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The Class Type G Distributions on $\mathbb{R}^d$ and Related Subclasses of Infinitely Divisible Distributions

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THE CLASS OF TYPE G DISTRIBUTIONS ON $\mathbb{R}^d$ AND RELATED SUBCLASSES OF INFINITELY DIVISIBLE DISTRIBUTIONS

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Abstract. Classes of infinitely divisible distributions obtained by iteration of Gaussian randomization of Lévy measures are introduced and studied.

Their relation to Urbanik-Sato nested classes of selfdecomposable distributions is also established.

1. Introduction

In our previous paper [MR00], we studied the class of type G distributions on $\mathbb{R}^d$ defined in the following way. A symmetric infinitely divisible probability distribution μ on $\mathbb{R}^d$ is of type G if its Lévy measure ν is of the form

$$\nu(A) = \mathbf{E}[\nu_0(Z^{-1}A)], \qquad A \in \mathcal{B}_0(\mathbb{R}^d), \tag{1.1}$$

where ν_0 is a Borel measure on $\mathbb{R}^d \setminus \{0\}$, Z is the standard normal random variable, and $\mathcal{B}_0(\mathbb{R}^d)$ is the class of all Borel sets A in $\mathbb{R}^d$ such that $A \subset \{|x| > \varepsilon\}$ for some $\varepsilon > 0$. Such kind of distributions combine Gaussian and Poissonian structures in a nontrivial way (see Section 5 in [MR00]). Denote by $TG(\mathbb{R}^d)$ the class of type G distributions on $\mathbb{R}^d$.

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A typical representative of the class $TG(\mathbb{R}^d)$ is a symmetric stable distribution. In this paper we will use the following convention. Given a class of measures H on $\mathbb{R}^d$, we will denote by $\widetilde{H}$ the subset of H consisting of symmetric measures. Denote by $S(\mathbb{R}^d)$ and $I(\mathbb{R}^d)$ the classes of stable and infinitely divisible distributions on $\mathbb{R}^d$, respectively. Therefore, we have $\widetilde{S}(\mathbb{R}^d) \subset TG(\mathbb{R}^d) \subset \widetilde{I}(\mathbb{R}^d)$. Our goal is to introduce and investigate the nested classes $TG_m(\mathbb{R}^d)$, $m \geq 1$, between $TG_0(\mathbb{R}^d) := TG(\mathbb{R}^d)$ and $\widetilde{S}(\mathbb{R}^d)$, using the procedure somewhat analogous to Urbanik-Sato construction of subclasses of selfdecomposable distributions.

In Section 2, we define the classes $TG_m(\mathbb{R}^d)$, $m \geq 1$, and show that they form a strictly descending sequence. In Section 3, we compare our nested subclasses of $TG_0(\mathbb{R}^d)$ and those of the class $L_0(\mathbb{R}^d)$ of selfdecomposable distributions introduced and studied by Urbanik [U72], [U73] and Sato [S80]. A necessary and sufficient condition for a type G distribution on $\mathbb{R}^1$ to be selfdecomposable was given in [R91]. We generalize this result to $\mathbb{R}^d$ and give an answer to the converse problem: When is a symmetric selfdecomposable distribution of type G? We also study related problems.

Every distribution $\mu \in TG_m(\mathbb{R}^d)$ has its predecessor $\mu_0 \in TG_{m-1}(\mathbb{R}^d)$, as defined in Section 2. In Section 4, we study the relationship between μ and μ_0 along the following lines : If μ belongs to a certain class of distributions, then does μ_0 belong to the same class? The answers are obtained for some important classes in Theorem 4.1. Section 5 contains some examples and Section 6 discusses open problems.

We conclude the introduction by stating a basic characterization theorem for type G distributions on $\mathbb{R}^d$, which has been proved in [MR00],

and will also be needed later in this paper.

Theorem A ([MR00]). *A symmetric probability measure μ on $\mathbb{R}^d$ is of type G if and only if it is infinitely divisible and its Lévy measure ν is either zero or represented as*

$$\nu(EB) = \int_B \lambda(dx) \int_E g_x(r^2)dr \qquad \textit{for} \quad E \in \mathcal{B}(\mathbb{R}_+),\ B \in \mathcal{B}(S),$$

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where λ *is a probability measure on* S *and* $g_x(r)$ *is a jointly measurable function which, for any fixed* x*, is completely monotone on* $(0,\infty)$ *and satisfies*

$$\int_0^\infty (1 \wedge r^2) g_x(r^2)\, dr = c \in (0,\infty)$$

with c *independent of* x*. This representation is unique in the sense that, if* $\nu \neq 0$ *and two pairs* (λ, g_x) *and* $(\widetilde{\lambda}, \widetilde{g}_x)$ *both satisfy the above conditions, then* $\lambda = \widetilde{\lambda}$ *and* $g_x = \widetilde{g}_x$ *for* λ*-a.e.* x*. Moreover,* λ *is a symmetric probability measure and* $g_x = g_{-x}$ λ*-a.e.*

2. Subclasses of the class of type G distributions

In the following, if μ is infinitely divisible, we denote its Lévy measure by $\nu(\mu)$.

We first rewrite the definition of type G distribution. In the definition (1.1), ν_0 is a Borel measure but since ν is a Lévy measure, ν_0 is also a Lévy measure (see Proposition 2.2 (i)–(ii) in [MR00] or Proposition 2 in [J90]). Moreover, ν_0 in (1.1) always can and will be taken symmetric. For any $\mu_0 \in \widetilde{I}(\mathbb{R}^d)$, define $K(\mu_0)$ as the infinitely divisible distribution μ having the same Gaussian component as μ_0 and Lévy measure ν given by (1.1) with $\nu_0 = \nu_0(\mu_0)$. The symmetric distribution μ_0 will be called the *predecessor* of μ (relative to the operation K). The predecessor is uniquely

defined. Indeed, suppose that μ has two predecessors μ_1 and μ_2. Then ν satisfies (1.1) with $\nu_0 = \nu_1(\mu_1)$ and $\nu_0 = \nu_2(\mu_2)$. By Proposition 2.2 (iii) in [MR00] $\nu_1 = \nu_2$, and since μ_1 and μ_2 have the same Gaussian part, $\mu_1 = \mu_2$. We have just shown that the operation K is one-to-one. If we write

$$K(H) = \{K(\mu_0) : \ \mu_0 \in H\}, \quad H \subset \widetilde{I}(\mathbb{R}^d),$$

then

$$TG(\mathbb{R}^d) = K(\widetilde{I}(\mathbb{R}^d)).$$

Put $TG_{-1}(\mathbb{R}^d) = \widetilde{I}(\mathbb{R}^d)$ and $TG_0(\mathbb{R}^d) = TG(\mathbb{R}^d)$. Define for $1 \le m < \infty$,

$$TG_m(\mathbb{R}^d) = K(TG_{m-1}(\mathbb{R}^d)),$$

and

$$TG_\infty(\mathbb{R}^d) = \bigcap_{m=0}^{\infty} TG_m(\mathbb{R}^d).$$

Theorem 2.1. $\widetilde{I}(\mathbb{R}^d) \supset TG_0(\mathbb{R}^d) \supset TG_1(\mathbb{R}^d) \supset \cdots \supset TG_m(\mathbb{R}^d) \supset TG_{m+1}(\mathbb{R}^d) \supset \cdots \supset TG_\infty(\mathbb{R}^d) \supset \widetilde{S}(\mathbb{R}^d)$.

Proof. By the definition,

$$TG_{-1}(\mathbb{R}^d) \supset TG_0(\mathbb{R}^d).$$

Suppose that $TG_{m-1}(\mathbb{R}^d) \supset TG_m(\mathbb{R}^d)$ for some $0 \le m < \infty$. If $\mu \in TG_{m+1}(\mathbb{R}^d)$, then $\nu(\mu)(A) = \mathbf{E}[\nu_0(Z^{-1}A)]$, where ν_0 is the Lévy measure of the predecessor $\mu_0 \in TG_m(\mathbb{R}^d)$. By the induction hypothesis, we have that $\mu_0 \in TG_{m-1}(\mathbb{R}^d)$. Hence $\mu \in TG_m(\mathbb{R}^d)$, concluding

$$TG_{m+1}(\mathbb{R}^d) \subset TG_m(\mathbb{R}^d).$$

The assertion $TG_m(\mathbb{R}^d) \supset TG_\infty(\mathbb{R}^d)$ is trivial from its definition.

We next show that $TG_\infty(\mathbb{R}^d) \supset \widetilde{S}(\mathbb{R}^d)$. If $\mu_0 \in \widetilde{S}(\mathbb{R}^d)$, then $\nu(A) = \mathbf{E}[\nu_0(Z^{-1}A)]$ is the Lévy measure of a symmetric stable distribution, where ν_0 is the Lévy measure of μ_0. Thus $K(\widetilde{S}(\mathbb{R}^d)) \subset \widetilde{S}(\mathbb{R}^d)$. Conversely, if $\mu \in \widetilde{S}(\mathbb{R}^d)$, then

$$\nu(\mu)(A) = \mathbf{E}[\nu_0(Z^{-1}A)],$$

where ν_0 is also the Lévy measure of a distribution in $\widetilde{S}(\mathbb{R}^d)$. For, since the Lévy measure of $\mu \in \widetilde{S}(\mathbb{R}^d)$ satisfies the condition $a^\alpha \nu(\mu)(A) = \nu(\mu)(a^{-1}A)$, for every $a > 0$ and $A \in \mathcal{B}_0(\mathbb{R}^d)$, where $\alpha \in (0, 2]$ is the index of stability, (1.1) holds with $\nu_0 = (\mathbf{E}[|Z|^\alpha])^{-1}\nu$. Hence $\widetilde{S}(\mathbb{R}^d) \subset K(\widetilde{S}(\mathbb{R}^d))$ and thus $K(\widetilde{S}(\mathbb{R}^d)) = \widetilde{S}(\mathbb{R}^d)$, namely, $\widetilde{S}(\mathbb{R}^d)$ is invariant under the operation K. We thus have, for each $m \ge 0$,

$$\widetilde{S}(\mathbb{R}^d) = K^m(\widetilde{S}(\mathbb{R}^d)) \subset K^m(\widetilde{I}(\mathbb{R}^d)) = TG_m(\mathbb{R}^d),$$

where K^m is the mth iteration of K. Thus $\widetilde{S}(\mathbb{R}^d) \subset \bigcap_{m\geq 0} TG_m(\mathbb{R}^d) = TG_\infty(\mathbb{R}^d)$. This completes the proof. $\square$

It might be asked whether the inclusions in Theorem 2.1 are strict or not. The answer is the following.

Theorem 2.2. *The inclusions in Theorem 2.1 are all strict, namely*

$$\widetilde{I}(\mathbb{R}^d) \supsetneqq TG_0(\mathbb{R}^d) \supsetneqq TG_1(\mathbb{R}^d) \supsetneqq \cdots \supsetneqq TG_m(\mathbb{R}^d) \supsetneqq TG_{m+1}(\mathbb{R}^d)$$
$$\supsetneqq \cdots \supsetneqq TG_\infty(\mathbb{R}^d) \supsetneqq \widetilde{S}(\mathbb{R}^d).$$

Proof. First note that $TG_{-1}(\mathbb{R}^d) \supsetneqq TG_0(\mathbb{R}^d)$, since the existence of non-type G infinitely divisible distribution is assured by Theorem A. Since the operation K is one-to-one we have

$$TG_{m-1}(\mathbb{R}^d) \setminus TG_m(\mathbb{R}^d) = K^m(TG_{-1}(\mathbb{R}^d) \setminus TG_0(\mathbb{R}^d)) \neq \emptyset,$$

proving

$$TG_{m-1}(\mathbb{R}^d) \supsetneqq TG_m(\mathbb{R}^d), \quad \forall m \geq 0. \tag{2.1}$$

We next show that

$$TG_m(\mathbb{R}^d) \supsetneqq TG_\infty(\mathbb{R}^d), \quad \forall m \geq 0.$$

If there exists an m_0 such that

$$TG_{m_0}(\mathbb{R}^d) = TG_\infty(\mathbb{R}^d),$$

then

$$TG_{m_0}(\mathbb{R}^d) = TG_{m_0+1}(\mathbb{R}^d) = \cdots = TG_\infty(\mathbb{R}^d),$$

which contradicts (2.1).

Finally the fact that $TG_\infty(\mathbb{R}^d) \supsetneqq \widetilde{S}(\mathbb{R}^d)$ follows from Corollary 3.1 in Section 3, and so the rest of the proof is postponed to the end of Section 3. $\square$

The class $TG_\infty(\mathbb{R}^d)$ has the following special property.

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Theorem 2.3. *$TG_\infty(\mathbb{R}^d)$ is the largest subclass of $\widetilde{I}(\mathbb{R}^d)$ invariant under operation K.*

Proof. By Theorem 2.1,

$$TG_m(\mathbb{R}^d) \supset TG_{m+1}(\mathbb{R}^d) = K(TG_m(\mathbb{R}^d)),$$

and hence

$$\bigcap_{m\ge 0} TG_m(\mathbb{R}^d) \supset \bigcap_{m\ge 0} K(TG_m(\mathbb{R}^d)) \supset K\left(\bigcap_{m\ge 0} TG_m(\mathbb{R}^d)\right).$$

Thus

$$TG_\infty(\mathbb{R}^d) \supset K(TG_\infty(\mathbb{R}^d)).$$

Let us show the converse inclusion. Let $\mu \in TG_\infty(\mathbb{R}^d)$. Then for any $m \ge 0$, $\mu \in TG_m(\mathbb{R}^d)$. Hence μ has the predecessor μ_0 in every class $TG_{m-1}(\mathbb{R}^d)$. Since the predecessor is uniquely defined,

$$\mu_0 \in \bigcap_{m\ge 0} TG_m(\mathbb{R}^d) = TG_\infty(\mathbb{R}^d),$$

and hence

$$\mu \in K(TG_\infty(\mathbb{R}^d)).$$

We thus conclude that

$$K(TG_\infty(\mathbb{R}^d)) = TG_\infty(\mathbb{R}^d).$$

We next show that $TG_\infty(\mathbb{R}^d)$ is the largest class among such classes. Suppose that $H(\subset \widetilde{I}(\mathbb{R}^d))$ satisfies that $K(H) = H$. As before, for each $m \ge 0$,

$$H = K^m(H) \subset K^m(\widetilde{I}(\mathbb{R}^d)) = TG_m(\mathbb{R}^d),$$

and thus

$$H \subset \bigcap_{m\ge 0} TG_m(\mathbb{R}^d) = TG_\infty(\mathbb{R}^d).$$

This completes the proof. □

In one dimensional case ($d = 1$), any random variable X with distribution μ in $TG_0(\mathbb{R}^1)$ can be characterized by

$$X \stackrel{d}{=} V^{1/2} Z, \tag{2.2}$$

where V is some nonnegative infinitely divisible random variable independent of Z and $\stackrel{d}{=}$ means equivalence in law. Then a natural question is how we can characterize X with μ in $TG_m(\mathbb{R}^1), m = 1, 2, ...$, or what type of restriction on V assures that μ belongs to $TG_m(\mathbb{R}^1)$.

To answer this question, we need a relationship between ν_0 in (1.1) and the Lévy measure ρ of V in (2.3).

Theorem 2.4. *For $A \in \mathcal{B}_0(\mathbb{R})$, let $A_1 = A \cap (-A)$ and $A_2 = A \setminus A_1$. Then*

$$\nu_0(A) = \rho(A_1^2) + \frac{1}{2}\rho(A_2^2),$$

where $A^2 = \{x^2 : x \in A\}$. Particularly,

$$\nu_0([x, \infty)) = \frac{1}{2}\rho([x^2, \infty)), \quad x > 0.$$

Proof. Let $\{V(t)\}$ be a Lévy process such that $V(1) \stackrel{d}{=} V$, $\{Z(t)\}$ the standard Brownian motion, and $X(t) = Z(V(t))$. $X(t)$ is a subordination, and $X(1) \stackrel{d}{=} X$. The Lévy measure of the subordination is given by

$$\nu(A) = \int_0^\infty P\{Z(t) \in A\}\rho(dt),$$

(see [Z58]). Hence

$$\begin{aligned}
\nu(A) &= \int_0^\infty P\{t^{1/2} Z \in A\}\rho(dt) \\
&= \mathbf{E}\left[\int_0^\infty \mathbf{1}_{\{t^{1/2} \in Z^{-1}A\}}\rho(dt)\right] \\
&= \mathbf{E}\left[\int_0^\infty \mathbf{1}_{\{t \in Z^{-2}A_1^2\}}\rho(dt)\right] + \frac{1}{2}\mathbf{E}\left[\int_0^\infty \mathbf{1}_{\{t \in Z^{-2}A_2^2\}}\rho(dt)\right] \\
&= \mathbf{E}[\rho(Z^{-2}A_1^2)] + \frac{1}{2}\mathbf{E}[\rho(Z^{-2}A_2^2)].
\end{aligned}$$

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Here if we put

$$\rho_0(A) = \rho(A_1^2) + \frac{1}{2}\rho(A_2^2),$$

then we have

$$\nu(A) = \mathbf{E}[\rho_0(Z^{-1}A)].$$

Note that ρ_0 is a symmetric measure.

On the other hand,

$$\nu(A) = \mathbf{E}[\nu_0(Z^{-1}A)].$$

Hence, from the uniqueness of ν_0 determined by ν among symmetric measures, it follows that $\rho_0 = \nu_0$, namely,

$$\nu_0(A) = \rho(A_1^2) + \frac{1}{2}\rho(A_2^2).$$

The proof is completed. □

We thus have the following equivalence from the definition of $TG_m(\mathbb{R}^d)$.

Theorem 2.5. *Let $m = 1, 2, \ldots$. Then the following are equivalent.*
(i) $\mu \in TG_m(\mathbb{R}^1)$.
(ii) The symmetric measure ν_0 determined by

$$\nu_0([x,\infty)) = \frac{1}{2}\rho([x^2,\infty)), \quad x > 0, \tag{2.4}$$

where ρ is the Lévy measure of V, is the Lévy measure of some $\mu_0 \in TG_{m-1}(\mathbb{R}^1)$.

3. The Urbanik-Sato nested subclasses of symmetric selfdecomposable distributions

Urbanik [U72], [U73] and Sato [S80] introduced and studied the nested classes $L_m(\mathbb{R}^d)$, $m = 0, 1, 2, \ldots, \infty$, between $I(\mathbb{R}^d)$ and $S(\mathbb{R}^d)$, which are defined in the following way.

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In general, for $H \subset I(\mathbb{R}^d)$, define

$$Q(H) = \{\mu \in I(\mathbb{R}^d) : \text{for any } a \in (0,1), \text{there exists } \rho_a \in H$$
$$\text{such that } \widehat{\mu}(\theta) = \widehat{\mu}(a\theta)\widehat{\rho}_a(\theta),\ \forall \theta \in \mathbb{R}^d\},$$

where $\widehat{\mu}$ is the characteristic function of μ.

Then, $L_0(\mathbb{R}^d)$ is defined as

$$L_0(\mathbb{R}^d) = Q(I(\mathbb{R}^d)),$$

and $L_m(\mathbb{R}^d), m = 1, 2, ...,$ are defined inductively as

$$L_m(\mathbb{R}^d) = Q(L_{m-1}(\mathbb{R}^d))$$

and

$$L_\infty(\mathbb{R}^d) = \bigcap_{m \geq 0} L_m(\mathbb{R}^d).$$

Then it was shown that

$$I(\mathbb{R}^d) \supset L_0(\mathbb{R}^d) \supset L_1(\mathbb{R}^d) \supset \cdots \supset L_\infty(\mathbb{R}^d) \supset S(\mathbb{R}^d).$$

Distributions in $L_0(\mathbb{R}^d)$ are called selfdecomposable. Throughout this paper, we are only concerned with symmetric distributions. Therefore we will consider classes $\widetilde{L}_m(\mathbb{R}^d)$. Now we have two sequences of nested classes between $\widetilde{I}(\mathbb{R}^d)$ and $\widetilde{S}(\mathbb{R}^d)$.

$$\text{(i)} \quad \widetilde{I}(\mathbb{R}^d) \supset TG_0(\mathbb{R}^d) \supset TG_1(\mathbb{R}^d) \supset \cdots \supset TG_\infty(\mathbb{R}^d) \supset \widetilde{S}(\mathbb{R}^d)$$

and

$$\text{(ii)} \quad \widetilde{I}(\mathbb{R}^d) \supset \widetilde{L}_0(\mathbb{R}^d) \supset \widetilde{L}_1(\mathbb{R}^d) \supset \cdots \supset \widetilde{L}_\infty(\mathbb{R}^d) \supset \widetilde{S}(\mathbb{R}^d).$$

Then a natural question is to compare two sequences. The following is due to Sato [S80].

Theorem B ([S80]). *A probability measure* $\mu \in I(\mathbb{R}^d)$ *is selfdecomposable, namely in* $L_0(\mathbb{R}^d)$ *if and only if its Lévy measure* ν *is either zero or represented as*

$$\nu(EB) = \int_B \lambda(dx) \int_E \frac{k_x(r)}{r} dr \qquad \text{for} \quad E \in \mathcal{B}(\mathbb{R}_+), \ B \in \mathcal{B}(S)$$

where λ *is a probability measure on* S *and* $k_x(r)$ *is, for any fixed* x*, a nonnegative nonincreasing right-continuous function of* r *satisfying*

$$\int_0^\infty (1 \wedge r^2) \frac{k_x(r)}{r} \, dr = c \in (0, \infty)$$

with c *independent of* x*, and for any* r*,* $k_x(r)$ *is a measurable function of* x*. This representation is unique in the sense that, if* $\nu \neq 0$ *and two pairs* (λ, k_x) *and* $(\widetilde{\lambda}, \widetilde{k}_x)$ *both satisfy the above conditions, then* $\lambda = \widetilde{\lambda}$ *and* $k_x = \widetilde{k}_x$ *for* λ*-a.e.* x*.*

A question when a given type G distribution on $\mathbb{R}^1$ is selfdecomposable was answered in [R91], namely, a type G distribution is selfdecomposable if and only if $x^{1/2} g_x(r)$ is nonincreasing with respect to r on $(0, \infty)$. The proof in [R91] did not use Theorem B, but once we have Theorems A and B, we can relate selfdecomposable and type G distributions in $\mathbb{R}^d$ using spectral forms of their Lévy measures (which are unique).

Theorem 3.1. *(i) Let* $\mu \in TG_0(\mathbb{R}^d)$*. Then* $\mu \in L_0(\mathbb{R}^d)$ *if and only if for* λ*-a.e.* x $r^{1/2} g_x(r)$ *is nonincreasing with respect to* r *on* $(0, \infty)$*.*

(ii) Let $\mu \in \widetilde{L}_0(\mathbb{R}^d)$*. Then* $\mu \in TG_0(\mathbb{R}^d)$ *if and only if for* λ*-a.e.* x $k_x(r^{1/2})/r^{1/2}$ *is complete monotone.*

Proof. Note that if $\mu \in TG_0(\mathbb{R}^d) \cap \widetilde{L}_0(\mathbb{R}^d)$, then in the representations of $\nu = \nu(\mu)$ given by Theorems A and B, the measures λ and constant c must be the same. Furthermore,

$$r g_x(r^2) = k_x(r) \tag{3.1}$$

for λ-a.e. x. The theorem follows from (3.1). $\square$

Sato [S80] also gave a necessary and sufficient condition for $\mu \in L_m(\mathbb{R}^d)$, $m = 1, 2, ..., \infty$. Define $h_x(s) = k_x(e^{-s})$, and call it the h-function of $\mu \in L_0(\mathbb{R}^d)$. For $\delta > 0$, let Δ_δ be the difference operator, $\Delta_\delta f(s) = f(s+\delta) - f(s)$, and Δ_δ^n be its nth iteration. We say that a function $f(s)$ is monotone of order n if

$$\Delta_\delta^j f(s) \geq 0 \quad \text{for } \delta > 0, s \in \mathbb{R}^1, \tag{3.2}$$

for any $j = 0, 1, ..., n$. When (3.2) holds for all integers j, f is called absolutely monotone. Then one of results by Sato [S80] is the following.

Theorem C ([S80]). *Let $m = 0, 1, 2, ..., \infty$. A probability measure μ belongs to $L_m(\mathbb{R}^d)$ if and only if $\mu \in L_0(\mathbb{R}^d)$ and h-function $h_x(s)$ of μ is monotone of order $m+1$ for λ-a.e. x, where λ is the spherical component of the Lévy measure of μ, and when $m = \infty$, being monotone of order $m+1$ is understood as being absolutely monotone.*

The next theorem is a direct consequence of Theorem C and the relation (3.1).

Theorem 3.2. *Let $\mu \in TG_0(\mathbb{R}^d)$, and $m = 0, 1, 2, ..., \infty$. Then $\mu \in L_m(\mathbb{R}^d)$ if and only if*

$$h_x(s) = e^{-s} g_x(e^{-2s})$$

is monotone of order $m+1$ (absolutely monotone when $m = \infty$) for λ-a.e. x.

In [MR00], we have shown that $TG_0(\mathbb{R}^d)$ is closed under convolution and weak convergence. By exactly the same argument, we can show the following.

Theorem 3.3. *The classes $TG_m(\mathbb{R}^d), m = 1, 2, ..., \infty$, are closed under convolution and weak convergence.*

Corollary 3.1. $TG_\infty(\mathbb{R}^d) \supset \widetilde{L}_\infty(\mathbb{R}^d)$.

Proof. It is known ([S80]) that $L_\infty(\mathbb{R}^d)$ is the smallest class containing the class $S(\mathbb{R}^d)$, closed under convolution and weak convergence, and thus $\widetilde{L}_\infty(\mathbb{R}^d)$ is the smallest class containing the class $\widetilde{S}(\mathbb{R}^d)$, closed under convolution and weak convergence. This fact combined with Theorem 3.3 for $m = \infty$ yields the conclusion. □

A consequence of Corollary 3.1 is that convolutions of symmetric stable distributions of different indices are of type G. This fact is pointed out in [R91] for the case $d = 1$.

Proof of Theorem 2.2 (continued). As stated above in the proof of Corollary 3.1, we know that $\widetilde{L}_\infty(\mathbb{R}^d) \supsetneqq \widetilde{S}(\mathbb{R}^d)$, because, for instance, convolutions of symmetric stable distributions of different indices are in $\widetilde{L}(\mathbb{R}^d)$ but not in $\widetilde{S}(\mathbb{R}^d)$. Thus by Corollary 3.1,

$$TG_\infty(\mathbb{R}^d) \supset \widetilde{L}_\infty(\mathbb{R}^d) \supsetneqq \widetilde{S}(\mathbb{R}^d).$$

This completes the proof of Theorem 2.2. □

4. Some invariant properties of type G distributions

The first two statements, (i) and (ii) of Theorem 4.1, give examples of invariant properties under the operation K. (iii) and (iv) show that selfdecomposability of $K(\mu_0)$ is inherited from its predecessor μ_0 but is not a K-invariant property (see Section 2 for the definition of K).

Theorem 4.1. *Suppose that $\mu \in TG_m(\mathbb{R}^d)$ and let $\mu_0 \in TG_{m-1}(\mathbb{R}^d)$ be its predecessor, $m \geq 0$. Then the following holds.*
(i) μ is operator stable if and only if μ_0 is operator stable.
(ii) μ is $\mathbb{R}^d$-valued semi-stable if and only if μ_0 is $\mathbb{R}^d$-valued semi-stable.
(iii) If μ_0 is selfdecomposable, then so is μ.
(iv) Let $m = 0$ and $d \geq 2$. Then there is a type G probability measure μ such that μ is selfdecomposable, but μ_0 is not selfdecomposable.

Proof. (i) The "if" part. If μ_0 is operator stable with some exponent matrix M, then its Lévy measure ν_0 satisfies that for any $a > 0$

$$a\nu_0(A) = \nu_0(b^{-M}A), \quad A \in \mathcal{B}_0(\mathbb{R}^d), \tag{4.1}$$

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where $t^M = \sum_{k=0}^{\infty} \frac{1}{k!}(\log t)^k M^k$, for $t > 0$ and a matrix M. Then we have

$$\nu(A) = \mathbf{E}[\nu_0(Z^{-1}A)] = \mathbf{E}[a^{-1}\nu_0(Z^{-1}a^{-M}A)] = a^{-1}\nu(a^{-M}A),$$

concluding that μ is operator stable.

The "only if" part. If μ is operator stable with some exponent M, then its Lévy measure ν satisfies the relation in (4.1) for ν instead of ν_0. Thus we have

$$\mathbf{E}[a\nu_0(Z^{-1}A)] = \mathbf{E}[\nu_0(Z^{-1}a^{-M}A)],$$

and by Proposition 2.3 in [MR00], we obtain

$$a\nu_0(\cdot) = \nu_0(a^{-M}\cdot),$$

concluding that μ_0 is operator stable.

(ii) The "if" part. If μ_0 is $\mathbb{R}^d$-valued semi-stable, then for some $r \in (0,1)$ and $\alpha \in (0,2]$,

$$r\nu_0(A) = \nu_0(r^{-1/\alpha}A), \quad A \in \mathcal{B}_0(\mathbb{R}^d). \tag{4.2}$$

Then obviously, ν satisfies (4.2) for the same r and α, which assures the semi-stability of μ. The "if" part can be shown as in the second half part of the proof of (i).

(iii) Since μ_0 is selfdecomposable, we have for each $a \in (0,1)$,

$$\nu_0(A) = \nu_0(aA) + \nu_0^a(A),$$

where ν_0^a is a Lévy measure. Thus the Lévy measure ν of μ satisfies

$$\nu(A) = \nu(aA) + \nu^a(A),$$

where ν^a is another Lévy measure. This implies the selfdecomposability of μ.

(iv) We use the same idea for Theorem 4.1 in [MR00]. Let $D_1 = \{x \in \mathbb{R}^d : 1 < |x| < 2\}$ and $D_2 = \{x \in \mathbb{R}^d : 0 < |x| < 1\}$, $d \geq 2$. Let

$$\rho_0(A) = \lambda_d(A \cap D_1) - \varepsilon\lambda_d(A \cap D_2), \quad 0 < \varepsilon < 1,$$

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and

$$\rho(A) = \mathbf{E}[\rho_0(Z^{-1}A)], \tag{4.3}$$

where λ_d is the Lebesgue measure In $\mathbb{R}^d$. Then we have shown in the proof of Theorem 4.1 in [MR00] that ρ_0 is not a measure, but ρ is a measure for sufficiently small $\varepsilon > 0$. Furthermore, these two ρ_0 and ρ satisfy conditions in (2.1) in Proposition 2.1 of [S98], and thus we can define

$$\nu_0(A) = \int_{\mathbb{R}^d} \rho_0(dx) \int_0^\infty \mathbf{1}_A(e^{-t}x)dt \tag{4.4}$$

and

$$\nu(A) = \int_{\mathbb{R}^d} \rho(dx) \int_0^\infty \mathbf{1}_A(e^{-t}x)dt. \tag{4.5}$$

A direct verification shows that (4.5) is the Lévy measure of some selfdecomposable distribution. In fact, it follows from [JV83] or by a reformulation in [SY84] of a theorem due to Urbanik [U69], that every Lévy measure of a selfdecomposable distribution can be written in the form (4.5) with ρ having the logarithmic moment. On the other hand, Sato [S98] showed that ν_0 is a Lévy measure, but the distribution whose Lévy measure is ν_0 in (4.4) is not selfdecomposable for ε small enough (see Proposition 2.2 of [S98]). It follows from (4.3) that

$$\begin{aligned}\nu(A) &= \int_{\mathbb{R}^d} \mathbf{E}[\rho_0(Z^{-1}dx)] \int_0^\infty \mathbf{1}_A(e^{-t}x)dt \\ &= \mathbf{E}\left[\int_{\mathbb{R}^d} \rho_0(dx) \int_0^\infty \mathbf{1}_{Z^{-1}A}(e^{-t}x)dt\right] \\ &= \mathbf{E}[\nu_0(Z^{-1}A)].\end{aligned}$$

Thus the infinitely divisible probability measure, whose Lévy measure is ν in (4.5), is of type G and satisfies our requirements in the statement (iv). This completes the proof of (iv). □

Related to Theorem 4.1 (iv), we want to know under what conditions in addition to the selfdecomposability of μ, μ_0 is selfdecomposable. To answer this question, we first prove the following. Note that if $H \subset \widetilde{I}(\mathbb{R}^d)$, then $Q(H) \subset \widetilde{I}(\mathbb{R}^d)$. Thus we can define $K(Q(H))$.

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Theorem 4.2. *For any* $H \subset \widetilde{I}(\mathbb{R}^d)$,

$$K(Q(H)) = Q(K(H)).$$

Proof. We first show that $K(Q(H)) \subset Q(K(H))$. Suppose $\mu \in K(Q(H))$. Then its Lévy measure ν is represented as in (1.1), and its predecessor μ_0 satisfies that for each $a \in (0,1)$, there exists $\rho^a \in H$ such that $\widehat{\mu}_0(\theta) = \widehat{\mu}_0(a\theta)\widehat{\rho_0^a}(\theta)$. Thus the respective Lévy measures ν_0 and ν_0^a of μ_0 and ρ_0^a satisfy

$$\nu_0(A) = \nu_0(aA) + \nu_0^a(A).$$

Hence we have

$$\nu(A) = \mathbf{E}[\nu_0(aZ^{-1}A)] + \mathbf{E}[\nu_0^a(Z^{-1}A)] = \nu(aA) + \xi^a(A),$$

implying that

$$\widehat{\mu}(\theta) = \widehat{\mu}(a\theta)\widehat{\eta^a}(\theta),$$

where $\eta^a \in I(\mathbb{R}^d)$ is the probability distribution with Lévy measure ξ^a and $\eta^a \in K(H)$. This concludes that $\mu \in Q(K(H))$.

We next show that $Q(K(H)) \subset K(Q(H))$. Suppose $\mu \in Q(K(H))$. Then for any $a \in (0,1)$ there exists $\rho_a \in K(H)$ such that

$$\widehat{\mu}(\theta) = \widehat{\mu}(a\theta)\widehat{\rho}_a(\theta),$$

If $\rho_a \in K(H)$, then its Lévy measure ν^a is represented as

$$\nu^a(A) = \mathbf{E}[\nu_0^a(Z^{-1}A)] \tag{4.6}$$

for some Lévy measure ν_0^a, depending on a, whose corresponding infinitely divisible distribution belongs to the class H. On the other hand, since ρ_a is of type G and ρ_a converges weakly to μ as $a \to 0$, μ is of type G (see Proposition 2.4 in [MR00]). Hence

$$\nu(A) = \mathbf{E}[\nu_0(Z^{-1}A)]$$

for some symmetric Lévy measure ν_0. Combining this with (4.6) we get

$$\mathbf{E}[\nu_0^a(Z^{-1}A)] = \mathbf{E}[\nu_0(Z^{-1}A) - \nu(aZ^{-1}A)]$$

for any $A \in \mathcal{B}_0(\mathbb{R}^d)$. By Proposition 2.3 in [MR00],

$$\nu_0^a(A) = \nu_0(A) - \nu(aA)$$

for any $A \in \mathcal{B}_0(\mathbb{R}^d)$. Hence μ_0, the predecessor of μ, is selfdecomposable and $\mu_0 \in Q(H)$. Consequently, $\mu \in K(Q(H))$ and the proof of Theorem 4.2 is complete. □

Theorem 4.3. *If μ is selfdecomposable, namely if for any $a \in (0,1)$, there exists $\rho_a \in I(\mathbb{R}^d)$ such that $\widehat{\mu}(\theta) = \widehat{\mu}(a\theta)\widehat{\rho}_a(\theta)$, and further if ρ_a is of type G, then μ_0, the predecessor of μ, is selfdecomposable.*

Proof. Applying Theorem 4.2 to the case $H = \widetilde{I}(\mathbb{R}^d)$, we have

$$K(\widetilde{L}_0(\mathbb{R}^d)) = Q(TG_0(\mathbb{R}^d)).$$

Therefore, the following two statements are equivalent:
(i) μ is selfdecomposable such that for any $a \in (0,1)$, $\widehat{\mu}(\theta) = \widehat{\mu}(a\theta)\widehat{\rho}_a(\theta)$, where ρ_a is of type G.
(ii) μ is of type and its predecessor μ_0 is selfdecomposable.
This equivalence concludes the statement of the theorem. □

5. Some examples

Here we give simple examples of $\mu \in TG_m(\mathbb{R}^1), m = 0, 1$. We start with a lemma due to [ShSr77].

Lemma 5.1. *Let Z be the standard normal random variable and Y be a positive random variable independent of Z. Then $|Z|^pY$ is infinitely divisible for any $p \geq 2$.*

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Example 5.1. *If $Z_1, ..., Z_n$ are i.i.d. standard normal random variables, then $Z_1 \cdots Z_n$ is of type G.*

Proof. $Z_1 \cdots Z_n \stackrel{d}{=} Z_1 |Z_2 \cdots Z_n|$ and $|Z_2 \cdots Z_n|^2$ is infinitely divisible by the above lemma. □

We are now going to show that the distribution of $Z_1 Z_2$ belongs to $TG_1(\mathbb{R}^1)$, by applying Theorem 2.5.

Example 5.2. *Let Z_1 and Z_2 be independent standard normal random variables. Then the distribution of $Z_1 Z_2$ is in $TG_1(\mathbb{R}^1)$.*

Proof. Since

$$X := Z_1 Z_2 \stackrel{d}{=} Z_1 (|Z_2|^2)^{1/2},$$

V in (2.3) is $|Z_2|^2$ in this case. By Theorem 2.5, it is enough to show that the symmetric measure ν_0 determined by (2.4) with the Lévy measure ρ of $V = |Z_2|^2$ is the Lévy measure of a type G distribution. Note that $|Z_2|^2$ is χ^2-distribution with freedom 1, thus is nonnegative infinitely divisible, and its Lévy measure ρ is of the form

$$\rho([x, \infty)) = \int_x^\infty \frac{e^{-u/2}}{u} du.$$

Then by Theorem 2.4, for $x > 0$,

$$\begin{aligned} \nu_0([x, \infty)) &= \frac{1}{2} \rho([x^2, \infty)) = \frac{1}{2} \int_{x^2}^\infty \frac{e^{-u/2}}{u} du \\ &= \frac{1}{4} \int_x^\infty \frac{e^{-v^{1/2}/2}}{v} dv = \int_x^\infty g(v^2) dv. \end{aligned}$$

By a characterization for type G distributions (see Theorem 1 of [R91], also see Theorem 2.5 of [MR00]), it is enough to check that

$$g(x) = x^{-1/2} e^{-x^{1/4}/2}$$

is completely monotone. However, this is true, (see again e.g., E 55.1, page 424 in [S99]). The proof is completed. □

6. Further problems

We conclude the paper by stating some further problems which naturally arise form the observations in this paper.

Problem 1 : In Theorem A, we gave a necessary and sufficient condition for that $\mu \in TG_0(\mathbb{R}^d)$. Namely, $\mu \in TG_0(\mathbb{R}^d)$ if and only if the radial component of its Lévy measure has a density involving a completely monotone function $g_x(\cdot)$. What additional conditions on $g_x(\cdot)$ assure that $\mu \in TG_m(\mathbb{R}^d)$?

Problem 2 : Related to Corollary 3.1, we conjecture that $TG_\infty(\mathbb{R}^d) = \widetilde{L}_\infty(\mathbb{R}^d)$, namely $TG_\infty(\mathbb{R}^d)$ is also the *smallest* class containing the class $\widetilde{S}(\mathbb{R}^d)$ of all symmetric stable distributions, closed under convolution and weak convergence.

Problem 3 : In Examples 5.1 and 5.2, we have shown that the distribution of the product $Z_1 \cdots Z_n$ is of type G and furthermore the distribution of Z_1Z_2 belongs to $TG_1(\mathbb{R}^d)$. Can one say more about the distribution of $Z_1 \cdots Z_n$?

Problem 4 : Type G distributions are continuous but are they absolutely continuous on their support?

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