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Type G Distributions on $\mathbb{R}^d$

by

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Abstract. This paper presents a systematic study of the class of multivariate distributions obtained by a Gaussian randomization of jumps of a Lévy process. This class, called the class of type G distributions, constitutes a closed convolution semigroup of the family of symmetric infinitely divisible probability measures. Spectral form of Lévy measures of type G distributions is obtained and it is shown that type G property can not be determined by one dimensional projections. Conditionally Gaussian structure of type G random vectors is exhibited via series representations.

1. Introduction

A real-valued random variable X is said to be of *type G* if

$$X \stackrel{d}{=} V^{1/2} Z, \tag{1.1}$$

where Z is the standard normal random variable and V is a nonnegative infinitely divisible random variable independent of Z. It is easy to see that X is symmetric and infinitely divisible.

There are several ways to generalize (1.1) to the multivariate case. One natural way is to replace the standard normal random variable Z by a mean zero Gaussian vector $\mathbf{W}$ in $\mathbb{R}^d$. This leads to the well studied subclass of elliptically symmetric measures defined as distributions of

$$\mathbf{X} \stackrel{d}{=} V^{1/2} \mathbf{W} \tag{1.2}$$

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(see Chmielewski [C81] for the review and further references on elliptically symmetric distributions). Such random vectors can also be related to a Lévy Gaussian process composed with a subordinator. Namely, let $\{\mathbf{W}(t)\}_{t\geq 0}$ be a Lévy Gaussian process in $\mathbb{R}^d$ with $\mathbf{W}(1) \stackrel{d}{=} \mathbf{W}$ (a Brownian motion, when $\mathbf{W}$ has the identity covariance matrix) and let $\{V(t)\}_{t\geq 0}$ be a nondecreasing Lévy process (a subordinator) with $V(1) \stackrel{d}{=} V$. Assume that these processes are independent of each other. Then $\mathbf{X}$ in (1.2) can also be represented as

$$\mathbf{X} \stackrel{d}{=} \mathbf{W}(V(1)).$$

Now we will describe another way of viewing (1.1) which leads to the definition of multivariate type G distributions. Let $\{V_0(t)\}_{t\geq 0}$ be a Lévy process whose quadratic variation $[V_0, V_0](t)$ satisfies

$$[V_0, V_0](1) \stackrel{d}{=} V.$$

Let $\{X(t)\}_{t\geq 0}$ be a Lévy process obtained from $\{V_0(t)\}_{t\geq 0}$ by multiplying jumps of $\{V_0(t)\}_{t\geq 0}$ by an independent sequence of i.i.d. standard normal random variables. This procedure will be called a Gaussian randomization of $\{V_0(t)\}_{t\geq 0}$. Then

$$X(1) \stackrel{d}{=} X \tag{1.3}$$

where X is given by (1.1). Indeed, the process $\{X(t)\}_{t\geq 0}$ has the same Gaussian component as $\{V_0(t)\}_{t\geq 0}$, say, a Brownian motion with variance $\sigma^2 \geq 0$, and has a symmetric Poissonian component with Lévy measure ν of $X(1)$ given by

$$\nu(A) = \mathbf{E}[\nu_0(Z^{-1}A)], \tag{1.4}$$

where ν_0 is Lévy measure of $V_0(1)$. (1.4) is obvious in the case when the Poissonian component of $\{V_0(t)\}_{t\geq 0}$ is a compound Poisson process, so that the procedure of Gaussian randomization can be clearly written. In general, this procedure is formalized by use of series representations of Lévy processes and (1.4) follows (see Section 5). Since

$$\mathbf{E}\left[e^{-\lambda V}\right] = \exp\left\{-\lambda\sigma^2 + \int (e^{-\lambda x^2} - 1)\, \nu_0(dx)\right\}, \quad \lambda \geq 0,$$

(1.3) can be easily checked. A multivariate Lévy process is said to be of type G if it can be obtained by a Gaussian randomization of some $\mathbb{R}^d$-valued Lévy process $\{\mathbf{V}_0(t)\}_{t\geq 0}$. Consequently, a *type G random vector* $\mathbf{X}(1)$ has Lévy measure given by (1.4). To obtain the analog of (1.1) in the multidimensional setting, we define $\mathbf{V}$ as a nonnegative definite random matrix by

$$\mathbf{V} = \Big([V_0^{(i)}, V_0^{(j)}](1)\Big)_{i,j=1,\ldots,d} \tag{1.5}$$

where $\mathbf{V}_0(t) = (V_0^{(1)}(t), \ldots, V_0^{(d)}(t))^T$, $(\cdot)^T$ stands for the transpose of a vector, and $[\cdot,\cdot](t)$ denotes the bracket process. Then

$$\mathbf{X}(1) \stackrel{d}{=} \mathbf{V}^{1/2}\mathbf{Z}, \tag{1.6}$$

where $\mathbf{Z}$ is the standard Gaussian random vector in $\mathbb{R}^d$ independent of $\mathbf{V}$ (see Section 5 for the proof). Random vectors possessing this kind of representation, with $\mathbf{V}$ being infinitely divisible nonnegative definite random matrix, have been recently introduced by Barndorff-Nielsen and Pérez-Abreu [BNPA00]. More generally, a random vector with all one dimensional projections of type G is said to be of type $\mathcal{G}$ (see [BNPA00]). The class of type G distributions considered here constitutes an important subclass of the latter class with rich infinitely divisible and Gaussian structures whose combination deserves a systematic study for its own.

Type G random vectors and stochastic processes were introduced by Marcus [M78] and further investigated by Rosiński [R90], [R91]. The well understood conditional Gaussian structure of type G processes made possible various extensions of Gaussian results to such class of infinitely divisible processes. This paper examines the class of type G distributions on $\mathbb{R}^d$ from the perspective of infinitely divisible laws.

Section 2 is devoted to basic properties of type G distributions and their Lévy-Khintchine representation. Theorem 2.5 gives the spectral representation of Lévy measure of a type G random vector. In Section 3 we define a type G stochastic process in a natural way, as the one with type G finite dimensional distributions. Theorem 3.2 proves that our definition is equivalent to the definition previously

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used in the literature. Section 4 examines whether the following *projection property* is true: if all one dimensional projections of a symmetric infinitely divisible random vector are of type G, then the random vector is of type G. The answer is negative; we construct an example of a symmetric infinitely divisible probability measure on $\mathbb{R}^d$, $d \geq 2$, which is not of type G on $\mathbb{R}^d$ but any lower dimensional projection is of type G on a lower dimensional space. In Section 5 we discuss a Gaussian randomization of Lévy processes using series representations and prove (1.5)–(1.6).

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2. Basic properties and Lévy-Khintchine representation of type G distributions.

For $\mathbf{x} \in \mathbb{R}^d$, $|\mathbf{x}|$ denotes the Euclidean norm of $\mathbf{x}$; $\mathcal{B}_0(\mathbb{R}^d)$ stands for the class of all Borel sets A in $\mathbb{R}^d$ such that $A \subset \{\mathbf{x} : |\mathbf{x}| > \epsilon\}$ for some $\epsilon > 0$. Following (1.4) of the Introduction, we define type G probability measures.

Definition 2.1. *A symmetric infinitely divisible probability measure μ on $\mathbb{R}^d$ is said to be of type G if its Lévy measure ν is of the form*

$$\nu(A) = \mathbf{E}[\nu_0(Z^{-1}A)], \qquad A \in \mathcal{B}_0(\mathbb{R}^d), \tag{2.1}$$

where ν_0 is a Borel measure on $\mathbb{R}^d \setminus \{0\}$ and Z is the standard normal random variable.

A symmetric α-stable distribution on $\mathbb{R}^d$ constitutes a fundamental example of type G probability measures. Indeed, the Lévy measure ν of symmetric α-stable distribution is symmetric and satisfies a condition $a^\alpha \nu(A) = \nu(a^{-1}A)$, for every $a > 0$ and $A \in \mathcal{B}_0(\mathbb{R}^d)$. Thus (2.1) holds for $\nu_0 = (\mathbf{E}[|Z|^\alpha])^{-1}\nu$. Other type G distributions can be constructed using Theorem 2.6 of this section.

Proposition 2.2. *Let ν and ν_0 be two σ–finite measures on $\mathcal{B}_0(\mathbb{R}^d)$ related by (2.1). Let $\widetilde{\nu}_0(A) = [\nu_0(A) + \nu_0(-A)]/2,\ \ A \in \mathcal{B}_0(\mathbb{R}^d)$, be the symmetrization of ν_0. Then*
(i) ν_0 is a Lévy measure if and only if ν is.
(ii) $\nu(A) = \mathbf{E}[\widetilde{\nu}_0(Z^{-1}A)] = \mathbf{E}[\widetilde{\nu}_0(|Z|^{-1}A)],\ \ A \in \mathcal{B}_0(\mathbb{R}^d)$.
(iii) $\widetilde{\nu}_0$ is determined uniquely by ν.

Proof. (i) follows from the inequalities

$$P(|Z|>1)\int_{\mathbb{R}^d}(1\wedge|\mathbf{x}|^2)\nu_0(d\mathbf{x}) \le \int_{\mathbb{R}^d}(1\wedge|\mathbf{x}|^2)\nu(d\mathbf{x}) \le \mathbf{E}[(1\vee Z^2)]\int_{\mathbb{R}^d}(1\wedge|\mathbf{x}|^2)\nu_0(d\mathbf{x}).$$

(ii) is obvious. To prove (iii) we consider a polar decomposition of $\widetilde{\nu}_0$,

$$\widetilde{\nu}_0(EB) = \int_B \lambda(d\mathbf{x})\rho_{\mathbf{x}}(E),$$

where λ is a probability measure on S, the unit sphere of $\mathbb{R}^d$, and $\{\rho_{\mathbf{x}}\}_{\mathbf{x}\in S}$ is a measurable family of Borel measures on $\mathbb{R}_+$; $E \in \mathcal{B}(\mathbb{R}_+)$, $B \in \mathcal{B}(S)$. In view of (2.1) we have for every $a > 0$ and $B \in \mathcal{B}(S)$,

$$\nu\big((a,\infty)B\big) = \mathbf{E}\big[\widetilde{\nu}_0\big((a|Z|^{-1},\infty)B\big)\big] = \int_B \lambda(d\mathbf{x})\int_0^\infty P(|Z|>au^{-1})\,\rho_{\mathbf{x}}(du).$$

Hence for $h > 0$,

$$h^{-1}\nu\big((a,a+h]B\big) = \int_B \lambda(d\mathbf{x})\int_0^\infty 2h^{-1}\{\Phi((a+h)u^{-1}) - \Phi(au^{-1})\}\,\rho_{\mathbf{x}}(du),$$

where Φ is the standard normal distribution function. Now we want to take $h \to 0$. The interchange of the integral with the limit is justified by the fact that

$$h^{-1}\{\Phi((a+h)u^{-1}) - \Phi(au^{-1})\} = \frac{1}{\sqrt{2\pi}}e^{-\frac{(a+\theta h)^2}{2u^2}}u^{-1} \le C(1\wedge u^2),$$

where $C < \infty$ depends only on a, and that $\widetilde{\nu}_0$ is a Lévy measure. Since a similar argument holds for $h < 0$, the function $a \to \nu\big((a,\infty)B\big)$ is differentiable and

$$\frac{d\nu\big((a,\infty)B\big)}{da} = \frac{-2}{\sqrt{2\pi}}\int_B \lambda(d\mathbf{x})\int_0^\infty e^{-\frac{a^2}{2u^2}}u^{-1}\,\rho_{\mathbf{x}}(du). \tag{2.2}$$

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Let ν_1 and ν_2 be two symmetric Lévy measures that can be put in the place of ν_0 in (2.1). We want to show that $\nu_1 = \nu_2$. Let $(\lambda_1, \rho_{\mathbf{x}}^{(1)})$ and $(\lambda_2, \rho_{\mathbf{x}}^{(2)})$ be their respective polar representations, as described above. Let $\lambda = (\lambda_1 + \lambda_2)/2$. We have $\lambda_1(d\mathbf{x}) = h_1(\mathbf{x})\lambda(d\mathbf{x})$ and $\lambda_2(d\mathbf{x}) = h_2(\mathbf{x})\lambda(d\mathbf{x})$ for some measurable nonnegative functions h_1 and h_2. Consequently, $(\lambda, h_1(\mathbf{x})\rho_{\mathbf{x}}^{(1)})$ and $(\lambda, h_2(\mathbf{x})\rho_{\mathbf{x}}^{(2)})$ are polar decompositions of ν_1 and ν_2, respectively. In view of (2.2), for each $a > 0$,

$$\int_0^\infty e^{-\frac{a^2}{2u^2}} u^{-1}\, h_1(\mathbf{x})\rho_{\mathbf{x}}^{(1)}(du) = \int_0^\infty e^{-\frac{a^2}{2u^2}} u^{-1}\, h_2(\mathbf{x})\rho_{\mathbf{x}}^{(2)}(du) \quad \lambda - a.e. \tag{2.3}$$

Hence (2.3) holds λ-a.e. for all rational $a > 0$, and by the continuity of both sides with respect to a, (2.3) holds λ-a.e for all $a > 0$. By the uniqueness of the Laplace transform

$$h_1(\mathbf{x})\rho_{\mathbf{x}}^{(1)} = h_2(\mathbf{x})\rho_{\mathbf{x}}^{(2)} \quad \lambda - a.e.$$

Since $(\lambda, h_1(\mathbf{x})\rho_{\mathbf{x}}^{(1)})$ and $(\lambda, h_2(\mathbf{x})\rho_{\mathbf{x}}^{(2)})$ are polar decompositions of ν_1 and ν_2, respectively, we have $\nu_1 = \nu_2$. □

The uniqueness described in Proposition 2.2 (ii) holds also for signed measures.

Proposition 2.3. *Let ν_1 and ν_2 be symmetric signed measures on $\mathbb{R}^d \setminus \{0\}$ such that $|\nu_1|$ and $|\nu_2|$ are Lévy measures. If*

$$\mathbf{E}[\nu_1(Z^{-1}A)] = \mathbf{E}[\nu_2(Z^{-1}A)] \quad \textit{for every } A \in \mathcal{B}_0(\mathbb{R}^d),$$

then $\nu_1 = \nu_2$.

Proof. If ν_1 and ν_2 are symmetric, then their total variations $|\nu_1|$ and $|\nu_2|$ are also symmetric. Therefore, in the orthogonal decomposition $\nu_i = \nu_i^+ - \nu_i^-$, the measures ν_i^+ and ν_i^- are symmetric, $i = 1, 2$. From our assumption and Proposition 2.2 (ii), $\nu_1^+ + \nu_2^- = \nu_2^+ + \nu_1^-$, implying $\nu_1 = \nu_2$. □

Denote by $\mathrm{TG}(\mathbb{R}^d)$ the class of all type G probability measures on $\mathbb{R}^d$.

Proposition 2.4. *The class $\mathrm{TG}(\mathbb{R}^d)$ of type G probability measures on $\mathbb{R}^d$ is closed under convolutions, linear mappings, and weak convergence.*

Proof. We only need to prove that $\mathrm{TG}(\mathbb{R}^d)$ is closed under weak convergence. Let $\mu = \lim_{n\to\infty} \mu_n$, where $\mu_n \in \mathrm{TG}(\mathbb{R}^d)$. Then μ is symmetric and infinitely divisible. Let ν be the Lévy measure of μ and let ν_n be the Lévy measure of μ_n. By our assumption we have

$$\nu_n(A) = \mathbf{E}[\nu_{0n}(Z^{-1}A)], \qquad A \in \mathcal{B}_0(\mathbb{R}^d)$$

for some symmetric Lévy measures ν_{0n}, $n \in \mathbb{N}$. Using (2.2) we get

$$\sup_n \int_{\mathbb{R}^d} (1 \wedge |\mathbf{x}|^2)\, \nu_{0n}(d\mathbf{x}) \le \{P(|Z| > 1)\}^{-1} \sup_n \int_{\mathbb{R}^d} (1 \wedge |\mathbf{x}|^2)\, \nu_n(d\mathbf{x}) = M < \infty.$$

Hence, for every $R > 0$,

$$\nu_{0n}(\{|\mathbf{x}| > R\}) \le R^{-2} \int_{\mathbb{R}^d} (1 \wedge |\mathbf{x}|^2)\, \nu_{0n}(d\mathbf{x}) \le R^{-2} M.$$

This implies that for every $r > 0$, the sequence $\{\nu_{0n}\}_{n\in\mathbb{N}}$ restricted to $\{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| > r\}$ is conditionally weakly compact. Using the diagonal method, we can find a subsequence $\{\nu_{0n_k}\}_{k\in\mathbb{N}}$ and a measure ν_0 on $\mathbb{R}^d \setminus \{0\}$ such that, for each $i \in \mathbb{N}$,

$$\lim_{k\to\infty} \nu_{0n_k|\{|\mathbf{x}|>i^{-1}\}} = \nu_{0|\{|\mathbf{x}|>i^{-1}\}}.$$

We will prove that ν_0 and ν are related by (2.1). To this end, let $f : \mathbb{R}^d \to [0, 1]$ be a continuous function such that $f(\mathbf{x}) = 0$ for all $|\mathbf{x}| < r$ and some $r > 0$. For arbitrary $R > r$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} f(\mathbf{x})\, \nu_{n_k}(d\mathbf{x}) &= \left(\int_{|Z|<R} + \int_{|Z|\ge R} \right) \int_{\mathbb{R}^d} f(Z\mathbf{x})\, \nu_{0n_k}(d\mathbf{x}) dP \\ &= I_k + J_k.\end{aligned}$$

If $|Z| < R$, then

$$\int_{\mathbb{R}^d} f(Z\mathbf{x})\, \nu_{0n_k}(d\mathbf{x}) \le \nu_{0n_k}(\{|\mathbf{x}| > rR^{-1}\}) \le r^{-2} R^2 M.$$

Thus

$$\lim_{k\to\infty} I_k = \int_{|Z|<R} \int_{\mathbb{R}^d} f(Z\mathbf{x})\, \nu_0(d\mathbf{x}) dP$$

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by the Dominated Convergence Theorem. If $|Z| \geq R$, then

$$\begin{aligned}\int_{\mathbb{R}^d} f(Z\mathbf{x})\,\nu_{0n_k}(d\mathbf{x}) &\leq \nu_{0n_k}(\{\mathbf{x}:\ |\mathbf{x}| \geq r|Z|^{-1}\}) \\ &\leq \int_{\mathbb{R}^d} (Z^2 r^{-2}|\mathbf{x}|^2 \wedge 1)\,\nu_{0n_k}(d\mathbf{x}) \\ &\leq (Z^2 r^{-2} \vee 1) \int_{\mathbb{R}^d} (|\mathbf{x}|^2 \wedge 1)\,\nu_{0n_k}(d\mathbf{x}).\end{aligned}$$

Hence

$$J_k \leq M \int_{|Z|\geq R} (Z^2 r^{-2} \vee 1)\,dP \quad \text{for all } k \in \mathbb{N}.$$

Consequently,

$$\begin{aligned}&\left| \int_{\mathbb{R}^d} f(\mathbf{x})\ \ \nu(d\mathbf{x}) - \int_\Omega \int_{\mathbb{R}^d} f(Z\mathbf{x})\,\nu_0(d\mathbf{x})dP \right| \\ &= \lim_{R\to\infty} \left| \int_{\mathbb{R}^d} f(\mathbf{x})\,\nu(d\mathbf{x}) - \int_{|Z|<R} \int_{\mathbb{R}^d} f(Z\mathbf{x})\,\nu_0(d\mathbf{x})dP \right| \\ &= \lim_{R\to\infty} \lim_{k\to\infty} \left| \int_{\mathbb{R}^d} f(\mathbf{x})\,\nu_{n_k}(d\mathbf{x}) - I_k \right| \\ &\leq \lim_{R\to\infty} M \int_{|Z|\geq R} (Z^2 r^{-2} \vee 1)\,dP = 0.\end{aligned}$$

This completes the proof. □

Theorem 2.5. *A symmetric probability measure μ on $\mathbb{R}^d$ is of type G if and only if it is infinitely divisible and its Lévy measure ν is either zero or it can be represented as*

$$\nu(EB) = \int_B \lambda(d\mathbf{x}) \int_E g_{\mathbf{x}}(r^2)dr, \tag{2.4}$$

$E \in \mathcal{B}(\mathbb{R}_+)$, $B \in \mathcal{B}(S)$, where λ is a probability measure on the unit sphere S of $\mathbb{R}^d$ and $g_{\mathbf{x}}(r)$ is a jointly measurable function such that, for any fixed $\mathbf{x} \in S$, $g_{\mathbf{x}}(\cdot)$ is completely monotone on $(0,\infty)$ and satisfies

$$\int_0^\infty (1 \wedge r^2) g_{\mathbf{x}}(r^2)\,dr = c \in (0,\infty) \tag{2.5}$$

with c *independent of* $\mathbf{x}$. *This representation is unique in the sense that, if* $\nu \neq 0$ *and two pairs* $(\lambda, g_{\mathbf{x}})$ *and* $(\widetilde{\lambda}, \widetilde{g}_{\mathbf{x}})$ *satisfy the above conditions, then* $\lambda = \widetilde{\lambda}$ *and* $g_{\mathbf{x}} = \widetilde{g}_{\mathbf{x}}$ *for* λ*-a.e. Moreover,* λ *in (2.5) is a symmetric probability measure and* $g_{\mathbf{x}} = g_{-\mathbf{x}}$ λ*-a.e.*

Proof. Suppose that μ is of type G with $\nu \neq 0$. By Proposition 2.2 (ii) we may assume that ν_0 is a symmetric Lévy measure. Let

$$\nu_0(EB) = \int_B \lambda_0(d\mathbf{x})\rho_{\mathbf{x}}(E) \tag{2.6}$$

be the polar decomposition of ν_0, where E and B as above. We choose $(\lambda_0, \rho_{\mathbf{x}})$ such that λ_0 is a symmetric probability measure and $\rho_{\mathbf{x}} = \rho_{-\mathbf{x}}$ for every $\mathbf{x}$ and also that

$$\lambda_0(\{\mathbf{x} \in S : \ \rho_{\mathbf{x}}(\mathbb{R}_+) = 0\}) = 0. \tag{2.7}$$

Define

$$k_{\mathbf{x}}(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{r}{2u^2}} u^{-1}\, \rho_{\mathbf{x}}(du), \quad r > 0. \tag{2.8}$$

Then

$$\begin{aligned}\int_B \lambda_0(d\mathbf{x}) \int_E k_{\mathbf{x}}(r^2)dr &= \sqrt{\frac{2}{\pi}} \int_B \lambda_0(d\mathbf{x}) \int_0^\infty \int_0^\infty \mathbf{1}_E(r) e^{-\frac{r^2}{2u^2}} u^{-1}\, dr \rho_{\mathbf{x}}(du) \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty \int_B \lambda_0(d\mathbf{x}) \int_0^\infty \mathbf{1}_E(uy)\rho_{\mathbf{x}}(du) e^{-\frac{y^2}{2}}\, dy = \nu(EB).\end{aligned}$$

This proves the identity (2.4) for $(\lambda_0, k_{\mathbf{x}})$. Since, for every $n \geq 1$,

$$\int_S \lambda_0(d\mathbf{x}) \int_{n^{-1}}^\infty k_{\mathbf{x}}(r^2)dr = \nu(\{\mathbf{x} : |\mathbf{x}| > n^{-1}\}) < \infty,$$

we get

$$\lambda_0(\{\mathbf{x} \in S : \ k_{\mathbf{x}}(r) = \infty \text{ for some } r > 0\}) = 0.$$

Hence (2.8) defines a real valued completely monotone function for λ_0-a.e. $\mathbf{x} \in S$. We will now modify $(\lambda_0, k_{\mathbf{x}})$ to fulfill (2.5). Let

$$c = \int_S \lambda_0(d\mathbf{x}) \int_0^\infty (1 \wedge r^2) k_{\mathbf{x}}(r^2)\, dr = \int_{\mathbb{R}^d} (1 \wedge |\mathbf{y}|^2)\, \nu(d\mathbf{y}),$$

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$c \in (0, \infty)$, and let

$$c_{\mathbf{x}} = \int_0^\infty (1 \wedge r^2) k_{\mathbf{x}}(r^2)\, dr.$$

$c_{\mathbf{x}} > 0$ λ_0-a.e. by (2.7). Define

$$g_{\mathbf{x}} = c c_{\mathbf{x}}^{-1} k_{\mathbf{x}}$$

and

$$\lambda(d\mathbf{x}) = c^{-1} c_{\mathbf{x}} \lambda_0(d\mathbf{x}).$$

$g_{\mathbf{x}}$ satisfies (2.5) λ_0-a.e. and λ is a symmetric probability measure. Finally, to ensure the validity of (2.5) for all $\mathbf{x}$'s, we replace $g_{\mathbf{x}}$ by some completely monotone function g satisfying (2.5) whenever $c_{\mathbf{x}} = \infty$ or (2.5) fails. Condition (2.5) implies uniqueness of $(\lambda, g_{\mathbf{x}})$. This ends the first part of the proof.

Assume now that ν has a representation (2.4) with (2.5). Since $g_{\mathbf{x}}(r)$ is jointly measurable and for each $\mathbf{x}$, it is completely monotone, there exists a measurable family $\{\rho_{\mathbf{x}}\}_{\mathbf{x} \in S}$ of Borel measures such that

$$g_{\mathbf{x}}(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{r}{2u^2}} u^{-1}\, \rho_{\mathbf{x}}(du), \quad r > 0.$$

A measure ν_0 defined by (2.6) with $\lambda_0 = \lambda$ satisfies (2.1). This completes the proof. □

The next theorem gives a general form of characteristic function of a type G distribution. This form was obtained for type G probability measures on Banach spaces in [R90] and its adaptation was used to define type G stochastic processes in [R91]. We provide it here for the sake of completeness.

Theorem 2.6. *Let $\psi : [0, \infty) \to [0, \infty)$ be a continuous function with completely monotone derivative on $(0, \infty)$ and such that $\psi(0) = \psi'(\infty) = 0$. Let ξ be a σ-finite measure on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} \psi(\langle \mathbf{y}, \mathbf{x} \rangle^2)\, \xi(d\mathbf{x}) < \infty$ for every $\mathbf{y} \in \mathbb{R}^d$, where $\langle \mathbf{y}, \mathbf{x} \rangle$ is an inner product of $\mathbf{y}$ and $\mathbf{x} \in \mathbb{R}^d$. Then there is a type G probability measure μ on $\mathbb{R}^d$ with no Gaussian component such that*

$$\widehat{\mu}(\mathbf{y}) = \exp\left\{ - \int_{\mathbb{R}^d} \psi\left(\langle \mathbf{y}, \mathbf{x} \rangle^2\right)\, \xi(d\mathbf{x}) \right\}, \quad \mathbf{y} \in \mathbb{R}^d. \tag{2.9}$$

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Conversely, characteristic function of any type G measure with no Gaussian component is of the form (2.9).

Proof. First we prove the converse part. If μ is of type G with no Gaussian component and Lévy measure ν, then

$$\widehat{\mu}(\mathbf{y}) = \exp\left[\int_{\mathbb{R}^d} \left(e^{-\langle \mathbf{y},\mathbf{x}\rangle^2/2} - 1\right) \nu_0(d\mathbf{x})\right]$$

where ν_0 is a measure in (2.1) corresponding to ν. Hence (2.9) holds with $\psi(u) = 1 - \exp(-u/2)$ and $\xi = \nu_0$.

To prove the direct part we notice that, by Feller [F71], Theorems 1-2 in Ch. 13.7, there exists a Borel measure ρ on $[0,\infty)$ such that $\int_{[0,\infty)}(s^{-1} \wedge 1)\rho(ds) < \infty$ and

$$\psi(u) = \int_{[0,\infty)} (1 - e^{-us})s^{-1}\,\rho(ds), \quad u \geq 0.$$

The assumption $\psi'(\infty) = 0$ gives $\rho(\{0\}) = 0$. Define a measure

$$\nu_0(A) = \int_{\mathbb{R}^d}\int_0^\infty \mathbf{1}_A((2s)^{1/2}\mathbf{x})s^{-1}\,\rho(ds)\xi(d\mathbf{x}), \quad A \in \mathcal{B}_0(\mathbb{R}^d)$$

and let ν be given by (2.1). Then ν is a symmetric measure satisfying

$$\begin{aligned}\int_{\mathbb{R}^d} (1 - \cos\langle \mathbf{y},\mathbf{x}\rangle)\,\nu(d\mathbf{x}) &= \int_{\mathbb{R}^d} (1 - e^{-\langle \mathbf{y},\mathbf{x}\rangle^2/2})\,\nu_0(d\mathbf{x}) \\ &= \int_{\mathbb{R}^d}\int_0^\infty (1 - e^{-\langle \mathbf{y},(2s)^{1/2}\mathbf{x}\rangle^2/2})s^{-1}\,\rho(ds)\xi(d\mathbf{x}) \\ &= \int_{\mathbb{R}^d} \psi(\langle \mathbf{y},\mathbf{x}\rangle^2)\,\xi(d\mathbf{x}) = M(\mathbf{y}) < \infty.\end{aligned}$$

Using this identity we get

$$\int_{|\langle \mathbf{y},\mathbf{x}\rangle|\leq 1} \langle \mathbf{y},\mathbf{x}\rangle^2\,\nu(d\mathbf{x}) \leq c_1 \int_{|\langle \mathbf{y},\mathbf{x}\rangle|\leq 1} (1 - \cos\langle \mathbf{y},\mathbf{x}\rangle)\,\nu(d\mathbf{x}) \leq c_1 M(\mathbf{y})$$

and

$$\begin{aligned}\int_{|\langle \mathbf{y},\mathbf{x}\rangle|>1} \nu(d\mathbf{x}) &\leq c_2 \int_{|\langle \mathbf{y},\mathbf{x}\rangle|>1} \left(1 - \frac{\sin\langle \mathbf{y},\mathbf{x}\rangle}{\langle \mathbf{y},\mathbf{x}\rangle}\right) \nu(d\mathbf{x}) \\ &= c_2 \int_0^1 \int_{\mathbb{R}^d} (1 - \cos\langle r\mathbf{y},\mathbf{x}\rangle)\,\nu(d\mathbf{x})dr \leq c_2 M(\mathbf{y})\end{aligned}$$

for some finite constants c_1 and c_2. Hence ν is a Lévy measure that generates a type G distribution μ by (2.9). This concludes the proof. □

3. Stochastic processes of type G.

Definition 3.1. *A real valued stochastic process $\{X(t)\}_{t\in T}$ is said to be of type G if all its finite dimensional distributions are of type G.*

In the sequel, we consider the path space $\mathbb{R}^T$ as a measurable space equipped with the cylindrical σ–field, $\mathbb{R}^{(T)}$ denotes the subspace of $\mathbb{R}^T$ consisting of functions with finite support, and $yx := \sum_{t\in T} y(t)x(t)$, $y \in \mathbb{R}^{(T)}$, $x \in \mathbb{R}^T$ is the natural bilinear form defining the duality of $\mathbb{R}^{(T)}$ and $\mathbb{R}^T$. A stochastic process $\{X(t)\}_{t\in T}$ is said to be separable in probability if there exists a countable set $T_0 \subset T$ with the property that for every $t \in T$ there exists $\{t_n\} \subset T_0$ such that $X(t_n) \to X(t)$ in probability as $n \to \infty$. Any continuous in probability stochastic process indexed by a separable metric space T is separable in probability. This includes all discrete time processes.

Theorem 3.2. *A separable in probability stochastic process $X = \{X(t)\}_{t\in T}$ is of type G if and only if there exist a nonnegative quadratic form β on $\mathbb{R}^{(T)}$, a σ–finite measure ξ on $\mathbb{R}^T$, and a continuous function $\psi : [0,\infty) \to [0,\infty)$ with completely monotone derivative on $(0,\infty)$, with $\psi(0) = \psi'(\infty) = 0$ and $\int_{\mathbb{R}^T} \psi(|yx|^2)\, \xi(dx) < \infty$ for every $y \in \mathbb{R}^{(T)}$, such that*

$$\mathbf{E}[\exp(yX)] = \exp\left\{-\frac{1}{2}\beta(y) - \int_{\mathbb{R}^T} \psi\left(|yx|^2\right)\, \xi(dx)\right\}, \quad y \in \mathbb{R}^{(T)}. \tag{3.1}$$

In particular, one can choose $\psi(u) = 1 - \exp(-u/2)$ with certain ξ.

Proof. By (3.1) and Theorem 2.6, finite dimensional distributions of the process X are of type G so that the process is of type G by the definition. Now we will prove the converse. We will use a result of Maruyama [M70] that there exist a nonnegative quadratic form β on $\mathbb{R}^{(T)}$ and a σ–finite symmetric measure ν on $\mathbb{R}^T$

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with $\int_{\mathbb{R}^T}(|x(t)|^2 \wedge 1)\,\nu(dx) < \infty$ for every $t \in T$, such that

$$\mathbf{E}[\exp(yX)] = \exp\left\{-\frac{1}{2}\beta(y) + \int_{\mathbb{R}^T} (\cos yx - 1)\ \nu(dx)\right\}, \quad y \in \mathbb{R}^{(T)}. \tag{3.2}$$

In fact, this result was proven in [M70] for continuous in probability infinitely divisible processes indexed by a separable metric space T but the same argument works under the assumption of the separability in probability. Let $\nu_{t_1,\dots,t_n}$ denote the projection of ν onto $\mathbb{R}^{t_1,\dots,t_n}$. Since X is of type G, there exists a unique symmetric Lévy measure $\nu_{0;t_1,\dots,t_n}$ on $\mathbb{R}^{t_1,\dots,t_n}$ such that

$$\nu_{t_1,\dots,t_n}(A) = \mathbf{E}\left[\nu_{0;t_1,\dots,t_n}(Z^{-1}A)\right], \quad A \in \mathcal{B}_0(\mathbb{R}^{t_1,\dots,t_n}). \tag{3.3}$$

Let $\mu_{0;t_1,\dots,t_n}$ be an infinitely divisible probability measure on $\mathbb{R}^{t_1,\dots,t_n}$ determined by

$$\widehat{\mu}_{0;t_1,\dots,t_n}(y) = \exp\left\{\int_{\mathbb{R}^{t_1,\dots,t_n}} (\cos yx - 1)\,\nu_{0;t_1,\dots,t_n}(dx)\right\}, \quad y \in \mathbb{R}^{t_1,\dots,t_n}.$$

The consistency of measures $\{\nu_{t_1,\dots,t_n}\}$, as the projections of ν, and the uniqueness of $\nu_{0;t_1,\dots,t_n}$ in (3.3) implies that $\{\nu_{0;t_1,\dots,t_n}\}$ are consistent, which in turn, yields the Kolmogorov's consistency of the family $\{\mu_{0;t_1,\dots,t_n}\}$. Therefore, there exists a symmetric infinitely divisible stochastic process $\{X_0(t)\}_{t\in T}$ with the family of finite dimensional distributions $\{\mu_{0;t_1,\dots,t_n}\}$. We will show that X_0 is also separable in probability. Let $T_0 \subset T$ a countable separant for X, and let $t \in T$. Choose $\{t_n\} \subset T_0$ such that $X(t_n) \to X(t)$ in probability as $n \to \infty$. Then

$$\begin{aligned} P(|Z| > 1) &\int_{\mathbb{R}^{t_n,t}} (1 \wedge |x(t_n) - x(t)|^2)\nu_{0;t_n,t}(dx) \\ &\le \int_{\mathbb{R}^{t_n,t}} (1 \wedge |x(t_n) - x(t)|^2)\nu_{t_n,t}(dx) \to 0, \end{aligned}$$

implying that $X_0(t_n) \to X_0(t)$ in probability as $n \to \infty$. Using the above quoted result of Maruyama, there exists a symmetric σ–finite measure ν_0 on $\mathbb{R}^T$ such that

$$\mathbf{E}[\exp(yX_0)] = \exp\left\{\int_{\mathbb{R}^T} (\cos yx - 1)\ \nu_0(dx)\right\}, \quad y \in \mathbb{R}^{(T)}.$$

The uniqueness of Lévy measures implies that the projection of ν_0 onto $\mathbb{R}^{t_1,\dots,t_n}$ coincides with $\nu_{0;t_1,\dots,t_n}$ on $\mathcal{B}_0(\mathbb{R}^{t_1,\dots,t_n})$. We have for every $y \in \mathbb{R}^{(T)}$ with the support $\{t_1,\dots,t_n\}$,

$$\begin{aligned}\int_{\mathbb{R}^T} (\cos yx - 1)\ \nu(dx) &= \int_{\mathbb{R}^{t_1,\dots,t_n}} (\cos yx - 1)\, \nu_{t_1,\dots,t_n}(dx) \\ &= \mathbf{E}\left[\int_{\mathbb{R}^{t_1,\dots,t_n}} (\cos Zyx - 1)\, \nu_{0;t_1,\dots,t_n}(dx)\right] \\ &= \int_{\mathbb{R}^T} \exp\left(-\frac{|yx|^2}{2} - 1\right)\ \nu_0(dx).\end{aligned}$$

Substituting this identity to (3.2) we get (3.1) with $\xi = \nu_0$ and $\psi(u) = 1 - \exp(-u/2)$. □

A symmetric stable process is a natural example of a type G process. It has the projection property: if all one dimensional projections yX, $y \in \mathbb{R}^{(T)}$, are symmetric stable random variables, then X is a symmetric stable process. Since one dimensional type G random variables have very simple structure, it is an important problem to determine whether type G processes have the projection property. Unfortunately, the answer is negative as it follows from the example constructed in the next section.

4. Non type G distributions with type G projections.

Let Q be a $k \times d$ matrix with $1 \le k \le d-1$. Define a lower dimensional projection of a probability measure μ on $\mathbb{R}^d$ by Q as

$$(Q\mu)(A) = \mu(\{\mathbf{x} \in \mathbb{R}^d : Q\mathbf{x} \in A\}), \quad A \in \mathcal{B}_0(\mathbb{R}^k).$$

If $\mu \in TG(\mathbb{R}^d)$, then it is easy to see that for any $k \times d$ matrix Q, $1 \le k \le d-1$, $Q\mu \in TG(\mathbb{R}^k)$. The following theorem says that the converse is not true.

Theorem 4.1. *Let $d \ge 2$. There is a probability measure μ on $\mathbb{R}^d$ satisfying the following three conditions.*

(i) $\mu \notin TG(\mathbb{R}^d)$,

(ii) $\mu \in I(\mathbb{R}^d)$,

(iii) for any $k \times d$ matrix Q with $1 \le k \le d-1$, $Q\mu \in TG(\mathbb{R}^k)$.

Proof. Let $D_1 = \{\mathbf{x} \in \mathbb{R}^d : 1 < |\mathbf{x}| < 2\}$ and $D_2 = \{\mathbf{x} \in \mathbb{R}^d : 0 < |\mathbf{x}| < 1\}$. Consider a signed measure

$$\nu_\varepsilon(A) = \lambda_d(A \cap D_1) - \varepsilon \lambda_d(A \cap D_2), \quad 0 < \varepsilon < 1,$$

where λ_d is the Lebesgue measure in $\mathbb{R}^d$. Linnik and Ostrovskii ([LO77], p.224) showed that for sufficiently small $\varepsilon > 0$, the function

$$\widehat{\mu}_\varepsilon(\mathbf{y}) = \exp\left\{\int_{\mathbb{R}^d \setminus \{0\}} (e^{i\mathbf{yx}} - 1)\nu_\varepsilon(d\mathbf{x})\right\}, \quad \mathbf{y} \in \mathbb{R}^d,$$

is the characteristic function of some probability measure μ_ε on $\mathbb{R}^d$ which is not infinitely divisible because ν_ε is not a measure. However, for any $k \times d$ matrix Q with $1 \le k \le d-1$, $Q\mu_\varepsilon$ is infinitely divisible. (They showed this fact only when Q is $1 \times d$ matrix, but Giné and Hahn [GH83] remarked that the same is true for any Q, $k \times d$ matrix, with $1 \le k \le d-1$.) We will need a much simpler fact that $Q\nu_\varepsilon$ is a measure for sufficiently small $\varepsilon > 0$ and every $k \times d$ matrix Q with $1 \le k \le d-1$. This fact is an obvious consequence of Linnik and Ostrovskii's result but it can also be verified directly.

We will first show that Gaussian randomization of ν_ε,

$$\nu(A) := \mathbf{E}[\nu_\varepsilon(Z^{-1}A)], \quad A \in \mathcal{B}_0(\mathbb{R}^k), \tag{4.1}$$

is a measure for sufficiently small $\varepsilon > 0$. We have

$$\begin{aligned}
\nu(A) &= \mathbf{E}[\lambda_d(Z^{-1}A \cap D_1) - \varepsilon\lambda_d(Z^{-1}A \cap D_2)] \\
&= \mathbf{E}\left[\int_{\mathbb{R}^d} \mathbf{1}_{Z^{-1}A}(\mathbf{x})\,(\mathbf{1}_{D_1}(\mathbf{x}) - \varepsilon\mathbf{1}_{D_2}(\mathbf{x}))\,d\mathbf{x}\right] \\
&= \mathbf{E}\left[\int_S \int_0^\infty E[\mathbf{1}_A(Zr\mathbf{y})]\,(\mathbf{1}_{D_1}(r\mathbf{y}) - \varepsilon\mathbf{1}_{D_2}(r\mathbf{y}))\,r^{d-1}dr f(\mathbf{y})d\mathbf{y}\right],
\end{aligned}$$

where $f(\mathbf{y})$ is a nonnegative function on S. Since ν_ε is rotation invariant, so is ν. Therefore, in order to prove that $\nu \ge 0$, it is enough to show that

$$\nu(\{r_1 < |\mathbf{x}| \le r_2\}) \ge 0 \tag{4.2}$$

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for $0 < r_1 < r_2 < \infty$. Also it is enough to show (4.2) for $0 < r_1 < r_2 < \infty$ with $r_2 \le 2r_1$. Indeed, if (4.2) holds for all such r_1, r_2, then for $r_2 > 2r_1$ we can find $m \in \mathbb{N}$ such that $2^m r_1 < r_2 \le 2^{m+1} r_1$. Hence

$$\nu(\{r_1 < |\mathbf{x}| \le r_2\}) = \sum_{n=1}^{m} \nu(\{2^{n-1} r_1 < |x| \le 2^n r_1\}) + \nu(\{2^m r_1 < |x| \le r_2\}) \ge 0.$$

Now suppose that $1 < r_2/r_1 \le 2$. Then we have

$$\nu(\{r_1 < |x| \le r_2\}) = 2c_d \int_0^\infty \{\Phi(\frac{r_2}{r}) - \Phi(\frac{r_1}{r})\}\{\mathbf{1}_{(1,2)}(r) - \varepsilon \mathbf{1}_{(0,1)}(r)\} r^{d-1} dr,$$

where c_d is the surface measure of the unit sphere of $\mathbb{R}^d$. Denote the last integral by I. By a change of variables $u = r_1/r$, we get

$$I = r_1^d \left[\int_{r_1/2}^{r_1} \{\Phi(\frac{r_2}{r_1} u) - \Phi(u)\} u^{-d-1} du \right.$$
$$\left. - \varepsilon \int_{r_1}^{\infty} \{\Phi(\frac{r_2}{r_1} u) - \Phi(u)\} u^{-d-1} du \right].$$

Put $K = r_2/r_1$, and recall that we are assuming $1 < K \le 2$. We have

$$\Phi(Ku) - \Phi(u) = \frac{1}{\sqrt{2\pi}} e^{-\gamma(u)^2 u^2/2} (K-1) u,$$

for some $\gamma(u)$ with $1 < \gamma(u) < K \le 2$. Hence

$$\begin{aligned}
\frac{\sqrt{2\pi}}{(K-1) r_1^d} I &= \int_{r_1/2}^{r_1} e^{-\gamma(u)^2 u^2/2} u^{-d} du - \varepsilon \int_{r_1}^{\infty} e^{-\gamma(u)^2 u^2/2} u^{-d} du \\
&\ge \int_{r_1/2}^{r_1} e^{-2u^2} u^{-d} du - \varepsilon \int_{r_1}^{\infty} e^{-u^2/2} u^{-d} du \\
&= \int_{r_1/2}^{r_1} e^{-2u^2} u^{-d} du - 2^{-d+1} \varepsilon \int_{r_1/2}^{\infty} e^{-2v^2} v^{-d} dv \\
&= (1 - 2^{-d+1}\varepsilon) \int_{r_1/2}^{r_1} e^{-2u^2} u^{-d} du - 2^{-d+1}\varepsilon \int_{r_1}^{\infty} e^{-2u^2} u^{-d} du.
\end{aligned}$$

Take $\varepsilon > 0$ such that $1 - 2^{-d+1}\varepsilon \ge 2^{-d+1}\varepsilon$. We have

$$\begin{aligned}
\frac{\sqrt{2\pi}}{(K-1) r_1^d} I &\ge 2^{-d+1} \varepsilon \left\{ \int_{r_1/2}^{r_1} e^{-2u^2} u^{-d} du - \int_{r_1}^{\infty} e^{-2u^2} u^{-d} du \right\} \\
&=: 2^{-d+1} \varepsilon J,
\end{aligned}$$

where

$$\begin{aligned} J &= \int_{r_1/2}^{r_1} e^{-2u^2} u^{-d} du - \int_{r_1}^{\infty} e^{-2u^2} u^{-d} du \\ &\geq e^{-2r_1^2} \left\{ \int_{r_1/2}^{r_1} u^{-d} du - \int_{r_1}^{\infty} u^{-d} du \right\} \\ &= \frac{1}{d-1} e^{-2r_1^2} r_1^{-d+1} (2^{d-1} - 2) \geq 0. \end{aligned}$$

This completes the proof of the claim in (4.1).

Fix now $\varepsilon > 0$ such that ν given by (4.1) is a measure and $Q\nu_\varepsilon$ is a measure for every $k \times d$ matrix Q with $1 \leq k \leq d-1$. Since ν is bounded, ν a Lévy measure of some infinitely divisible distribution μ on $\mathbb{R}^d$. $\mu \notin TG(\mathbb{R}^d)$ because (4.1) and Proposition 2.3. However, for any $k \times d$ matrix Q, $1 \leq k \leq d-1$, $Q\mu$ has Lévy measure $Q\nu$ given by

$$Q\nu(A) = \mathbf{E}[Q\nu_\varepsilon(Z^{-1}A)].$$

Thus $Q\nu$ is of the form (2.1), implying $Q\mu \in TG(\mathbb{R}^k)$. □

5. Conditionally Gaussian representations of type G random vectors.

Let $\mathbf{X}$ be a type G random vector in $\mathbb{R}^d$ with Gaussian component having covariance matrix Σ and with Lévy measure ν satisfying (2.1), where ν_0 is symmetric. Since a verification of formulas (1.5)–(1.6) is elementary and intuitive when $\nu_0(\mathbb{R}^d) < \infty$, we will first assume that it is the case. Let $\{\mathbf{Y}_j\}_{j\geq 1}$ be i.i.d. random vectors in $\mathbb{R}^d$ with common distribution $\nu_0/\nu_0(\mathbb{R}^d)$ and let $\{N(t)\}_{t\geq 0}$ be a Poisson process with rate $\nu_0(\mathbb{R}^d)$. Let $\{Z_j\}_{j\geq 1}$ be i.i.d. standard normal random variables. Finally, let $\{\mathbf{W}(t)\}_{t\geq 0}$ be a Lévy Gaussian process in $\mathbb{R}^d$ with covariance matrix Σ. Assume that $\{\mathbf{Y}_j\}_{j\geq 1}$, $\{N(t)\}_{t\geq 0}$, $\{Z_j\}_{j\geq 1}$, and $\{\mathbf{W}(t)\}_{t\geq 0}$ are independent of each other. It is elementary to check that $\mathbf{X}$ can be represented as

$$\mathbf{X} \stackrel{d}{=} \mathbf{W}(1) + \sum_{j=1}^{N(1)} Z_j \mathbf{Y}_j \tag{5.1}$$

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and the corresponding Lévy process of type G is given by

$$\mathbf{X}(t) = \mathbf{W}(t) + \sum_{j=1}^{N(t)} Z_j \mathbf{Y}_j.$$

Clearly the process $\{\mathbf{X}(t)\}$ is obtained by a Gaussian randomization, as described in the Introduction, of the process $\{\mathbf{V}_0(t)\}$ given by

$$\mathbf{V}_0(t) = \mathbf{W}(t) + \sum_{j=1}^{N(t)} \mathbf{Y}_j.$$

Substituting this to (1.5) yields

$$\mathbf{V} = \Sigma + \sum_{j=1}^{N(1)} \mathbf{Y}_j \mathbf{Y}_j^T$$

(use, e.g., Theorem 4.52 of Chapter I of [JS87]). Let $\mathbf{Z}$ be the standard Gaussian vector in $\mathbb{R}^d$ independent of $\mathbf{V}$. For any $\mathbf{x} \in \mathbb{R}^d$ we have

$$\begin{aligned}
\mathbf{E}\big[\exp(i\langle \mathbf{x}, \mathbf{V}^{1/2}\mathbf{Z}\rangle)\big] &= \mathbf{E}\big[\exp(-\frac{1}{2}\|\mathbf{V}^{1/2}\mathbf{x}\|^2)\big] = \mathbf{E}\big[\exp(-\frac{1}{2}\langle \mathbf{x}, \mathbf{V}\mathbf{x}\rangle)\big] \\
&= \exp(-\frac{1}{2}\langle \mathbf{x}, \Sigma\mathbf{x}\rangle)\mathbf{E}\left[\exp\left\{-\frac{1}{2}\sum_{j=1}^{N(1)}\langle \mathbf{x}, \mathbf{Y}_j\rangle^2\right\}\right] \\
&= \exp\left\{-\frac{1}{2}\langle \mathbf{x}, \Sigma\mathbf{x}\rangle + \int (e^{-\frac{1}{2}\langle \mathbf{x},\mathbf{y}\rangle^2} - 1)\,\nu_0(d\mathbf{y})\right\} \\
&= \exp\left\{-\frac{1}{2}\langle \mathbf{x}, \Sigma\mathbf{x}\rangle + \int (e^{i\langle \mathbf{x},\mathbf{y}\rangle} - 1)\,\nu(d\mathbf{y})\right\}.
\end{aligned}$$

This proves (1.6) in the case $\nu_0(\mathbb{R}^d) < \infty$.

Now we will remove the restriction of finiteness of ν_0. Let $\mathbf{X}$ be as above and let ν_0 be an arbitrary symmetric Lévy measure. Let ν_1 be a probability measure on $\mathbb{R}^d$ equivalent to ν_0. Define $g(x) = (d\nu_0/d\nu_1)(x)$. If $\nu_0(\mathbb{R}^d) < \infty$, we take $\nu_1 = \nu_0/\nu_0(\mathbb{R}^d)$ and $g(x) = \nu_0(\mathbb{R}^d)$. Let $\{\mathbf{Y}_j\}$ be i.i.d. random vectors in $\mathbb{R}^d$ with common distribution ν_1 and let $\{\Gamma_j\}$ be the sequence of arrival times in a Poisson process with rate 1. Assume that $\{\mathbf{Y}_j\}$, $\{\Gamma_j\}$, $\{Z_j\}$ and $\{\mathbf{W}(t)\}$ are independent of each other. Then

$$\mathbf{X} \stackrel{d}{=} \mathbf{W}(1) + \sum_{\{j:\ g(\mathbf{Y}_j) \geq \Gamma_j\}} Z_j \mathbf{Y}_j, \tag{5.2}$$

where the series converges a.s. We immediately see that when g is constant, (5.2) reduces to (5.1). The series in (5.2) is a special case of series studied in Rosiński [R90]. Formula (5.2) and further statements concerning random series of this type can be verified using Theorem 2.4 in [R90]. Simpler to use reference is Theorem 4.1 in Rosiński [R00]. Let now $\{U_j\}$ be i.i.d. uniform random variables independent of the previous random sequences and processes. Then

$$\mathbf{X}(t) = \mathbf{W}(t) + \sum_{\{j:\ g(\mathbf{Y}_j) \geq \Gamma_j\}} Z_j \mathbf{Y}_j I(U_j \leq t), \quad t \in [0, 1]$$

is a type G Lévy process such that $\mathbf{X}(1) \stackrel{d}{=} \mathbf{X}$. $\{\mathbf{X}(t)\}$ is obtained by a Gaussian randomization of $\{\mathbf{V}_0(t)\}$ given by

$$\mathbf{V}_0(t) = \mathbf{W}(t) + \sum_{\{j:\ g(\mathbf{Y}_j) \geq \Gamma_j\}} \mathbf{Y}_j I(U_j \leq t), \quad t \in [0, 1].$$

Consequently, (1.5) becomes

$$\mathbf{V} = \Sigma + \sum_{\{j:\ g(\mathbf{Y}_j) \geq \Gamma_j\}} \mathbf{Y}_j \mathbf{Y}_j^T.$$

and the verification of (1.6) goes along the same lines as above.

Another representation of type G random vectors is given in Rosiński [R91]. It is based on the spherical decomposition of ν_0. Using that representation in (1.5), instead of (5.2), one obtains a different representation of the random matrix $\mathbf{V}$ in (1.5). A representation given in this paper is more natural under the present setting because it reduces to the standard compound Poisson representation when ν_0 is finite.

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