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Multi-dimensional Schwarzian derivative revisited

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Abstract

We give intrinsic formula for the multi-dimensional Schwarzian derivative on a manifold M endowed with a projective connection. This Schwarzian derivative is naturally related to the space of second-order linear differential operators acting on tensor densities on M.

1 Introduction

Let $\mathcal{F}_\lambda$ be the space of tensor densities of degree λ on the circle S^1, and $\mathrm{Diff}(S^1)$ be the group of diffeomorphisms of S^1 preserving orientation. The space $\mathcal{F}_\lambda$ admits naturally a $\mathrm{Diff}(S^1)-$module structure. Consider the space of Sturm-Liouville operators $A = -2d^2/dx^2 + u(x) : \mathcal{F}_{-1/2} \to \mathcal{F}_{3/2}$, where $u(x) \in \mathcal{F}_2$ is the potential, x is an affine parameter on S^1. This space was already studied as a module over the group $\mathrm{Diff}(S^1)$ (see e.g., [9, 14]). Classical Schwarzian derivative is related to the space of Sturm-Liouville operators by the following way: the action of a diffeomorphism $f \in \mathrm{Diff}(S^1)$ on the Sturm-Liouville operator A is an operator with potential $u \circ f^{-1} \cdot (f^{-1})'^2 + S(f^{-1})$, where

$$S(f) = \left(\frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 \right)(dx)^2, \tag{1.1}$$

is the so-called Schwarzian derivative.

Said differently, the space of Sturm-Liouville operators viewed as a module over the group $\mathrm{Diff}(S^1)$ is a non-trivial deformation of the quadratic differentials $\mathcal{F}_2$. This deformation is generated by a non-trivial 1-cocycle on $\mathrm{Diff}(S^1)$ with values in $\mathcal{F}_2$, given actually by the Schwarzian derivative (1.1) (see [2]). On the other hand, the space of Sturm-Liouville operators and the module $\mathcal{F}_2$ are isomorphic as $\mathrm{PSL}_2(\mathbb{R})-$module; it follows from the fact that the kernel of the Schwarzian derivative is the subgroup $\mathrm{PSL}_2(\mathbb{R})$.

Consider now $\mathcal{D}^2_{\lambda,\mu}(S^1)$ the space of second-order linear differential operators acting from $\lambda-$densities to $\mu-$densities. This space admits naturally a two parameter family of $\mathrm{Diff}(S^1)-$modules (see [2, 7]). Recall that the space of Sturm-Liouville operators is a submodule of $\mathcal{D}^2_{-\frac{1}{2},\frac{3}{2}}(S^1)$. For $\delta := \mu - \lambda$ generic, the module $\mathcal{D}^2_{\lambda,\mu}(S^1)$ is a non-trivial deformation of the module $\mathcal{M}_\delta := \mathcal{F}_\delta \oplus \mathcal{F}_{\delta-1} \oplus \mathcal{F}_{\delta-2}$ (cf. [4, 7]). On the other hand, these

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modules are isomorphic as $\mathrm{PSL}_2(\mathbb{R})$−module. The problem of deformation of the module $\mathcal{M}_\delta$ with respect to the Lie group $\mathrm{PSL}_2(\mathbb{R})$ is related to the cohomology group

$$\mathrm{H}^1(\mathrm{Diff}(S^1), \mathrm{PSL}_2(\mathbb{R}); \mathrm{End}(\mathcal{M}_\delta)). \tag{1.2}$$

(i.e. Consider cochains on $\mathrm{Diff}(S^1)$ vanishing on the Lie group $\mathrm{PSL}_2(\mathbb{R})$ see [6]). Restrict the coefficients of the group of cohomology (1.2) to the submodule $\mathcal{D}(\mathcal{M}_\delta) \subset \mathrm{End}(\mathcal{M}_\delta)$ (i.e. linear differential operators on $\mathcal{M}_\delta$). This group of cohomology was calculated in [2], it is one dimension, for δ generic, generated by a 1-cocycle given as a multiplication operator by the Schwarzian derivative (1.1).

In this paper we are interested to generalize, in higher dimension, classical Schwarzian derivative (1.1) in the sense described above. Let us explicate this approach. Let M be a manifold of any dimension, and $\mathrm{Diff}(M)$ (resp. $\mathrm{Vect}(M)$) be the group of diffeomorphisms of M (resp. Lie algebra of vector fields on M). Denote by $\mathcal{F}_\lambda$ the space of λ−densities on M. Consider $\mathcal{D}^2_{\lambda,\mu}(M)$ the space of second-order linear differential operators: $\mathcal{F}_\lambda \to \mathcal{F}_\mu$. This space admits naturally a two parameter family of $\mathrm{Diff}(M)$−modules (resp. $\mathrm{Vect}(M)$−modules) (see [1, 3, 5, 10, 11]). Consider now $\mathcal{S}_2(M)$ the space of symmetric contravariant tensor fields of degree less than two on M. One can define a one parameter family of $\mathrm{Diff}(M)$−modules on $\mathcal{S}_2$ by taking $\mathcal{S}_{2,\delta}(M) := \mathcal{S}_2(M) \otimes \mathcal{F}_\delta$. Following [1, 5, 10], the space $\mathcal{D}^2_{\lambda,\mu}(M)$ can be viewed as a non-trivial deformation of the module $\mathcal{S}_{2,\delta}(M)$, where $\delta = \mu - \lambda$. The problem of deformation of the module $\mathcal{S}_{2,\delta}(M)$ is related to the cohomology group

$$\mathrm{H}^1(\mathrm{Diff}(M), \mathrm{End}(\mathcal{S}_{2,\delta}(M))) \qquad (\textit{resp. } \mathrm{H}^1(\mathrm{Vect}(M), \mathrm{End}(\mathcal{S}_{2,\delta}(M)))).$$

Suppose now $M = \mathbb{R}^n$ endowed with a projective structure. Then the problem of deformation with respect to the Lie algebra $\mathrm{sl}_{n+1}(\mathbb{R})$ is given by the relative cohomology group

$$\mathrm{H}^1(\mathrm{Vect}(\mathbb{R}^n), \mathrm{sl}_{n+1}(\mathbb{R}); \mathrm{End}(\mathcal{S}_{2,\delta}(M))), \tag{1.3}$$

For $\delta = 0$, the group of cohomology (1.3) was calculated in [11] (restricting coefficients on the space of differential operators on $\mathcal{S}_{2,\delta}$). This group is generated by two 1-cocycles, one of the two 1-cocycles is the infinitesimal multi-dimensional Schwarzian derivative. Our goal is to integrate these 1-cocyles on the group $\mathrm{Diff}(\mathbb{R}^n)$ and find which of these 1-cocycles inherits naturally all properties of the classical Schwarzian derivative.

The aim of this paper is to give an intrinsic formula for the multi-dimensional Schwarzian derivative introduced in [3]. We explain, as done in the projectively flat case (see [3]), that this derivative is related to the space of second-order linear differential operators acting on tensor densities.

2 Deformation of the space of symbols

Let M be an oriented manifold of dimension n. Fix an affine connection ∇ on M. In this section we will recall some results on the projectively equivariant quantization (see [1, 10]).

2.1 Space of linear differential operators as a module

Let $\mathcal{F}_\lambda(M)$, or $\mathcal{F}_\lambda$ for simplify, be the space of tensor densities on M. This space admits naturally a Diff(M)-module structure:

Let $f \in \mathrm{Diff}(M)$ and $\phi \in \mathcal{F}_\lambda$. In a local coordinates (x^i), the action is given by

$$f^*\phi = \phi \circ f^{-1} \cdot (J_{f^{-1}})^\lambda, \tag{2.1}$$

where $J_f = |Df/Dx|$ is the Jacobian of f.
Consider $\mathcal{D}_{\lambda,\mu}(M)$ the space of linear differential operators acting on tensor densities

$$A : \mathcal{F}_\lambda \to \mathcal{F}_\mu. \tag{2.2}$$

The action of Diff(M) on $\mathcal{D}_{\lambda,\mu}(M)$ depends on two parameters λ and μ. This action is given by the equation

$$f_{\lambda,\mu}(A) = f^* \circ A \circ f^{*-1}, \tag{2.3}$$

where f^* is the action (2.1) of Diff(M) on $\mathcal{F}_\lambda$.

The formulæ (2.1) and (2.3) do not depend on the choice of a system of coordinates.

Denote by $\mathcal{D}^2_{\lambda,\mu}(M)$ the space of second-order linear differential operators with the Diff$(M)-$module structure given by (2.3). The space $\mathcal{D}^2_{\lambda,\mu}(M)$ is in fact a Diff(M)-submodule of $\mathcal{D}_{\lambda,\mu}(M)$.

In a local coordinates (x^i), one can write $A \in \mathcal{D}^2_{\lambda,\mu}(M)$:

$$A = a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(x)\frac{\partial}{\partial x^i} + c(x), \tag{2.4}$$

with the coefficients $a^{ij}(x), b^i(x), c(x) \in C^\infty(M)$, where $i, j = 1, \ldots, n$. (The summation is understood in repeated indices).

2.2 Space of symbols

The space of symbols, Pol(T^*M), is the space of functions on the cotangent bundle T^*M polynomial on the fibers. This space is naturally isomorphic to the space $\mathcal{S}(M)$ of symmetric contravariant tensor fields on M. In a local coordinates (x^i, ξ_i), one can write $P \in \mathcal{S}(M)$ in the form

$$P = \sum_{l \geq 0} P^{i_1,\ldots,i_l}\xi_{i_1}\cdots\xi_{i_l},$$

with $P^{i_1,\ldots,i_l}(x) \in C^\infty(M)$.

One defines a one parameter family of Diff$(M)-$modules on the space of symbols by

$$\mathcal{S}_\delta(M) := \mathcal{S}(M) \otimes \mathcal{F}_\delta.$$

Let us explicate this action.

Take $f \in \mathrm{Diff}(M)$ and $P \in \mathcal{S}_\delta(M)$. Then, in a local coordinates (x^i), one has

$$f_\delta(P) \;=\; f^*P \cdot (J_{f^{-1}})^\delta, \tag{2.5}$$

where $J_f = |Df/Dx|$ is the Jacobian of f, and f^* is the natural action of Diff(M) on $\mathcal{S}(M)$.

We have then a filtration of Diff(M)−module given by

$$\mathcal{S}_\delta(M)=\bigoplus_{k=0}^{\infty}\mathcal{S}_\delta^k(M),$$

where $S_\delta^k(M)$ is the space of contravariant tensor fields of degree k endowed with the Diff(M)− module structure (2.5).

We are interested to study the space of contravariant tensor fields of degree less than two noted $\mathcal{S}_{\delta,2}(M)$ (i.e. $\mathcal{S}_{\delta,2}(M)=\mathcal{S}_\delta^2(M)\oplus\mathcal{S}_\delta^1(M)\oplus\mathcal{S}_\delta^0(M)$).

2.3 Projectively equivariant quantization map

The problem of the equivariant quantization is to find a map between $\mathcal{S}_\delta(M)$ and the space $\mathcal{D}_{\lambda,\mu}(M)$, where $\delta=\mu-\lambda$, equivariant with respect to the action (2.5) and (2.3). In the one dimensional case $M=S^1$, there exists such quantization map, for δ generic, equivariant with respect to the action of the subgroup $\mathrm{PSL}_2(\mathbb{R})\subset\mathrm{Diff}(S^1)$ (cf. [4, 7]). In higher dimension $n>1$, there exists a quantization map

$$Q:\mathcal{S}_{\delta,2}(M)\to\mathcal{D}^2_{\lambda,\mu}(M),$$

given as follows: for $\delta\neq 1,\frac{n+3}{n+1},\frac{n+2}{n+1}$, and for each $P=P^{ij}\xi_i\xi_j+P^i\xi_i+P_0\in\mathcal{S}_{\delta,2}$ one associates a linear differential operator given by

$$Q(P)=P^{ij}\nabla_i\nabla_j+(\alpha_1\nabla_iP^{ij}+P^j)\nabla_j+\alpha_2\nabla_i\nabla_jP^{ij}+\alpha_3\nabla_iP^i+\alpha_4R_{ij}P^{ij}+P_0,\quad(2.6)$$

where R_{ij} is the components of Ricci tensor of the connection ∇ (cf. [1, 11]). The constants $\alpha_1,\dots,\alpha_4$ are given by

$$\alpha_1=\frac{2+2\lambda(n+1)}{2+(n+1)(1-\delta)},\qquad \alpha_3=\frac{\lambda}{1-\delta},$$
$$\alpha_2=\frac{\lambda(n+1)(1+\lambda(n+1))}{((1-\delta)(1+n)+1)((1-\delta)(1+n)+2)},\quad \alpha_4=\frac{\lambda(\mu-1)(n+1)^2}{(1-n)((1-\delta)(1+n)+1)}.$$

The quantization map (2.6) has the following properties:
(i) It depends only on the projective class of the connection ∇ (see section 3.3).
(ii) If $M=\mathbb{R}^n$ endowed with a projective structure (i.e. coordinates change are given by projective transformations) the map (2.6) is unique, equivariant with respect to the subgroup $\mathrm{SL}_{n+1}(\mathbb{R})\subset\mathrm{Diff}(\mathbb{R}^n)$.

Remark 2.1 In the one-dimensional case, classical Schwarzian derivative appears as an obstruction to the equivariant quantization on the full group Diff(S^1) (cf. [3]). In the same way, we will see that the multi-dimensional Schwarzian derivative appears as an obstruction to the equivariant quantization (2.6) on the full group Diff(M).

3 Introducing the Schwarzian derivative

3.1 Cohomology of the Lie algebra of vector fields

Consider the standard $\mathrm{SL}_{n+1}(\mathbb{R})$-action on $\mathbb{R}^n$. Let $\mathcal{D}(\mathcal{S}_{2,\delta}(\mathbb{R}^n))$ be the space of linear differential operators on $\mathcal{S}_{2,\delta}(\mathbb{R}^n)$. This space is decomposed, as a Vect$(\mathbb{R}^n)$− module, into direct sum

$$\mathcal{D}(\mathcal{S}_{2,\delta}(\mathbb{R}^n)) = \bigoplus_{k,m=0}^{2} \mathcal{D}(\mathcal{S}_\delta^k(\mathbb{R}^n), \mathcal{S}_\delta^m(\mathbb{R}^n)),$$

where $\mathcal{D}(\mathcal{S}_\delta^k(\mathbb{R}^n), \mathcal{S}_\delta^m(\mathbb{R}^n)) \subset \mathrm{Hom}(\mathcal{S}_\delta^k(\mathbb{R}^n), \mathcal{S}_\delta^m(\mathbb{R}^n))$.

For $\delta = 0$, the first group of differential cohomology of $\mathrm{Vect}(\mathbb{R}^n)$, with coefficients in the space $\mathcal{D}(\mathcal{S}^k(\mathbb{R}^n), \mathcal{S}^m(\mathbb{R}^n))$ of linear differential operators from $\mathcal{S}^k(\mathbb{R}^n)$ to $\mathcal{S}^m(\mathbb{R}^n)$, vanishing on the Lie algebra $\mathrm{sl}_{n+1}(\mathbb{R})$, was calculated in [11]. For $n \geq 2$ the result is as follows

$$\mathrm{H}^1(\mathrm{Vect}(\mathbb{R}^n), \mathrm{sl}_{n+1}(\mathbb{R}); \mathcal{D}(\mathcal{S}^k(\mathbb{R}^n), \mathcal{S}^m(\mathbb{R}^n))) = \begin{cases} \mathbb{R}, & k-m=2, \\ \mathbb{R}, & k-m=1, m\neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

For $k = 2$, the 1-cocycles generating the group of cohomology (3.1) are integrated on the group $\mathrm{Diff}(\mathbb{R}^n)$ (see [3]). We recall in the following section explicit formulæ of these 1-cocylces.

3.2 Explicit formulæ for the 1-cocycles on $\mathrm{Diff}(\mathbb{R}^n)$

Let $f(x^1, \ldots, x^n) = (f^1(x), \ldots, f^n(x)) \in \mathrm{Diff}(\mathbb{R}^n)$. The following $(2,1)-$tensors

$$\ell_0(f) = \left(\frac{\partial^2 f^l}{\partial x^i \partial x^j} \frac{\partial x^k}{\partial f^l} - \frac{1}{n+1} \left(\delta_j^k \frac{\partial \log J_f}{\partial x^i} + \delta_i^k \frac{\partial \log J_f}{\partial x^j} \right) \right) dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k} \tag{3.2}$$

where $J_f = |Df/Dx|$ is the Jacobian, is introduced in the literature as the multi-dimensional Schwarzian derivative. It is well known that the map

$$f \mapsto \ell_0(f^{-1}), \tag{3.3}$$

defines a non-trivial 1-cocyle on $\mathrm{Diff}(\mathbb{R}^n)$ with values in the space of tensor fields of type $(2,1)$, vanishing on the group $\mathrm{SL}_{n+1}(\mathbb{R})$ (cf. [13]).

For $k = 2$, and $m = 1$, the 1-cocyle on $\mathrm{Diff}(\mathbb{R}^n)$ with values in $\mathcal{D}(\mathcal{S}^2(\mathbb{R}^n), \mathcal{S}^1(\mathbb{R}^n))$, vanishing on the group $\mathrm{SL}_{n+1}(\mathbb{R})$, is an operator given by contracting a $2-$order contravariant tensor fields with the 1-cocycle (3.3).

Remark 3.1 As explained in [3], the tensor (3.2) cannot be considered as a multi-dimensional analogue of the Schwarzian derivative, actually in the one dimensional case $\ell_0 \equiv 0$.

For $k = 2$, and $m = 0$, consider the differential operator $S(f) : \mathcal{S}^2(\mathbb{R}^n) \to \mathcal{S}^0(\mathbb{R}^n)$, where

$$S(f)_{ij} = \ell_0(f)_{ij}^k \frac{\partial}{\partial x^k} + \frac{\partial^3 f^k}{\partial x^i \partial x^j \partial x^l} \frac{\partial x^l}{\partial f^k} - \frac{n+3}{n+1} \frac{\partial^2 J_f}{\partial x^i \partial x^j} J_f^{-1} + \frac{n+2}{n+1} \frac{\partial J_f}{\partial x^i} \frac{\partial J_f}{\partial x^j} J_f^{-2}, \tag{3.4}$$

and $\ell_0(f)_{ij}^k$ are the components of the tensor (3.2). The map

$$f \mapsto S(f^{-1}),$$

defines a 1-cocycle on $\mathrm{Diff}(\mathbb{R}^n)$ with values in $\mathcal{D}(\mathcal{S}^2(\mathbb{R}^n), \mathcal{S}^0(\mathbb{R}^n))$, vanishing on the subgroup $\mathrm{SL}_{n+1}(\mathbb{R})$.

Remark 3.2 The operator (3.4) was introduced in [3] as the multi-dimensional Schwarzian derivative since in the one-dimensional case it coincides (up to sign) with the Classical Schwarzian derivative (1.1).

3.3 Global definition of the 1-cocycles

Let M be a manifold of dimension n. Fix a symmetric affine connection Γ on M. Let us recall the notion of projective connection which allows us to globalize the 1-cocycles (3.2) and (3.4).

A projective connection is an equivalent class of symmetric affine connections giving the same unparameterized geodesics.

Following [8], the symbol of the projective connection is given by the expression

$$\Pi^k_{ij} = \Gamma^k_{ij} - \frac{1}{n+1}\left(\delta^k_i \Gamma_j + \delta^k_j \Gamma_i\right), \tag{3.5}$$

where Γ^k_{ij} are the Christoffel symbols of the connection Γ and $\Gamma_i = \Gamma^j_{ij}$.
Two affine connection Γ and $\tilde{\Gamma}$ are projectively equivalent if the corresponding symbols (3.5) coincide.
A projective connection on M is called *flat* if in a neighborhood of each point there exists a local coordinate system $(x^1, \ldots, x^n)$ such that the symbols Π^k_{ij} are identically zero (see [8] for a geometric definition). Every flat projective connection defines a projective structure on M.
Let Π and $\tilde{\Pi}$ be two projective connections on M. Then the difference $\Pi - \tilde{\Pi}$ is a well-defined $(2,1)-$tensor fields. Therefore, it is clear that a projective connection on M leads to the following 1-cocycle on Diff(M):

$$\ell(f^{-1}) = \left((f^{-1})^* \Pi^k_{ij} - \Pi^k_{ij}\right) dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k} \tag{3.6}$$

vanishing on (locally) projective diffeomorphisms. This formula is independent on the choice of the coordinate system. The 1-cocycle (3.6) globalizes the 1-cocycle (3.3) in any manifold.

As in section (3.2), we define a 1-cocycle on Diff(M) with values in $\mathcal{D}(\mathcal{S}^2(M), \mathcal{S}^1(M))$ by contracting a 2−order contravariant tensor fields with the 1-cocycle (3.6).

3.4 The main definition

Let us introduce the formula of the multi-dimensional Schwarzian derivative by the expression

$$S(f)_{ij} = \ell(f)^k_{ij} \nabla_k - \frac{2 - \delta(n+1)}{n-1} \nabla_k \left(\ell(f)^k_{ij}\right) + \frac{(n+1)(1-\delta)}{n-1} \ell(f)^k_{im} \ell(f)^m_{kj}, \tag{3.7}$$

where $\ell(f)^k_{ij}$ are the components of the $(2,1)-$tensor (3.6). This expression is a linear differential operator from $\mathcal{S}^2_\delta(M)$ to $\mathcal{S}^0_\delta(M)$.

Remark 3.3 Even if the cohomology group (3.1) is calculated only for $\delta = 0$, the formula of the Schwarzian derivative (3.7) is given for δ not necessarily zero.

Theorem 3.4 *(i) The map $f \mapsto S(f^{-1})$ defines a non-trivial 1-cocycle on* $\mathrm{Diff}(M)$ *with values in $\mathcal{D}(\mathcal{S}^2_\delta(M), \mathcal{S}^0_\delta(M))$.*

(ii) The operator (3.7) depends only on the projective class of the connection.

Proof. To prove that the map $f \mapsto S(f^{-1})$ is a 1-cocycle one has to check the 1-cocycle condition

$$S(f \circ g) = g^* S(f) + S(g), \tag{3.8}$$

for all $f, g \in \mathrm{Diff}(M)$, and g^* is the natural action on $\mathcal{D}(\mathcal{S}^2_\delta(M), \mathcal{S}^0_\delta(M))$. A simple calculation show that the 1-cocycle given by the operator (3.7) satisfies the equation (3.8). Let us prove that this 1-cocycle is not trivial. Suppose that there exists an operator $A : \mathcal{S}^2_\delta(M) \to \mathcal{S}^0_\delta(M)$ such that

$$S(f) = f^* A - A. \tag{3.9}$$

Since the operator (3.7) is first-order then the operator A is also first-order.
The coefficients of the operator $A = \left(t^k_{ij}\partial_k + u_{ij}\right)$ transform under coordinates change as follows:

$$\begin{aligned} t^k_{ij}(y) &= t^c_{ab}(x)\frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j}\frac{\partial y^k}{\partial x^c}, \\ u_{ij}(y) &= u_{ab}(x)\frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j} - 2t^c_{ab}(x)\frac{\partial^2 y^k}{\partial x^c \partial x^l}\frac{\partial x^a}{\partial y^k}\frac{\partial x^b}{\partial y^{(i}}\frac{\partial x^l}{\partial y^{j)}}, \end{aligned}$$

where round brackets mean symmetrization.
The coefficients (t^k_{ij}) are actually a tensor fields of type (2,1). Comparing the coefficients of degree one in (3.9), one obtains $\ell(f)^k_{ij} = (f^*t)^k_{ij} - t^k_{ij}$. It follows that the 1-cocycle $\ell(f)$ is trivial which is absurd (cf. [13]).

To prove (ii) denote by S^∇_{ij} the operator (3.7) written with the connection ∇. Let $\tilde\nabla$ be an other connection which is projectively equivalent to ∇. A direct computation gives

$$\begin{aligned} \nabla_k(P^{ij}) &= \tilde\nabla_k(P^{ij}) + 2\frac{1}{n+1}P^{l(i}\delta^{j)}_k(\Gamma_l - \tilde\Gamma_l) + \frac{2-\delta(n+1)}{n+1}P^{ij}(\Gamma_k - \tilde\Gamma_k), \\ \nabla_k \ell(f)^k_{ij} &= \tilde\nabla_k \ell(f)^k_{ij} + \frac{n-1}{n+1}\ell(f)^k_{ij}(\Gamma_k - \tilde\Gamma_k), \end{aligned}$$

for all $P^{ij} \in \mathcal{S}^2_\delta$.
It is now easy to see that one obtains from (3.7), according to those formulæ, that $S^\nabla_{ij}(f) = S^{\tilde\nabla}_{ij}(f)$. ■

4 Relation to the modules of differential operators

The space of second-order linear differential operators $\mathcal{D}^2_{\lambda,\mu}(M)$ endowed with the structure of $\mathrm{Diff}(M)$−module given by (2.3) and the corresponding space of symbols $\mathcal{S}_{\delta,2}(M)$, where $\delta = \mu - \lambda$, endowed with the structure of $\mathrm{Diff}(M)$−module given by (2.5) are not isomorphic as $\mathrm{Diff}(M)$−module (see section 2.3). We will see in the proposition below that the module $\mathcal{D}^2_{\lambda,\mu}(M)$ is a non-trivial deformation of the module $\mathcal{S}_{\delta,2}(M)$ in the sense of the Neijenhuis & Richardson's theory of deformation (see [12]). Let us give explicitly this deformation generated by the 1-cocycles (3.6) and (3.7). Namely, we are looking for the operator $\bar{f}_\delta = Q^{-1} \circ f_{\lambda,\mu} \circ Q$ such that the diagram below is commutative

$$\begin{array}{ccc} \mathcal{S}_{\delta,2}(M) & \xrightarrow{\bar{f}_\delta} & \mathcal{S}_{\delta,2}(M) \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathcal{D}^2_{\lambda,\mu}(M) & \xrightarrow{f_{\lambda,\mu}} & \mathcal{D}^2_{\lambda,\mu}(M) \end{array} \tag{4.1}$$

The following statement, whose proof is straightforward, shows how the cocycles (3.6) and (3.7) are related to the module of second-order linear differential operators.

Proposition 4.1 *If* $\dim M \geq 2$, *for all* $\delta \neq 1, \frac{n+3}{n+1}, \frac{n+2}{n+1}$, *the deformation of the space of symbols* $\mathcal{S}_{\delta,2}(M)$ *by the space of second-order linear differential operators* $\mathcal{D}^2_{\lambda,\mu}(M)$ *as* $\mathrm{Diff}(M)$*−module is given as follows:*

$$\begin{aligned} \left(\bar{f}_\delta\, P\right)^{ij} &= \left(f_\delta\, P\right)^{ij} \\ \left(\bar{f}_\delta\, P\right)^{i} &= \left(f_\delta\, P\right)^{i} + \frac{(2\lambda+\delta-1)(n+1)}{2+(n+1)(1-\delta)}\,\ell^i_{kl}(f^{-1})(f_\delta\, P)^{kl} \\ (\bar{f}_\delta\, P)_0 &= (f_\delta\, P)_0 - \frac{\lambda(\lambda+\delta-1)(n+1)}{(1-\delta)((1-\delta)(n+1)+1)}\,S_{kl}(f^{-1})(f_\delta\, P)^{kl} \end{aligned} \tag{4.2}$$

where $P = P^{ij}\xi_i\xi_j + P^i\xi_i + P_0 \in \mathcal{S}_{\delta,2}(M)$, *and* f_δ *is the action (2.5).*

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