1^i m_2^j n_1^i n_2^j \leq (m_1+m_2+n_1+n_2)^{2p}.\end{aligned}$$

Using the inequality $(u+v)^{u+v} \leq e^{u+v}u^u v^v$ in Lemma 3.2 to the term $(\tau_0(p+k_1+k_2))^{\tau_0(p+k_1+k_2)}$, we have

$$\begin{aligned}||F_1 * F_2||_{\tau,s} \leq \sum |a_{k_1m_1n_1}||b_{k_2m_2n_2}| \cdot \sum_{p\geq 0} \frac{1}{p!}(m_1+n_1+m_2+n_2)^{2p} \\ \times e^{\tau_0(p+k_1+k_2)}(\tau_0(k_1+k_2))^{\tau_0(k_1+k_2)}(\tau_0 p)^{\tau_0 p} \\ \times(\tau_1(m_1+m_2+n_1+n_2-2p))^{\tau_1(m_1+m_2+n_1+n_2-2p)} \\ \times s_0^{\tau_0(p+k_1+k_2)} s_1^{\tau_1(m_1+m_2+n_1+n_2-2p)}.\end{aligned}$$

Plugging $u^u v^v \leq (u+v)^{u+v}$ into the term involving $(\tau_0 p)^{\tau_0 p}$, we have

$$\begin{aligned}||F_1 * F_2||_{\tau,s} \leq \sum |a_{k_1m_1n_1}||b_{k_2m_2n_2}| \cdot \sum_{p\geq 0} \frac{1}{p!}(m_1+n_1+m_2+n_2)^{2p} \\ \times e^{\tau_0(p+k_1+k_2)}(\tau_0(k_1+k_2))^{\tau_0(k_1+k_2)} \\ \times((\tau_0-2\tau_1)p+\tau_1(m_1+m_2+n_1+n_2))^{((\tau_0-2\tau_1)p+\tau_1(m_1+m_2+n_1+n_2))} \\ \times s_0^{\tau_0(p+k_1+k_2)} s_1^{\tau_1(m_1+m_2+n_1+n_2-2p)}.\end{aligned} \tag{4.3}$$

Using the assumption $\tau_0 \le 2\tau_1 - 1$, we get

$$\begin{aligned}
||F_1 * F_2||_{\tau,s} &\le \sum |a_{k_1m_1n_1}||b_{k_2m_2n_2}| \cdot \sum_{p\ge 0} \frac{1}{p!}(m_1+n_1+m_2+n_2)^p \\
&\quad \times e^{\tau_0(p+k_1+k_2)}(\tau_0(k_1+k_2))^{\tau_0(k_1+k_2)} \\
&\quad \times \tau_1^{\tau_1(m_1+n_1+m_2+n_2)}(m_1+m_2+n_1+n_2)^{((\tau_0-2\tau_1+1)p+\tau_1(m_1+m_2+n_1+n_2))} \\
&\quad \times s_0^{\tau_0(p+k_1+k_2)} s_1^{\tau_1(m_1+m_2+n_1+n_2-2p)} \\
&\le \sum |a_{k_1m_1n_1}||b_{k_2m_2n_2}| \cdot \exp[e^{\tau_0}(m_1+n_1+m_2+n_2)s_0^{\tau_0}s_1^{-2\tau_1}] \\
&\quad e^{\tau_0(k_1+k_2))}(\tau_0(k_1+k_2))^{\tau_0(k_1+k_2)} \\
&\quad \times(\tau_1(m_1+m_2+n_1+n_2))^{\tau_1(m_1+m_2+n_1+n_2))} \\
&\quad \times s_0^{\tau_0(k_1+k_2)} s_1^{\tau_1(m_1+m_2+n_1+n_2)}.
\end{aligned} \tag{4.4}$$

Using $(u+v)^{u+v} \le e^{u+v}u^uv^v$ again, we see that

$$\begin{aligned}
||F_1 * F_2||_{\tau,s} &\le \sum |a_{k_11m_1n_1}||b_{k_2m_2n_2}| \cdot (\tau_0(k_1+k_2))^{\tau_0(k_1+k_2)} \\
&\quad \times(\tau_1(m_1+m_2))^{\tau_1(m_1+m_2))}(\tau_1(n_1+n_2))^{\tau_1(n_1+n_2))} \\
&\quad \times(es_0)^{\tau_0(k_1+k_2)}e^{\tau_1(m_1+m_2+n_1+n_2))} \\
&\quad \times s_1^{\tau_1(m_1+m_2+n_1+n_2)} \exp[e^{\tau_0}(m_1+n_1+m_2+n_2)s_0^{\tau_0}s_1^{-2\tau_1}].
\end{aligned} \tag{4.5}$$

By the definition of seminorms, we remind the following equality:

$$\begin{aligned}
||F_1 \circledcirc F_2||_{\tau,\sigma} &= \sum |a_{k_1m_1n_1}||b_{k_2m_2n_2}|(\tau_0(k_1+k_2))^{\tau_0(k_1+k_2)} \\
&\quad \times(\tau_1(m_1+m_2))^{\tau_1(m_1+m_2)}(\tau_1(n_1+n_2))^{\tau_1(n_1+n_2)} \\
&\quad \times\sigma_0^{\tau_0(k_1+k_2)}\sigma_1^{\tau_1(m_1+m_2+n_1+n_2)}.
\end{aligned} \tag{4.6}$$

Therefore, we have

$$||F_1 * F_2||_{\tau,s} \le ||F_1 \circledcirc F_2||_{\tau,\sigma}, \tag{4.7}$$

where $\sigma = (s_0e,\ s_1\exp(1+\tau_1^{-1}(es_0)^{\tau_0}s_1^{-2\tau_1}))$.

The properties (i)–(vi) in Theorem 2.1 are easily obtained (cf. Theorem 5.1 below).

If $\mho = (Hol, \tau_*)$ and $\tau_1 \ge \frac{1}{2}$, then the system of seminorms is obtained by neglecting $(\tau_0k)^{\tau_0k}$ part and setting $s_0^{\tau_0} = t$. Minding this, we have the following:

Theorem 4.2 *Assume* $\tau_* = (\tau_1, \tau_1)$, $\tau_1 \ge \frac{1}{2}$. *Then,* $(S^3_{Hol,\tau_*}, *)$ *is a* Z*-central,* Z*-regulated Fréchet analytic algebra.*

Proof. By following the computations as in 4.1, we see Theorem 4.2. In particular, we neglect $(\tau_0(k_1+k_2))^{\tau_0(k_1+k_2)}$ and put $\tau_0 = 1$ in (4.3). Then, the same computations in (4.4) and (4.5) gives the following estimates:

$$||F_1 * F_2||_{\tau_*,s_*,r_0} \leq ||F_1 \circledcirc F_2||_{\tau_*,s_{1*},r_0}, \tag{4.8}$$

where $s_{1*} = (\exp \tau_1 s_1^{-2a})s_1$.

In this case, there is no restriction of τ_1. By the definition of (3.19), the product $*$ is well-defined for any $F_1, F_2 \in \mathcal{T}^3_{\infty,\tau_*}$. Then, we have

Theorem 4.3 *Assume $\tau_* = (\tau_1, \tau_1)$, $\tau_1 > 0$. Then, $(S^3_{\infty,\tau_*}, *)$ is a Z-central, Z-regulated Fréchet formal algebra.*

4.2 Remarks on the star product.

We remark the assumption in Theorem 3.1 is best possible in the following sense, which give the necessity part in Theorem 2.1.

Proposition 4.1 *Assume $\tau = (\tau_0, \tau_1, \tau_1)$ with $\tau_0 > 0, \tau_1 > 0, \tau_0 > 2\tau_1 - 1$. Then, $*$ does not give a Fréchet algebra structure on S^3_τ.*

Proof. We set

$$U_\circledcirc(Z,X) = \sum_{n=0}^{\infty} \frac{(Z\circledcirc)^n (X\circledcirc)^n}{n^{\alpha n}}, \quad U_\circledcirc(Z,Y) = \sum_{n=0}^{\infty} \frac{(Z\circledcirc)^n (Y\circledcirc)^n}{n^{\alpha n}}. \tag{4.9}$$

If $\alpha > \tau_0 + \tau_1$, then $U_\circledcirc(Z,X), U_\circledcirc(Z,Y) \in S^3_\tau$ and we have the product

$$\begin{aligned} &U_\circledcirc(Z,X) * U_\circledcirc(Z,Y) \\ &= \sum \frac{1}{n^{\alpha n}} \frac{1}{m^{\alpha m}} \cdot \sum_{p=0}^{min(n,m)} p!(\frac{i}{2})^p \binom{n}{p} \binom{m}{p} (Z\circledcirc)^{n+m+p} (X\circledcirc)^{n-p} (Y\circledcirc)^{m-p} \end{aligned} \tag{4.10}$$

Then, we get

$$||U_\circledcirc(Z,X) * U_\circledcirc(Z,Y)||_{\tau,s} \geq \sum_{l=0}^{\infty} \frac{1}{l^{2\alpha l}} \cdot \frac{1}{2^l l!} (l!)^2 (3\tau_0 l)^{3\tau_0 l} s_0^{3\tau_0 l}. \tag{4.11}$$

If we choose $\alpha = \tau_0 + \tau_1 + \varepsilon$ for a sufficiently small $\varepsilon > 0$, the right hand member is a power series of $s_0^{3\tau_0}$ with no positive radius of convergence.

Thus, we have (4.10) diverges for any $s_0 > 0$. Similar computation gives the following:

Corollary 4.1 *Assume $\tau_* = (\tau_1, \tau_1)$ satisfies $0 < \tau_1 < \frac{1}{2}$. Then $a * b$ diverges for some elements in S^3_{Hol,τ_*}.*

Note that if $\mho = (\infty, \tau_*), \tau_1 > 0$, there is no complementary case. Hence, the argument in this section gives the "only if" part of Theorem 2.1.

5 Quotient of $\mathcal{T}^3_{\mho}$

As in §4, let $\mathcal{T}^3$ and $\mathcal{T}^2$ be the free tensor algebras with generators $X_0 = Z, X_1 = X, X_2 = Y$ and with $X_1 = X, X_2 = Y$, respectively. Let $\mathcal{T}^3_{\mho}$ be the completion of $\mathcal{T}^3$ via one of systems of seminorms given by Definition 3.1. Let $\mathcal{I}^3$ be the two sided ideal of relations in $\mathcal{T}^3$ generated by

$$X\circledast Z - Z\circledast X,\ \ Y\circledast Z - Z\circledast Y, \text{ and } X\circledast Y - Y\circledast X - iZ.$$

Let $\mathcal{A}^3$ be the quotient algebra $\mathcal{T}/\mathcal{I}$. We denote by $\hat{*}$ the product of $\mathcal{A}^3$ Denote by $\mathcal{I}^3_{\mho}$ the closure of $\mathcal{I}$ in $\mathcal{T}^3_{\mho}$.

In spite of Proposition 4.1 and Corollary 4.1, we see that $\mathcal{A}^3_{\mho} = \mathcal{T}^3_{\mho}/\mathcal{I}^3_{\mho}$ is a Fréchet algebra.

5.1 Algebra structure of quotient

First, we observe the algebra structure of $(\mathcal{A}^3_{\mho}, \hat{*})$. We remark first the following:

Theorem 5.1 *For any $\mho$, $(\mathcal{A}^3_{\mho}, \hat{*})$ is a Z-central, Z-regulated, analytic Fréchet algebra.*

Proof. Let $T = \sum t_\alpha X_\alpha \in \mathcal{T}^3_{\mho}$. We remark for every X_α the following:
(i) If X_α does not contain Z, then X_α can be viewed as

$$X_\alpha = Q_\alpha + P_\alpha, \text{ where } Q_\alpha \in S^2_{\mho_*},\ P_\alpha \in Z\circledast\mathcal{T}^3_{\mho} + \mathcal{I}^3_{\mho}, \tag{5.1}$$

and moreover the seminorms of X_α and S_α are equal.
(ii) If X_α contains Z, then X_α can be viewed as

$$X_\alpha = P_\alpha, \text{ where } P_\alpha \in Z\circledast\mathcal{T}^3_{\mho} + \mathcal{I}^3_{\mho}. \tag{5.2}$$

Thus, T is written as

$$T = \sum t_\alpha Q_\alpha + Z\circledast P' + R, \tag{5.3}$$

where $Q \in \mathcal{T}^3_{\mho}, R \in \mathcal{I}^3_{\mho}$. Repeat this computation for Q.

Reminding

$$\mathcal{T}^3_{\mho}/(Z\circledast\mathcal{T}^3_{\mho} + \mathcal{I}^3_{\mho}) \cong (S^2_{\mho_*}, \circ), \tag{5.4}$$

we have (iii) in Theorem 2.1.

The other properties in Theorem 2.1 are obvious.

5.2 Properties for $\mathcal{A}^3_\tau$

We study algebraic properties on $\mathcal{A}^3_\tau = \mathcal{T}^3_\tau/\mathcal{I}^3_\tau$ where $\tau = (\tau_0, \tau_1, \tau_1)$. We denote by $\hat{*}$ the induced product from the closure of free tensor algebra $\mathcal{T}^3_\tau$. We first show the following:

Theorem 5.2 *Assume for $\tau = (\tau_0, \tau_1, \tau_1)$ and $0 < \tau_0 \leq 2\tau_1 - 1$. Then, we have*

$$\mathcal{T}^3_\tau = S^3_\tau \oplus \mathcal{I}^3_\tau \quad (\textit{direct sum}). \tag{5.5}$$

Moreover $(S^3_\tau, *)$ is isomorphic to $(\mathcal{A}^3_\tau, \hat{*})$.

Proof. Tracing how the symmetric product $\circ$ is defined on $\mathcal{T}^3$ by using $\otimes$ in (3.9), and how the $\odot$-product is defined on S^3 (3.12), we define the $\hat{\odot}$-product on the quotient algebra $\mathcal{A}^3 = \mathcal{T}^3/\mathcal{I}^3$ by using the $\hat{*}$-product instead of $\otimes$. Since $a \circ (b \circ c) - (a \circ b) \circ c = \frac{1}{4}[b, [a, c]]$, we see that if a, b, c are generators then $X \hat{\odot} Y = Y \hat{\odot} X$. Hence the $\hat{\odot}$-product is a commutative product (see [5]) without any artificial definition. Remark that

$$[X, Y]_{\hat{*}} = X \hat{*} Y - Y \hat{*} X = iZ, \ [X, Z]_{\hat{*}} = [Y, Z]_{\hat{*}} = 0,$$

in $\mathcal{A}^3$ and also $X \hat{*} Y = X \hat{\odot} Y + \frac{i}{2} Z$. Then the quotient algebra $\mathcal{A}^3$ is naturally isomorphic to S^3 with the Moyal product $*$ and the natural projection $\mathcal{T}^3 \to \mathcal{T}^3/\mathcal{I}^3$ is naturally translated to the mapping given by the replacement of the $\otimes$-product by the $*$-product.

Let π be the homomorphism $\mathcal{T}^3$ to S^3 defined by

$$\pi(X_{\alpha_1} \otimes \cdots \otimes X_{\alpha_n}) = X_{\alpha_1} * \cdots * X_{\alpha_n}. \tag{5.6}$$

Notice $S^3 \subset \mathcal{T}^3$ and

$$X \odot Y = X \circ Y = (X \otimes Y + Y \otimes X)/2.$$

Then we see

$$\pi(X \odot Y) = (X * Y + Y * X)/2 = X \hat{\odot} Y.$$

Similarly, it is easy to see that the replacement of $\otimes$ by $*$ gives the identity on S^3, i.e. $\pi | S^3$ is the identity. Hence, we have $\mathcal{T}^3 = \mathcal{S}^3 \oplus \mathcal{I}^3$.

We now show that π extends continuously to the map from $(\mathcal{T}^3_\tau, \otimes)$ to $(S^3_\tau, *)$. Let Y^k and X^k denote by $Y * \cdots * Y$ and $X * \cdots * X$. We first note that

$$(5.7) \qquad Y^m * X^n = \sum_{k=0}^{m} \binom{m}{k} \mathrm{ad}(Y)_*^k (X)^n * Y^{m-k},$$

where $\mathrm{ad}(Y)_*(X)^n = [Y, X^n]_*$. Using $\mathrm{ad}(Y)_*(X)^n = -inZ * X^{n-1}$, we have

$$(5.8) \qquad Y^m * X^n = \sum_{l=0}^{min\{m,n\}} (-1)^l \frac{m!}{l!(m-l)!} \frac{n!}{(n-l)!} Z^l * X^{n-l} * Y^{m-l}.$$

Let $\alpha = (\alpha_1, \cdots \alpha_n)$,$\beta = (\beta_1, \cdots, \beta_n)$ be n-tuples of nonnegative integers. By (5.8), we have

$$\begin{aligned}(5.9) \quad & Y^{\alpha_1} * X^{\beta_1} * Y^{\alpha_2} * X^{\beta_2} * \cdots * Y^{\alpha_n} * X^{\beta_n} \\ &= \sum_{k=(k_1,\cdots,k_n)} (-i)^{k_1+\cdots+k_n} \frac{\alpha_1!}{k_1!(\alpha_1-k_1)!} \frac{\beta_1!}{(\beta_1-k_1)!} \times \cdots \\ &\qquad \cdots \times \frac{(\alpha_1+\cdots+\alpha_n-(k_1+\cdots+k_{n-1}))!}{k_n!(\alpha_1+\cdots+\alpha_n-(k_1+\cdots+k_n))!} \frac{\beta_n!}{(\beta_n-k_n)!} \\ &\qquad \times Z^{|k|} * X^{|\beta|-|k|} * Y^{|\alpha|-|k|},\end{aligned}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $|\beta| = \beta_1 + \cdots + \beta_n$ and $|k| = k_1 + \cdots + k_n$.

Note that $\binom{a-n}{b}$ is monotone decreasing in n. Using

$$\begin{aligned}(5.10) \qquad & \binom{\alpha_1+\alpha_2-k_1}{\alpha_2} \cdots \binom{\alpha_1+\cdots+\alpha_n-(k_1+\cdots+k_{n-1})}{\alpha_n} \\ &\leq \frac{(\alpha_1+\cdots+\alpha_n)!}{\alpha_1!\cdots\alpha_n!},\end{aligned}$$

we have

$$\begin{aligned}(5.11) \ & \frac{\alpha_1!}{k_1!(\alpha_1-k_1)!} \frac{\beta_1!}{(\beta_1-k_1)!} \times \cdots \\ &\qquad \cdots \times \frac{(\alpha_1+\cdots+\alpha_n-(k_1+\cdots+k_{n-1}))!}{k_n!(\alpha_1+\cdots+\alpha_n-(k_1+\cdots+k_n))!} \frac{\beta_n!}{(\beta_n-k_n)!} \\ &= \frac{1}{k_1!\cdots k_n!} \cdot \frac{\alpha_1!\cdots\alpha_n!}{(|\alpha|-|k|)!} \frac{\beta_1!}{(\beta_1-k_1)!} \cdots \frac{\beta_n!}{(\beta_n-k_n)!} \\ &\qquad \times \binom{\alpha_1+\alpha_2-k_1}{\alpha_2} \cdots \binom{\alpha_1+\cdots+\alpha_n-(k_1+\cdots+k_{n-1})}{\alpha_n} \\ &\leq \frac{1}{k_1!\cdots k_n!} \frac{|\alpha|!}{(|\alpha|-|k|)!} \frac{\beta_1!}{(\beta_1-k_1)!} \cdots \frac{\beta_n!}{(\beta_n-k_n)!}.\end{aligned}$$

Plugging (5.16) into (5.15) and using Theorem 4.1, we have

$$(5.12)\ ||Y^{\alpha_1}*X^{\beta_1}*Y^{\alpha_2}*X^{\beta_2}*\cdots*Y^{\alpha_n}*X^{\beta_n}||_{\tau,s} \leq \sum_{k=(k_1,\cdots,k_n)} \frac{1}{k_1!\cdots k_n!}\frac{|\alpha|!}{(|\alpha|-|k|)!}\frac{\beta_1!}{(\beta_1-k_1)!}\cdots\frac{\beta_n!}{(\beta_n-k_n)!} \times(\tau_0|k|)^{\tau_0|k|}\cdot(\tau_1(|\alpha|+|\beta|-2|k|))^{\tau_1(|\alpha|+|\beta|-2|k|)} \times s_0^{\tau_0|k|}(s_1')^{\tau_1(|\alpha|+|\beta|-2k)},$$

for some $s' = s'(\tau_0,\tau_1,s_0,s_1)$. Similar as in the computations in Theorem 4.1, we have

$$(5.13)\qquad ||Y^{\alpha_1}*X^{\beta_1}*Y^{\alpha_2}*X^{\beta_2}*\cdots*Y^{\alpha_n}*X^{\beta_n}||_{\tau,s} \leq \sum_{k=(k_1,\cdots,k_n)}\binom{\beta_1}{k_1}\cdots\binom{\beta_n}{k_n}|\alpha|^{|k|} \times(\tau_0|k|)^{\tau_0|k|}\cdot(\tau_1(|\alpha|+|\beta|-2|k|))^{\tau_1(|\alpha|+|\beta|-2|k|)} \times s_0^{\tau_0|k|}(s_1')^{\tau_1(|\alpha|+|\beta|-2k)}.$$

Using $2\tau_1-1\geq\tau_0$, we have

$$|\alpha|^{|k|}|k|^{\tau_0|k|}(|\alpha|+|\beta|-2|k|)^{\tau_1(|\alpha|+|\beta|-2|k|)}\leq(|\alpha|+|\beta|)^{\tau_1(|\alpha|+|\beta|)}.$$

Plugging the above into (5.13), we have

$$||Y^{\alpha_1}*X^{\beta_1}*Y^{\alpha_2}*X^{\beta_2}*\cdots*Y^{\alpha_n}*X^{\beta_n}||_{\tau,s} \leq \sum_{k=(k_1,\cdots,k_n)}\binom{\beta_1}{k_1}\cdots\binom{\beta_n}{k_n}|\alpha|^{|k|} \times(\tau_0)^{\tau_0|k|}s_0^{\tau_0|k|}(s_1')^{-2\tau_1 k}(\tau_1(|\alpha|+|\beta|))^{\tau_1(|\alpha|+|\beta|)}(s_1')^{\tau_1(|\alpha|+|\beta|)} \leq(1+\tau_0 s_0^{\tau_0}(s_1')^{-\tau_1})(\tau_1(|\alpha|+|\beta|))^{\tau_1(|\alpha|+|\beta|)}.$$

Thus we have

$$(5.14)\ ||Y^{\alpha_1}*X^{\beta_1}*Y^{\alpha_2}*X^{\beta_2}*\cdots*Y^{\alpha_n}*X^{\beta_n}||_{\tau,s}\leq C^{\tau_1(|\alpha|+|\beta|)}(\tau_1(|\alpha|+|\beta|))^{\tau_1(|\alpha|+|\beta|)},$$

for some constant C.

For such an $\tilde{F}\in\mathcal{T}^3$, we consider the element $F=\sum a_{\delta_0\alpha\beta}Z^{\delta_0}\otimes Y^{\alpha_1}\otimes X^{\beta_1}\otimes\cdots\otimes Y^{\alpha_n}\otimes X^{\beta_n}$ by pushing all Z's to the left hand side in each term. Then, we have $||\tilde{F}||_{\tau,s}=||F||_{\tau,s}$ and $\pi(\tilde{F})=\pi(F)$. Since $\pi(F)=\sum a_{\delta_0\alpha\beta}Z^{\delta_0}*Y^{\alpha_1}*X^{\beta_1}*\cdots*Y^{\alpha_n}*X^{\beta_n}$, we have $||\pi(F)||_{\tau,s}\leq||F||_{\tau,\hat{s}}$ for some $\hat{s}=\hat{s}(\tau_0,\tau_1,s_0,s_1)$.

Thus, π extends to a continuous homomorphism from $(\mathcal{T}_\tau^3,\otimes)$ to $(S_\tau^3,*)$.

Taking its completion, we have the exact sequence

$$0 \to \mathcal{I}^3_\tau \overset{j_*}{\to} \mathcal{T}^3_\tau \overset{\pi}{\to} S^3_\tau \to 0$$

such that $\pi|S^3_\tau = \mathrm{id}$. Hence $\mathcal{T}^3_\tau = S^3_\tau \oplus \mathcal{I}^3_\tau$. It follows $S^3_\tau = \mathcal{T}^3_\tau/\mathcal{I}^3_\tau$ and then $(\mathcal{A}^3_\tau, \hat{*}) \cong (S^3_\tau, *)$, which gives Theorem5.2. □

By the same procedure as above, we have the following:

Theorem 5.3 *Assume $\tau_* = \tau_1, \frac{1}{2} \leq \tau_1$. Then,*

$$\mathcal{T}^3_{Hol,\tau_*} = S^3_{Hol,\tau_*} \oplus \mathcal{I}^3_{Hol,\tau_*} \qquad (\textit{direct sum}). \tag{5.15}$$

Moreover, $(S_{Hol,\tau^3}, *)$ is isomorphic to $(\mathcal{A}_{Hol,\tau^3}, *_\tau)$.

Let $S^2_{\mho_*}$ be the completion of S^2 in $\mathcal{T}^2_{\mho_*}$. Reminding that $\mathcal{T}^3_{\infty,\tau_*}$ coincides with $S^2_{\tau_*}[[Z]]$, we have easily

Theorem 5.4 *Let $\tau_* = \tau_1 > 0$. Then, we have*

$$\mathcal{T}^3_{\infty,\tau_*} = S^3_{\infty,\tau_*} \oplus \mathcal{I}^3_{\infty,\tau_*} \qquad (\textit{direct sum}). \tag{5.16}$$

Moreover, $(\mathcal{A}^3_{\infty,\tau_*}, *)$ is isomorphic to $(S^3_{\infty,\tau_*}, *)$.

6 Degeneration of algebraic structure

In §3, it is shown that, $(S^3_\mho, *)$ is a Fréchet algebra under certain assumptions on the weights τ and τ_*. In this section, we study algebraic structure of $(S^3_\mho, *)$ where the $*$-product diverges for some elements.

Recall the Fréchet algebra $(\mathcal{A}^3_\tau, *)$ for $\tau = (\tau_0, \tau_1, \tau_1), \tau_0 > 2\tau_1 - 1, \tau_0, \tau_1 > 0$. If $\tau_1 > \frac{1}{2}, \tau_0 > 2\tau_1 - 1$, then we see easily that

$$\mathcal{A}_{\tau_0',\tau_1} \supset \mathcal{A}_{\tau_0,\tau_1} \supset \mathcal{A}_{\tau_0,\tau_1'} \text{ where } \tau_0' = 2\tau_1 - 1, \tau_1' = \frac{1}{2}(\tau_0 + 1).$$

By Theorem 5.2 we have $\mathcal{A}_{\tau_0'\tau_1} \cong S^3_{\tau_0',\tau_1}, \mathcal{A}_{\tau_0,\tau_1'} \cong S^3_{\tau_0,\tau_1'}$, but Proposition 4.1 shows that $\mathcal{A}_{\tau_0,\tau_1} \ncong S^3_{\tau_0,\tau_1}$, since $S^3_{\tau_0,\tau_1}$ is not closed in $*$ by Proposition 4.1. It follows that $\mathcal{T}^3_{\tau_0,\tau_1} \neq S^3_{\tau_0,\tau_1} \oplus \mathcal{I}^3_{\tau_0,\tau_1}$.

6.1 Collapsing of algebras

Next, we consider the case $\frac{1}{2} > \tau_1 > 0$. In this case, the algebra $\mathcal{A}_{\tau_0,\tau_1}$ collapses to an almost formal algebra in Z.

Theorem 6.1 *Assume* $\tau = (\tau_0, \tau_1, \tau_1), \tau_0 \geq 0, \frac{1}{2} > \tau_1 > 0$. *(If* $\tau_0 = 0$, *then we read this as Hol or* ∞.*) Then, for any* $a \neq 0$, *there exist* $R_a \in \mathcal{I}^3_\tau$ *and* $H_a \in \mathcal{T}^3_\tau$ *such that*

$$1 = (a - Z){\otimes}H_a + R_a. \tag{6.1}$$

Proof. Set $X^\bullet = \frac{1}{X}{\odot}(1 - e_\odot^{\frac{2i}{a}X\odot Y})$, $X^\circ = \frac{1}{X}{\odot}(1 - e_\odot^{\frac{-2i}{a}X\odot Y})$ where $e_\odot^{tX\odot Y} = \sum_{l=0}^{\infty} \frac{t^l}{l!}(X{\odot}Y)^l$, and $\frac{1}{X}$ takes the factorization by X for the power series for $1 - e_\odot^{\frac{2i}{a}X\odot Y}$. Computing the following identity which is the associativity in the free tensor algebra

$$(X^\bullet{\otimes}X){\otimes}X^\circ - X^\bullet{\otimes}(X{\otimes}X^\circ) = 0. \tag{6.2}$$

Since the computations modulo $\mathcal{I}^3_\tau$ is that of the $*$-product (3.19), the Moyal product formula gives

$$X^\circ - X^\bullet - (1 - \frac{Z}{a}){\otimes}(e_\odot^{-\frac{2i}{a}X\odot Y}{\otimes}X^\circ - X^\bullet{\otimes}e_\odot^{\frac{2i}{a}X\odot Y}) \in \mathcal{I}^3_\tau. \tag{6.3}$$

Hence

$$\frac{1}{X}{\odot}(e_\odot^{\frac{2i}{a}X\odot Y} - e_\odot^{\frac{-2i}{a}X\odot Y}) \in \langle Z - a\rangle + \mathcal{I}^3_\tau,$$

where $\langle Z - a\rangle$ is the two sided ideal generated by $Z - a$. Thus, we have

$$\begin{aligned} &X{\otimes}(\frac{1}{X}{\odot}(e_\odot^{\frac{2i}{a}X\odot Y} - e_\odot^{\frac{-2i}{a}X\odot Y})) \\ &= (e_\odot^{\frac{2i}{a}X\odot Y} - e_\odot^{\frac{-2i}{a}X\odot Y}) - Z\cdot\frac{2}{a}{\otimes}(e_\odot^{\frac{2i}{a}X\odot Y} - e_\odot^{\frac{-2i}{a}X\odot Y}) \in \langle Z - a\rangle + \mathcal{I}^3_\tau, \end{aligned}$$

Thus we have

$$e_\odot^{\frac{2i}{a}X\odot Y} - e_\odot^{\frac{-2i}{a}X\odot Y} - 2(e_\odot^{\frac{2i}{a}X\odot Y} - e_\odot^{\frac{-2i}{a}X\odot Y}) \in \langle Z - a\rangle + \mathcal{I}^3_\tau.$$

Take the conjugation given by Lemma 3.1 in the above computations. Note that the above computations are only contained in terms of the symmetric product. If we take the conjugation of the above relations, we only replace a and i by $\bar{a}$ and $-i$, respectively. Then, we have the same relation to the above by replacing a by the complex conjugation $\bar{a}$. We have

$$e_\odot^{\pm\frac{2i}{a}X\odot Y} \in \langle Z - a\rangle + \mathcal{I}^3_\tau.$$

Since $\partial_X f$, $\partial_Y f$ can be written by using commutator bracket, this shows that

$$(X^m \circ Y^n) * \partial_X^k \partial_Y^l e_{\odot}^{\pm\frac{2i}{a} X \odot Y} * (X^{m'} \circ Y^{n'}) \in \langle Z - a \rangle + \mathcal{I}_\tau^3.$$

Hence, we have $(X^m \circ Y^n) \circ e_{\odot}^{-\frac{2i}{a} X \odot Y} \in \langle Z - a \rangle + \mathcal{I}_\tau^3$. It follows

$$\Big(\sum_{k=0}^{m} \frac{1}{k!} (\frac{2i}{a})^k (X \circ Y)^k \Big) \circ e_{\odot}^{-\frac{2i}{a} X \odot Y} \in \langle Z - a \rangle + \mathcal{I}_\tau^3.$$

Taking $m \to \infty$, we have $1 \in \overline{\langle Z - a \rangle + \mathcal{I}_\tau^3}$ where $\overline{\langle Z - a \rangle + \mathcal{I}_\tau^3}$ is a closure of the two sided ideal generated by $Z - a$ and $\underline{\mathcal{I}_\tau^3}$ in $\mathcal{T}_\tau$. This $\underline{\text{means}}$ that 1 is contained in the closed two-sided ideal $\overline{\langle Z - a \rangle}$ in $\mathcal{A}_\tau^3$. Since $\overline{\langle Z - a \rangle} = (Z - a) * \mathcal{A}_\tau^3$, there is $\tilde{H}_a \in \mathcal{A}_\tau^3$ such that $(Z - a) * \tilde{H}_a = 1$. This gives Theorem. □

Theorem 6.1 gives the following:

Theorem 6.2 *Under the same assumption as in Theorem 6.1, any element $a - Z$ for $a \neq 0$ in the Fréchet algebra $(\mathcal{A}_\tau^3, *_\tau)$ has an inverse, and* $\bigcap_{k \geq 0} Z^k * \mathcal{A}_\tau^3 = \{0\}$.

Proof. Theorem 2.1 gives that $\bigcap_{k \geq 0} Z^k * \mathcal{T}_\tau^3 = \{0\}$. It follows $\bigcap_{k \geq 0} Z^k * \mathcal{A}_\tau^3 = \{0\}$. □

For a polynomial $p(Z)$ of Z, we define a family of seminorms:

$$(6.4) \qquad ||p(Z)||_{\tau_0, s} = \sum |a_k| (\tau_0 k)^{\tau_0 k} s^{\tau_0 k}, \quad p(Z) = \sum a_k Z^k.$$

Denote by $\mathcal{Z}_{\tau_0}$ the completion of the polynomial ring by the system of seminorms (6.4). $\mathcal{Z}_{\tau_0}$ is a closed algebra of $\mathcal{T}_\tau^3$, $\tau = (\tau_0, \tau_1, \tau_1)$. We get $\mathcal{Z}_{\tau_0} \cap \mathcal{I}_\tau^3 = \{0\}$ by Theorem5.1. However, $\mathcal{Z}_{\tau_0} + \mathcal{I}_\tau^3$ is not a closed subalgebra in $\mathcal{T}_\tau^3$ as we see below.

Consider the algebra $(\mathcal{A}_\tau^3, *_\tau)$. We denote by $\mathfrak{Z}$ the closure of the algebra generated by Z and 1 in $\mathcal{A}_\tau^3$. Then $\mathfrak{Z}$ is a commutative Fréchet algebra, and $a - Z$, $a \neq 0$ is invertible. Now, we get

$$(6.5) \qquad \mathcal{Z}_{\tau_0} \cong \mathcal{Z}_{\tau_0} / (\mathcal{Z}_{\tau_0} \cap \mathcal{I}_{\tau_0}^3) \subset \overline{\mathcal{Z}_{\tau_0} + \mathcal{I}_{\tau_0}^3} / \mathcal{I}_{\tau_0}^3 = \mathfrak{Z}.$$

where $\overline{\mathcal{Z}_{\tau_0} + \mathcal{I}_{\tau_0}^3}$ denotes the closure of $\mathcal{Z}_{\tau_0} + \mathcal{I}_{\tau_0}^3$. Thus, $\mathcal{Z}_{\tau_0}$ is contained in $\mathfrak{Z}$ and $\mathfrak{Z}$ is viewed as a completion of $\mathcal{Z}_{\tau_0}$ by taking a weaker topology than the previous one. Remark that $Z - a$, $a \neq 0$, is not invertible in $\mathcal{Z}_{\tau_0}$, but this is invertible in $\mathfrak{Z}$. Hence we see that $\mathcal{Z}_{\tau_0} + \mathcal{I}_\tau^3$ is not a closed subalgebra in $\mathcal{T}_\tau^3$.

The following shows that $\mathcal{A}_\tau^3$ is almost formal.

Proposition 6.1 $\mathfrak{Z}$ *is contained densely in the space of formal power series ring* $\mathbb{C}[[Z]]$ *and also contains the space of* $\mathbb{C}((Z))$ *of convergent power series in* Z.

Proof. Since $\mathcal{Z}_\tau$ does not clash, $\mathfrak{Z}$ is contained densely in $\mathbb{C}[[Z]]$. Let D_n be the disk with the radius $\frac{1}{n}$ with the boundary C_n with the center at the origin. Let $f(\theta)$ be a continuous function on C_n. By the completeness of $\mathfrak{Z}$, we have

$$\hat{f} = \frac{1}{2\pi i}\int_{C_n} f(\theta)(\theta - Z)^{-1}d\theta \in \mathfrak{Z}. \tag{6.6}$$

$\hat{f}$ is holomorphic on D_n and extends continuously to C_n. Conversely, such function is written as the form. Moving n, we see that $\mathfrak{Z}$ contains every function which converges on an appropriate disk with the center at the origin. □

Remark Z is central in the algebra $(\mathcal{A}^3_\tau, *)$. Hence, it seems possible to insert any number to Z. However, if $\tau_0 \geq 0$ and $\tau_1 < \frac{1}{2}$, the algebra $(\mathcal{A}^3_\tau, *)$ collapses to $\{0\}$, if we insert to Z a non-zero number a.

7 Extensions by breaking symmetry

In previous sections, we saw that $(\mathcal{E}_{Hol,2}(\mathbb{C}^3); *)$ is an associative algebra, but the $*$-product did not extend to $(\mathcal{E}_{Hol,p}(\mathbb{C}^3); *)$ for $p > 2$. Note that exponential functions of quadratic form of x, y are not contained in $\mathcal{E}_{Hol,2}(\mathbb{C}^3)$, but in $\mathcal{E}_{Hol,p}(\mathbb{C}^3)$ for every $p > 2$. In this section we give a certain class of entire functions on $\mathbb{C}^3$ which contains the function $e^{-(x^2+y^2)}$ and is closed under the $*$-product. Remark first that $\mathcal{E}_{Hol,p}(\mathbb{C}^3)$ has the following $SL(2;\mathbb{C})$-symmetry: The transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = P \begin{bmatrix} x \\ y \end{bmatrix}, \quad z' = z, \qquad P \in SL(2;\mathbb{C}) \tag{7.1}$$

leaves the space $\mathcal{E}_{Hol,p}(\mathbb{C}^3)$ invariant for every $p > 0$ and gives an *isomorphism* φ_P in the sense that $\varphi_P(f) * \varphi_P(g)$ is defined and $\varphi_P(f * g) = \varphi_P(f) * \varphi_P(g)$, whenever $f * g$ is well-defined.

Our class of functions constructed in this section has not the $SL(2;\mathbb{C})$-symmetry, but $SL(2;\mathbb{R})$-symmetry.

7.1 Functions of S_a^b-type

We introduce a class of entire functions on $\mathbb{C}^3$ originally given by Gel'fand-Shilov [2] with a slight modification.

For $0 < a, b < 1$ and $A, B > 0$, an entire function $f(z, x, y)$ on $\mathbb{C}^3$ is called a function of *type* $S_{a,A}^{b,B}$, if the variables z, x, y are restricted in the real, then f satisfies the following inequality for every fixed $z \in \mathbb{R}$and for all non-negative integers $k = (k_1, k_2), \ell = (\ell_1, \ell_2)$, where $|k| = k_1 + k_2, |\ell| = \ell_1 + \ell_2$:

$$(7.2) \qquad |x^{k_1} y^{k_2} (\partial_x^{\ell_1} \partial_y^{\ell_2} f)(z, x, y)| \leq C A^{|k|} B^{|\ell|} |k|^{a|k|} |\ell|^{b|\ell|}, \quad (z, x, y) \in \mathbb{R}^3.$$

We denote by $S_{a,A}^{b,B}(\mathbb{C}^3)$ the space of all functions of $S_{a,A}^{b,B}$-type. $S_{a,A}^{b,B}(\mathbb{C}^3)$ is invariant only if the matrix P in (7.1) is restricted in $SL(2, \mathbb{R})$. Thus the space $S_{a,A}^{b,B}(\mathbb{C}^3)$ looses the symmetry of $SL(2, \mathbb{C})$.

It is obvious that if $A \leq A', B \leq B'$, then

$$S_{a,A}^{b,B}(\mathbb{C}^3) \subset S_{a,A'}^{b,B'}(\mathbb{C}^3).$$

$S_{a,A}^{b,B}(\mathbb{C}^3)$ is a Fréchet space defined by a system of countable seminorms. We define

$$S_a^b(\mathbb{C}^3) = \bigcup_{A,B>0} S_{a,A}^{b,B}(\mathbb{C}^3)$$

with the direct limit topology, and also

$$T_a^b(\mathbb{C}^3) = \bigcap_{A,B>0} S_{a,A}^{b,B}(\mathbb{C}^3)$$

with the projective limit topology. $T_a^b(\mathbb{C}^3)$ is a Fréchet space defined by a system of countable seminorms.

$$T_a^b(\mathbb{C}^3) \subset S_{a,A}^{b,B}(\mathbb{C}^3) \subset S_a^b(\mathbb{C}^3) \subset T_{a+\varepsilon}^{b+\varepsilon}(\mathbb{C}^3)$$

for every $\varepsilon > 0$.

We denote

$$T_{a+}^{b+}(\mathbb{C}^3) = \bigcap_{\varepsilon>0} T_{a+\varepsilon}^{b+\varepsilon}(\mathbb{C}^3).$$

The following is not hard by tracing Ch.IV, 2.3 of [2]:

Proposition 7.1 *If* $0 < b < 1$, *every function* $f \in S_a^b(\mathbb{C}^3)$ *is an entire function on* $\mathbb{C}^3$ *having the following estimate, if* (z, x, y) *is restricted in the real line:*

$$|f(z, x + i\xi, y + i\eta)| \leq C \exp(-\alpha |(x, y)|^{1/a} + \beta |(\xi, \eta)|^{1/(1-b)})$$

for some $\alpha, \beta > 0$. Conversely, if $f(z,x,y)$ defined on $\mathbb{C}\times\mathbb{R}^2$ is holomorphic in $z \in \mathbb{C}$ and for every fixed $z \in \mathbb{C}$, f satisfies the inequality (7.2) for $0 < b < 1$, then such f can be extended uniquely as an entire function having the estimate given above.

Since the following estimate

$$\begin{aligned}\exp\big(-\alpha|(x,y)|^{1/a} + \beta|(\xi,\eta)|^{1/(1-b)}\big) \leq & C\exp\beta|(x+i\xi, y+i\eta)|^{1/(1-b)} \\ \leq & C\exp s|(x+i\xi, y+i\eta)|^{p}\end{aligned}$$

holds for every $p > \frac{1}{1-b}$ and every $s > 0$, we have $S^b_a(\mathbb{C}^3) \subset \mathcal{E}_{Hol,\frac{1}{1-b}+}(\mathbb{C}^3)$ and the inclusion is continuous.

It is also known

Proposition 7.2 ([2] p.227) *If $a + b < 1$, then $S^b_a(\mathbb{C}^3) = \{0\}$*

Remark that $e^{-(x^2+y^2)} \in S^{1/2}_{1/2}(\mathbb{C}^3)$, and $\in \mathcal{E}_{2+}(\mathbb{C}^3)$.

Let $\mathcal{F}$ denote the Fourier transform:

$$(\mathcal{F}f)(z,s,t) = \int f(z,x,y)e^{i(sx+ty)}\bar{d}x\bar{d}y.$$

It is known in [2] p.205

Lemma 7.1 *For $0 < a, b < 1$ and $a + b > 1$, the Fourier transform has the property $\mathcal{F}S^{b,B}_{a,A}(\mathbb{C}^3) = S^{a,A}_{b,B}(\mathbb{C}^3)$.*

7.2 $*$-product

Lemma7.1 shows that $S^a_a(\mathbb{C}^3)$ is invariant under the Fourier transform, and Proposition7.2 shows only the case $\frac{1}{2} \leq a$ is nontrivial. By Proposition7.1 we treat the case $a < 1$, where

$$S^a_a(\mathbb{C}^3) \subset \mathcal{E}_{Hol,\frac{1}{a}}(\mathbb{C}^3).$$

It is easy to see by (3.8) that $f \in S^a_a(\mathbb{C}^3)$, if and only if there are $A_f(z), C_f(z) > 0$ such that if x, y are restricted in $\mathbb{R}$, the estimate

$$|x^{k_1}y^{k_2}(\partial_x^{\ell_1}\partial_y^{\ell_2}f)(z,x,y)| \leq C_f A_f{}^{|k|+|\ell|}(|k|+|\ell|)^{a(|k|+|\ell|)}$$

holds for any non-negative integers $k = (k_1, k_2), \ell = (\ell_1, \ell_2)$. Similarly, $f \in T^a_a(\mathbb{C}^3)$, if and only if for every $\varepsilon > 0$ there is a constant $C_\varepsilon(z) > 0$ such that if x, y are restricted in $\mathbb{R}$ the estimate

$$|x^{k_1}y^{k_2}(\partial_x^{\ell_1}\partial_y^{\ell_2}f)(z,x,y)| \leq C_\varepsilon\ \varepsilon^{|k|+|\ell|}(|k|+|\ell|)^{a(|k|+|\ell|)}$$

holds for any non-negative integers $k = (k_1, k_2), \ell = (\ell_1, \ell_2)$.

Theorem 7.1 *If* $\frac{1}{2} \le a < 1$, $T^a_a(\mathbb{C}^3)$ *and* $S^a_a(\mathbb{C}^3)$ *are closed under the* $*$*-product, and form topological associative algebras.*

Proof. Using Fourier transform, we write as follows:

$$f(z,x,y) = \int \hat{f}(z,s,t) e^{i(sx+ty)} \bar{d}s \bar{d}t,$$
$$g(z,x,y) = \int \hat{g}(z,s',t') e^{i(s'x+t'y)} \bar{d}s' \bar{d}t'.$$

Since

$$e^{i(sx+ty)} * e^{i(s'x+t'y)} = e^{\frac{iz}{2}(st'-ts')} e^{i(s+s')x+i(t+t')y},$$

we have

$$\begin{aligned} & f(z,x,y) * g(z,x,y) \\ (7.3) \qquad & = \iint \hat{f}(z,s,t)\hat{g}(z,s',t') e^{\frac{iz}{2}(st'-ts')} e^{i(s+s')x+i(t+t')y} \bar{d}s\bar{d}t\bar{d}s'\bar{d}t'. \end{aligned}$$

Changing variables gives

$$f(z,x,y)*g(z,x,y) = \iint \hat{f}(z,s,t)\hat{g}(z,s'-s,t'-t) e^{\frac{iz}{2}(st'-ts')} e^{is'x+it'y} \bar{d}s\bar{d}t\bar{d}s'\bar{d}t'.$$

If we set $f(z,x,y) * g(z,x,y) = h(z,x,y) = \int \hat{h}(z,s',t') e^{is'x+it'y} \bar{d}s'\bar{d}t'$, then

$$\hat{h}(z,s,t) = \int \hat{f}(z,s',t')\hat{g}(z,s-s',t-t') e^{\frac{iz}{2}(s't-t's)} \bar{d}s'\bar{d}t'.$$

Remarking that the estimate in Proposition 7.1 holds also for $\hat{f}, \hat{g}$ via Lemma 7.1, we see that $\hat{h}(z,s,t)$ is well-defined and a holomorphic function in $z \in \mathbb{C}$. Hence for the proof that $S^a_a(\mathbb{C}^3)$ are closed under the $*$-product, we have only to show that $h(z,x,y) \in S^{a,KA}_{a,KA}(\mathbb{C}^3)$ for some $K > 0$, supposing $f, g \in S^{a,A}_{a,A}(\mathbb{C}^3)$.

A rough estimate gives

$$\begin{aligned} & |x^{k_1} y^{k_2} \partial_x^{\ell_1} \partial_y^{\ell_2} h(z,x,y)| \\ & \le \int \left|\partial_s^{k_1}\partial_t^{k_2}\left(\hat{f}(z,s',t')\hat{g}(z,s-s',t-t') s^{\ell_1} t^{\ell_2} e^{\frac{iz}{2}(s't-t's)}\right)\right| \bar{d}s'\bar{d}t'. \end{aligned}$$

To estimate the integral, we use estimates :

$$\begin{aligned} \left|\hat{f}(z,s',t')|s'|^{m_1}|t'|^{m_2}(1+|s'|^2+|t'|^2)\right| & \le A^{|m|+2}(|m|+2)^{a(|m|+2)} \\ & \le (eA)^2 (eA)^{|m|}|m|^{a|m|}, \end{aligned}$$

$$\left|\partial_s^{j_1}\partial_t^{j_2}\hat{g}(z,s-s',t-t')|s-s'|^{m_1}|t-t'|^{m_2}\right| \leq A^{|m|+|j|}(|m|+|j|)^{a(|m|+|j|)},$$

Then, setting $C = \int(1+|s'|^2+|t'|^2)^{-1}\,ds'dt'$, we have

$$\begin{aligned}&|u^{k_1}v^{k_2}\partial_u^{\ell_1}\partial_v^{\ell_2}h(u,v)|\\ &\leq \sum_{|i|+|j|=|\ell|} C2^{|k|+|\ell|}(1+|z|)^{|k|}A^{|k|+2}A^{|k|+|\ell|}(|k|+2)^{a(|k|+|i|)}(|k|+|j|)^{a(|k|+|j|)}\\ &\leq C'(2(1+|z|)A^2)^{|k|+|\ell|}(|k|+|\ell|)^{a(|k|+|\ell|)}.\end{aligned}$$

This yields the desired estimate.

The associativity follows easily from the product formula (7.3).

□

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