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Projectively Invariant Cocycles of Holomorphic Vector Fields on an Open Riemann Surface

by

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Abstract

Let Σ be an open Riemann surface and $\mathrm{Hol}(\Sigma)$ be the Lie algebra of holomorphic vector fields on Σ. We fix a projective structure (i.e. a local $\mathrm{SL}_2(\mathbb{C})$−structure) on Σ. We calculate the first group of cohomology of $\mathrm{Hol}(\Sigma)$ with coefficients in the space of linear holomorphic operators acting on tensor densities, vanishing on the Lie algebra $\mathrm{sl}_2(\mathbb{C})$. The result is independent on the choice of the projective structure. We give explicit formulæ of 1-cocycles generating this cohomology group.

1 Introduction

The first group of cohomology of the Lie algebra of (formal) vector fields on the circle S^1 with coefficients in the space $\mathrm{Hom}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$, where $\mathcal{F}_\lambda$ is the space of tensor densities of degree λ on S^1, was first calculated in [6]. This group of cohomology measures all extensions of exact sequences $0 \to \mathcal{F}_\mu \to \cdot \to \mathcal{F}_\lambda \to 0$ of modules. The first group of cohomology of the Lie algebra $\mathrm{Vect}(S^1)$ of (smooth) vectors fields on the circle with coefficients in the space of linear differential operators acting from λ−densities to μ−densities, vanishing on the subalgebra $\mathrm{sl}_2(\mathbb{R}) \subset \mathrm{Vect}(S^1)$, was calculated in [2]. This group of cohomology appears as an obstruction to the equivariant quantization (see [2], [8]). The computation is based on the following observation: any 1-cocycle vanishing on the subalgebra $\mathrm{sl}_2(\mathbb{R})$ is an $\mathrm{sl}_2(\mathbb{R})$−invariant operator. The $\mathrm{sl}_2(\mathbb{R})$−invariant differential operators acting on tensor densities, which are called "Transvectants", were classified by Gordan (see [7, 11]). To find the 1-cocycles generating the group of cohomology means, therefore, to determine which from the Transvectants are 1-cocycles.

In this paper, we study the complex analog of the above group of cohomology on an open Riemann surface Σ endowed with a flat projective structure.

The aim of this paper is to describe the first group of cohomology

$$H^1(\mathrm{Hol}(\Sigma), \mathrm{sl}_2(\mathbb{C}); \mathcal{D}_{\lambda,\mu}), \tag{1.1}$$

of holomorphic vector fields on Σ with coefficients in the space of linear differential operators acting on tensor densities, vanishing on the Lie algebra $\mathrm{sl}_2(\mathbb{C})$. We give explicit formulæ of 1-cocycles generating the group (1.1). These 1-cocycles are the complex analog of the 1-cocycles given in [2].

The main tool of this paper is the existence of affine and projective connection on any open Riemann surface (see [10, 13]). These notions has been recently used in [16] to compute the second group of cohomology of the Lie algebra $\mathrm{Hol}(\Sigma)$ with coefficients in the space of λ−densities.

2 Affine and Projective Structure

Let Σ be a Riemann surface, and let $\{U_\alpha, z_\alpha\}$ be an atlas of Σ.

A holomorphic affine connection is a family of holomorphic functions $\Gamma(z_\alpha)$ on U_α such that for non-empty $U_\alpha \cap U_\beta$, we have

$$\Gamma(z_\beta)\,\frac{dz_\beta}{dz_\alpha} = \Gamma(z_\alpha) + \frac{d^2 z_\beta}{dz_\alpha^2}\frac{dz_\alpha}{dz_\beta}.$$

Affine connection exists in any open Riemann surface in contrast with the compact case where affine connection exists only if the genus of Σ is one (see [10]).

A holomorphic projective connection is a family of holomorphic functions $R(z_\alpha)$ on U_α such that for non-empty $U_\alpha \cap U_\beta$, we have

$$R(z_\beta)\,\left(\frac{dz_b}{dz_\alpha}\right)^2 = R(z_\alpha) + S(z_\beta, z_\alpha),$$

where $S(z_\beta, z_\alpha) = \left(\frac{d^2 z_\beta}{dz_\alpha^2}\right)' - \frac{1}{2}\left(\frac{d^2 z_\beta}{dz_\alpha^2}\right)^2$ is the Schwarzian derivative.

Recall that any holomorphic affine connection Γ defines naturally a holomorphic projective connection given in a local coordinates z by

$$R(z) = \frac{d\Gamma(z)}{dz} - \frac{1}{2}\Gamma^2(z). \tag{2.1}$$

For any projective connection R there exits locally an affine connection Γ satisfying (2.1).

We say that two affine connections are projectively equivalent if they define the same projective connection.

Let us define the notion of projective structure.

A Riemann surface admits a projective structure if there exists an atlas of charts $\{U_\alpha, z_\alpha\}$ such that the coordinate change $z_\alpha \circ z_\beta^{-1}$ are projective transformations.

In this case, the Lie algebra $\mathrm{sl}_2(\mathbb{C})$, in each chart of the projective structure, is generated by the following vector fields

$$\frac{d}{dz},\ z\frac{d}{dz},\ z^2\frac{d}{dz}. \tag{2.2}$$

There exists a 1-1 correspondence between projective structures and projective connections on an open Riemann surface (cf. [10]). We use implicitly this correspondence along this paper.

3 Hol(Σ)-module Structures on the Space of Differential Operators

The modules of linear differential operators on the space of tensor densities on a (real) smooth manifold has been studied in series of recent papers (see [1, 2, 3, 4, 8, 9, 14, 15]). Note that this space viewed as a module over the Lie algebra of vector fields has already been studied in the classical monograph [17].

Let us give the definition of the natural two-parameter family of modules over the Lie algebra of holomorphic vector fields on the space of linear differential operators.

3.1 Tensor Densities

Let Σ be an open Riemann surface. Fix an affine connection Γ on it.

Space of tensor densities on Σ, noted $\mathcal{F}_\lambda$, is the space of sections of the line bundle $(T^*\Sigma)^{\otimes\lambda}$, where $\lambda \in \mathbb{C}$. This bundle is of course trivial, since any (holomorphic) bundle on an open Riemann surface is holomorphically trivial.

Fix a global section dz^λ on $\mathcal{F}_\lambda$. Any $\lambda-$density can be written in the form $\phi\, dz^\lambda$. Let us recall the definition of a covariant derivative of tensor densities. Let ∇ be the covariant derivative associated to the affine connection Γ. If $\phi \in \mathcal{F}_\lambda$, then $\nabla\,\phi \in \Omega^1(\Sigma)\otimes\mathcal{F}_\lambda$ given by the formula

$$\nabla\,\phi = \frac{d\phi}{dz} - \lambda\Gamma\phi.$$

The standard action of Hol(Σ) on $\mathcal{F}_\lambda$ reads as follows (cf. [16]):

$$L_X^\lambda(\phi) = X\nabla\phi + \lambda\phi\,\nabla X, \tag{3.1}$$

where $X = X(z)\dfrac{d}{dz} \in \mathrm{Hol}(\Sigma)$.

3.2 Hol(Σ)$-$Module of Differential Operators

Consider differential operators acting on tensor densities:

$$A : \mathcal{F}_\lambda \to \mathcal{F}_\mu. \tag{3.2}$$

In local coordinates z, any operator A can be written in the form

$$A = a_k(z)\frac{d^k}{dz^k} + \cdots + a_0(z),$$

where a_i, for $i = 1, k$, are holomorphic functions on z.
A two-parameter family of actions of Hol(Σ) on the space of differential operators is defined by

$$L_X^{\lambda,\mu}(A) = L_X^\lambda \circ A - A \circ L_X^\mu. \tag{3.3}$$

Denote by $\mathcal{D}_{\lambda,\mu}(\Sigma)$ the space of operators (3.2) endowed with the defined Hol(Σ)-module structure (3.3).

4 Main Result

Assume Σ endowed with a projective structure, this defines locally an action of the Lie group $\mathrm{SL}_2(\mathbb{C})$ on Σ.

Consider cochains on $\mathrm{Hol}(\Sigma)$ with values in $\mathcal{D}_{\lambda,\mu}$, vanishing on the Lie algebra $\mathrm{sl}_2(\mathbb{C})$. One, therefore, obtains the so-called relative cohomology of the Lie algebra $\mathrm{Hol}(\Sigma)$, namely

$$H^1(\mathrm{Hol}(\Sigma), \mathrm{sl}_2(\mathbb{C}); \mathcal{D}_{\lambda,\mu}(\Sigma)),$$

(see [5]).

The purpose of this paper is the following:

Theorem 4.1 *The first group of cohomology $H^1(\mathrm{Hol}(\Sigma), \mathrm{sl}_2(\mathbb{C}); \mathcal{D}_{\lambda,\mu}(\Sigma))$ is one-dimensional in the following cases:*
(a) $\mu - \lambda = 2, \lambda \neq -1/2$,
(b) $\mu - \lambda = 3, \lambda \neq -1$,
(c) $\mu - \lambda = 4, \lambda \neq -1/2$,
(d) $(\lambda, \mu) = (-4, 1), (0, 5)$.
Otherwise, this cohomology group is trivial.

This Theorem generalizes the result of [2] in the case of the circle S^1.

Note that this result does not depend on the choice of the projective structure.

5 Construction of the 1-cocycles

In this section, we give explicit formula for the 1-cocycles generating the nontrivial cohomology classes from Theorem 4.1. Given a projective structure on Σ, we prove that there is a canonical choice of the 1-cocycles vanishing on $\mathrm{sl}_2(\mathbb{C})$.

Fix (locally) an affine connection Γ related to the projective structure. Denote by R the projective connection associated to Γ (see section 2.)

Lemma 5.1 *The following linear differential operators*

$$\begin{aligned}
\mathcal{I}_3 &= R, \\
\mathcal{I}_4 &= R\nabla - \frac{\lambda}{2}\nabla R, \\
\mathcal{I}_5 &= R\nabla^2 - \frac{2\lambda+1}{2}\nabla R\nabla + \frac{\lambda(2\lambda+1)}{10}\nabla^2 R + \frac{\lambda(\lambda+3)}{5}R^2, \\
\mathcal{I}_6 &= R\nabla^3 - \frac{3}{2}\nabla R\nabla^2 + \left(\frac{3}{10}\nabla^2 R + \frac{4}{5}R^2\right)\nabla, \\
\mathcal{I}'_6 &= R\nabla^3 + \frac{9}{2}\nabla R\nabla^2 + \left(\frac{63}{10}\nabla^2 R + \frac{4}{5}R^2\right)\nabla + \frac{14}{5}\nabla^3 R + \frac{8}{5}R\nabla R,
\end{aligned}$$

are globally defined in $\mathcal{D}_{\lambda,\lambda+2}(\Sigma)$, $\mathcal{D}_{\lambda,\lambda+3}(\Sigma)$, $\mathcal{D}_{\lambda,\lambda+4}(\Sigma)$, $\mathcal{D}_{0,5}(\Sigma)$, $\mathcal{D}_{-4,1}(\Sigma)$, respectively, and depend only on the projective class of the connection Γ.

Proof. Since the surface Σ is projectively flat, the connection R defines a 2-density on Σ. Then the operators of the above Lemma are globally defined. Let us prove that the operators depend only on the projective class of the connection Γ.

Let $\mathcal{I}_4 = R\nabla + \alpha\nabla R$. Denote by $\tilde{\mathcal{I}}_4$ the operator $\mathcal{I}_4$ written with respect to a connection $\tilde{\Gamma}$ which is projectively equivalent to Γ. After an easy calculation one has

$$\mathcal{I}_4 = \tilde{\mathcal{I}}_4 + (2\alpha + \lambda)(\tilde{\Gamma} - \Gamma)R.$$

Then $\mathcal{I}_4 = \tilde{\mathcal{I}}_4$ if and only if $\alpha = -\lambda/2$. The proof is analogous for the operators $\mathcal{I}_5, \mathcal{I}_6$ and $\mathcal{I}'_6$.

Theorem 5.2 *(i) For every λ, there exist unique (up to constant) 1-cocycles*

$$\begin{aligned} \mathcal{J}_3 &: \mathrm{Hol}(\Sigma) \to \mathcal{D}_{\lambda,\lambda+2} \\ \mathcal{J}_4 &: \mathrm{Hol}(\Sigma) \to \mathcal{D}_{\lambda,\lambda+3} \\ \mathcal{J}_5 &: \mathrm{Hol}(\Sigma) \to \mathcal{D}_{\lambda,\lambda+4} \end{aligned}$$

vanishing on $\mathrm{sl}_2(\mathbb{C})$. *They are given by the formulæ:*

$$\begin{aligned} \mathcal{J}_3(X) &= \nabla^3 X - L_X^{\lambda,\lambda+2}(\mathcal{I}_3), \\ \mathcal{J}_4(X) &= \nabla^3 X\nabla - \frac{\lambda}{2}\nabla^4 X - L_X^{\lambda,\lambda+3}(\mathcal{I}_4), \\ \mathcal{J}_5(X) &= \nabla^3 X\nabla^2 - \frac{2\lambda+1}{2}\nabla^4 X\nabla + \frac{\lambda(2\lambda+1)}{10}\nabla^5 X - L_X^{\lambda,\lambda+4}(\mathcal{I}_5). \end{aligned}$$

For $(\lambda,\mu) = (0,5), (-4,1)$, respectively, the 1-cocycles vanishing on $\mathrm{sl}_2(\mathbb{C})$ *are given by*

$$\begin{aligned} \mathcal{J}_6^0(X) &= \nabla^3 X\,\nabla^3 - \frac{3}{2}\nabla^4 X\,\nabla^2 + \frac{3}{10}\nabla^5 X\nabla - L_X^{0,5}(\mathcal{I}_6), \\ \mathcal{J}_6^{-4}(X) &= \nabla^3 X\,\nabla^3 + \frac{9}{2}\nabla^4 X\,\nabla^2 + \frac{63}{10}\nabla^5 X\nabla + \frac{14}{5}\nabla^6 X - L_X^{-4,1}(\mathcal{I}'_6). \end{aligned}$$

(ii) The 1-cocycles $\mathcal{J}_3$, $\mathcal{J}_4$ and $\mathcal{J}_5$ are nontrivial for every λ except $\lambda = -1/2, \lambda = -1$ and $\lambda = -3/2$, respectively. The 1-cocycles $\mathcal{J}_6^0$ and $\mathcal{J}_6^{-4}$ are nontrivial.
(iii) These 1-cocycles are independent on the choice of the projective structure.

6 $\mathrm{sl}_2(\mathbb{R})$–invariant operators on S^1

For almost all λ and μ, there exits unique (up to constant) $\mathrm{sl}_2(\mathbb{R})$–invariant bilinear differential operators $J_m^{\lambda,\mu} : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \to \mathcal{F}_{\lambda+\mu+m}$ given by

$$J_m^{\lambda,\mu}(\phi,\psi) = \sum_{i+j=m} (-1)^i m! \binom{2\lambda+m-1}{i}\binom{2\mu+m-1}{j}\phi^{(i)}\psi^{(j)}, \tag{6.1}$$

called "Transvectants" (see [9, 11]).

Let us recall the results of [2, 9].

The first group of cohomology $H^1(\mathrm{Vect}(S^1), \mathrm{sl}_2(\mathbb{R}); \mathcal{D}_{\lambda,\mu}(S^1))$ is one-dimensional in the following cases:
(a) $\mu - \lambda = 2, \lambda \neq -1/2$,
(b) $\mu - \lambda = 3, \lambda \neq -1$,
(c) $\mu - \lambda = 4, \lambda \neq -1/2$,
(d) $(\lambda,\mu) = (-4,1), (0,5)$.
Otherwise, this cohomology group is trivial (see [2]).

the relation (7.2) implies that the 1-cocycle J_3 is trivial which is absurd (see section 6). For $\lambda = -1/2$, take $A = -2\nabla^2 \in \mathcal{D}_{-\frac{1}{2},\frac{3}{2}}(\Sigma)$. One can easily check that $\mathcal{J}_3(X) = L_X^{-1/2,3/2}(A)$.
With the same arguments we prove the non-triviality of the 1-cocyles $\mathcal{J}_k$, for $k = 4, 5, 6$. Theorem 5.2 (ii) is proven.

7.2 Proof of Theorem 4.1

Let us prove that the dimension of the group of cohomology $H^1(\mathrm{Hol}(\Sigma), \mathrm{sl}_2(\mathbb{C}); \mathcal{D}_{\lambda,\mu}(\Sigma))$ is bounded by the dimension of the group of cohomology $H^1(\mathrm{Vect}(S^1), \mathrm{sl}_2(\mathbb{R}); \mathcal{D}_{\lambda,\mu}(S^1))$. Let C and C' be two 1-cocycles in $H^1(\mathrm{Hol}(\Sigma), \mathrm{sl}_2(\mathbb{C}); \mathcal{D}_{\lambda,\mu}(\Sigma))$. We will prove that C and C' are cohomologous. Denote $\tilde{C}$ and $\tilde{C}'$ the restriction of C and C' on a neighborhood of a point of Σ. The operators $\tilde{C}$ and $\tilde{C}'$ define 1-cocycle in $H^1(\mathrm{Vect}(S^1), \mathrm{sl}_2(\mathbb{R}); \mathcal{D}_{\lambda,\mu}(S^1))$. These 1-cocycles are equal (up to constant); since the unique $\mathrm{sl}_2(\mathbb{R})$−invariant linear differential operators are given as in (6.2). It follows that the 1-cocyle C and C' are cohomologous. Now from the construction of the 1-cocyles given in Theorem (5.2) follows Theorem (4.1).

8 Final Remark

The group of cohomology $H^1(\mathrm{Diff}(S^1), \mathrm{PSL}_2(\mathbb{R}); \mathcal{D}_{\lambda,\mu})$ is one-dimension, for generic λ, generated by the following 1-cocyles

$$
\begin{aligned}
\mathcal{S}_\lambda(f) &= S(f), \\
\mathcal{T}_\lambda(f) &= S(f)\frac{d}{dx} - \frac{\lambda}{2}S(f)', \\
\mathcal{U}_\lambda(f) &= S(f)\frac{d^2}{dx^2} - \frac{2\lambda+1}{2}S(f)'\frac{d}{dx} + \frac{\lambda(2\lambda+1)}{10}S(f)'' - \frac{\lambda(\lambda+3)}{5}S(f)^2, \\
\mathcal{V}_0(f) &= S(f)\frac{d^3}{dx^3} - \frac{3}{2}S(f)'\frac{d^2}{dx^2} + \left(\frac{3}{10}S(f)'' + \frac{4}{5}S(f)^2\right)\frac{d}{dx} \\
\mathcal{V}_{-4} &= S(f)\frac{d^3}{dx^3} + \frac{9}{2}S(f)'\frac{d^2}{dx^2} + \left(\frac{63}{10}S(f)'' + \frac{4}{5}S(f)^2\right)\frac{d}{dx} + \frac{14}{5}S(f)''' + \frac{8}{5}S(f)S(f)',
\end{aligned}
\tag{8.1}
$$

where $S(f)$ is the Schwarzian derivative (see [2]). It is a remarkable fact to see that the 1-cocycles (8.1) on the group $\mathrm{Diff}(S^1)$ have the same expression (up to change of sign) than the operators of the Lemma 5.1 if one replaces the connection R by $-S(f)$.
Since the group of biholomorphic maps on a Riemann surface is finite dimension (see [12]), this group does not integrate the Lie algebra of holomorphic vector fields. In some sense, the cohomology group (1.1) contains informations coming from the cohomology of the diffeomorphisms of S^1 and the cohomology of the Lie algebra of vector fields on S^1.

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