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Spectra of Arithmetic Infinite Graphs and Their Application

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ABSTRACT. We discuss some properties of certain finite volume infinite graphs defined arithmetically, from spectral point of view. These graphs are constructed from principal congruence subgroups over function fields and known to be Ramanujan diagrams.

1. INTRODUCTION

The notation and an explicit construction of Ramanujan graphs have first appeared in [LPS] as the nice connection between the objects which created from the intrinsic interest in number theory and our motivation of constructing efficient communication networks in the real world. Let X be a k-regular connected graph with n vertices and $\lambda_0 = k > \lambda_1 \geq \cdots \geq \lambda_{n-1} \geq -k$ the eigenvalues of its adjacency operator, and we put $\lambda(X) = \max_{\lambda_i} |\lambda_i|$, where λ_i runs through all eigenvalues distinct from $\pm k$. Then the graph X is called a Ramanujan graph if $\lambda(X) \leq 2\sqrt{k-1}$. Roughly speaking, this is a regular finite graph with nontrivial eigenvalues small. We note that the number $2\sqrt{k-1}$ comes from the well-known result of Alon and Boppana about the lower bound of $\lambda(X)$ when n is large. Such small eigenvalue bound forces the graph X to have high magnification and small diameter, hence these graphs give good communication networks. Here the magnification of a graph, which is like Cheeger's constant of a Riemannian manifold, is defined so that it measures a speed of transmission of information. However, it is a difficult task to give an explicit example of Ramanujan graphs, especially when the size of graphs is large. There are three systematic methods, which are number-theoretic, to construct an infinite family of Ramanujan graphs explicitly (see [L2] for example).

Morgenstern [M1] introduced the notation of diagrams, which are finite volume graphs, and Ramanujan diagrams as a generalization of Ramanujan graphs. Moreover by the result of Drinfeld he showed in [M2] that the quotients graphs X_Γ given from principal congruence subgroups Γ of $PGL(2)$ over

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function fields over finite fields $\mathbb{F}_q$, described in Section 1, are Ramanujan diagrams.

In this survey paper we discuss some properties of these finite volume infinite graphs X_Γ from the viewpoint of spectral graph theory and number theory. Part of the material can be found in [N1] and [N2]. Section 2 provides the Selberg trace formula of Γ and a determinant expression of the Ihara-Selberg zeta function of X_Γ. They can be regarded as a generalization of the well-known works on finite graphs. In Section 3 we investigate the limit distributions of the eigenvalues of X_Γ. Finally in Section 4 we discuss the asymptotic distribution of the primitive hyperbolic conjugacy classes of Γ and its application to number theory. More investigation of related topics to Section 4 will appear elsewhere.

2. Bruhat-Tits trees and Ramanujan diagrams

Our aim in this section is to exhibit certain arithmetic infinite graphs mentioned in Introduction, which we will treat in this paper.

The Ramanujan graphs which appeared for the first time in [LPS], [Ma] and implicitly [Ih], are quotients of type $\Gamma\backslash PGL(2,F)/K$, where F is a local non-archimedian field, K a maximal compact subgroup and Γ a lattice, i.e. discrete subgroup of finite covolume in $PGL(2,F)$. In the case of $\mathrm{char}(F)=0$, it was shown by Ihara and (more generally) Tamagawa (cf. [Sel, II. 1. 5]) that every lattice is uniform, i.e. cocompact. On the other hand, when $\mathrm{char}(F)=p>0$, there are many non-uniform (i.e. finite covolume but not cocompact) lattices. These comprise not only arithmetic lattices, which are presented in [Sel, II] for example, but also non-arithmetic lattices [Lu1]. This is unlike the case of F-rank ≥ 2, which is the work of Margulis. In this paper we treat the following typical examples of arithmetic non-uniform lattices: principal congruence subgroups of $PGL(2,\mathbb{F}_q[t])$ in $G=PGL(2,\mathbb{F}_q((\frac{1}{t})))$ (see (1)). These discrete subgroups are seen as the counterpart of that of $PSL(2,\mathbb{Z})$ in $PGL(2,\mathbb{R})$ in view of the number-theoretic analogy between algebraic number fields and function fields over finite fields.

Let $\mathbb{F}_q$ be the finite field with q elements, $\mathbb{F}_q[t]$ the ring of polynomials in t over $\mathbb{F}_q$, and $k=\mathbb{F}_q(t)$ its quotient field. Let $k_\infty=\mathbb{F}_q((\frac{1}{t}))$ the field of Laurent formal power series in $\frac{1}{t}$ over $\mathbb{F}_q$, which is the completion of k with respect to the norm $|\cdot|_\infty$ at $\frac{1}{t}$ (infinity). If an element a in k_∞ is written as $\sum_{i=n}^{\infty} a_i t^{-i}\,(a_n\neq 0)$, then $|a|_\infty=q^{-n}$. We denote by r_∞ the ring of local integers $\mathbb{F}_q[[\frac{1}{t}]]$, whose elements can be written as Taylor series in $\frac{1}{t}$ over $\mathbb{F}_q$. Throughout this paper we put $G=PGL(2,k_\infty)$ and $K=PGL(2,r_\infty)$. Note that K is a maximal compact subgroup of G.

One of the symmetric spaces associated with G is the Bruhat-Tits building of G. This is in fact the right coset space G/K to which we add the structure as a $(q+1)$-regular tree $\mathcal{T}=\mathcal{T}_{q+1}$ (cf. [Se1, II.1.1] [St]). More precisely, this coset space is the set of vertices $V(\mathcal{T})$ of the tree $\mathcal{T}$, which are represented by the following set of matrices:

$$\left\{ \begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix} \in G \,\middle|\, n \in \mathbb{Z}, x \in k_\infty, x \bmod t^n r_\infty \right\},$$

and from the way of construction of $\mathcal{T}$ the neighbors of a vertex $gK\,(g\in G)$ are $q+1$ vertices $gs_iK\,(i=1,\cdots,q+1)$, where

$$\{s_1,\cdots,s_{q+1}\} = \left\{ \begin{pmatrix} \frac{1}{t} & \alpha \\ 0 & 1 \end{pmatrix} \,\middle|\, \alpha \in \mathbb{F}_q \right\} \cup \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence the group G acts on the tree $\mathcal{T}$ from the left as a group of automorphisms. This action induces an action of G on the boundary $\partial\mathcal{T}$ of $\mathcal{T}$, which is just the usual fractional linear transformation of G on $\mathbb{P}^1(k_\infty)$.

Let Γ be a principal congruence subgroup $\Gamma(A)\,(A\in\mathbb{F}_q[t])$ of $PGL(2,\mathbb{F}_q[t])$, which is defined by

(1)
$$\Gamma(A)=\{\gamma\in PGL(2,\mathbb{F}_q[t])\,|\,\text{for some representative } \tilde{\gamma} \text{ of } \gamma, \tilde{\gamma}\equiv I \pmod{A}\}.$$

The full modular group $\Gamma(1)=PGL(2,\mathbb{F}_q[t])$ is a non-uniform lattice in G (cf. [Se1, II.1.6]), so $\Gamma(A)$, which is of finite index in $\Gamma(1)$, is also a non-uniform lattice in G. If we let Γ act on the tree $\mathcal{T}$, noting that Γ has no inversions, it naturally gives rise to a quotient graph $\Gamma\backslash\mathcal{T}$ and this graph $\Gamma\backslash\mathcal{T}$ has the shape of the union of a finite graph together with finitely many infinite half lines (cf. [Se1, II.1.6] [M2]). In fact, we are lead to have a graph of groups $(\Gamma,\Gamma\backslash\mathcal{T})$, where we attach the stabilizer Γ_v to a vertex $[v]=\Gamma\cdot v\in V(\Gamma\backslash\mathcal{T})$, and the stabilizer Γ_e to an edge $[e]=\Gamma\cdot e\in E(\Gamma\backslash\mathcal{T})$ (more precisely, see [Se1, I.4, I.5]). Now this graph of groups $(\Gamma,\Gamma\backslash\mathcal{T})$ gives the structure of a $(q+1)$-regular diagram $(\Gamma\backslash\mathcal{T},w)$ under the weights $w([v])=|\Gamma_v|^{-1}$ and $w([e])=|\Gamma_e|^{-1}$ (for the definition of diagrams, see [M1]). We should note that the measure defined by this weight function w are just the atomic one on $V(\Gamma\backslash\mathcal{T})$ induced from the Haar measure m on G normalized so that $m(K)=1$.

In this paper we consider $\mathbb{C}$-valued functions defined on the set of vertices $V(\mathcal{T})$ of $\mathcal{T}$. If a function f on $V(\mathcal{T})$ satisfies $f(\gamma g)=f(g)$ for all $\gamma\in\Gamma$ and $g\in V(\mathcal{T})=G/K$, then f is called an automorphic function for Γ, namely, just functions on the quotient graph $\Gamma\backslash\mathcal{T}$. Next we define an operator for $\mathcal{T}$, which is called the adjacency operator, by

$$(Tf)(v)=\sum_{d(v,u)=1} f(u), \qquad f\colon V(\mathcal{T})\to\mathbb{C},$$

where d denotes the natural distance on $\mathcal{T}$, i.e. if u and v are adjacent in $\mathcal{T}$ we let $d(u,v) = 1$. It induces an operator for $\Gamma\backslash\mathcal{T}$ and this induced operator can be represented as

$$(Tf)(v) = \sum_{e=(v,u)\in E(\Gamma\backslash\mathcal{T})} \frac{w(e)}{w(v)} f(u), \qquad f\colon V(\Gamma\backslash\mathcal{T}) \to \mathbb{C},$$

where $e \doteq (v,u)$ denotes the edge from v to u. It is known that T is self-adjoint and $\|T\| \leq q+1$.

As described in Introduction, Morgenstern introduced the notation of Ramanujan diagram [M1], and by virtue of the function field analogue of the Ramanujan conjecture proved by Drinfeld [Dr], he deduced the following result:

Theorem 2.1. [M2, Theorem 2.1] *The diagrams* $(\Gamma(A)\backslash\mathcal{T}, w)$ $(A \in \mathbb{F}_q[t])$ *are Ramanujan diagrams.*

We note that this theorem has the implication that if λ is a nontrivial eigenvalue of T on $L^2(\Gamma(A)\backslash\mathcal{T}, w)$ (i.e. $\lambda \neq \pm(q+1)$), then $|\lambda| \leq 2\sqrt{q}$.

In [M2] an explicit structure of $\Gamma(A)\backslash\mathcal{T}$ are given and by Theorem 2.1 some finite subgraph of $\Gamma(A)\backslash\mathcal{T}$ is shown to be a good bounded concentrator, from which we get a family of superconcentrators of density 66.

3. Trace formulas and Ihara-Selberg zeta functions

In this section we discuss an explicit trace formula of $\Gamma = \Gamma(A)$ and express the Ihara-Selberg zeta function of Γ in a determinant form with T. From this section we let q be odd.

The quotient graph $\Gamma\backslash\mathcal{T}$ is an infinite graph, so that there will be continuous spectra of T. Let σ be the number of inequivalent cusps of $\Gamma\backslash\mathcal{T}$. It is known that the continuous spectra are furnished by the Eisenstein series $E_i(g,s)$ at each cusp κ_i $(i = 1,\cdots,\sigma)$ (see [L1] [N1] for the definition) and parametrized by the interval $[-2\sqrt{q}, 2\sqrt{q}]$. The Eisenstein series $E_i(g,s)$ is invariant under Γ, so it can be expanded as a Fourier series at each cusp κ_j. In the case of principal congruence subgroups $\Gamma(A)$, Li [L1] obtains an explicit form of the Fourier series in terms of the L-functions associated to characters χ on $\mathbb{F}_q[t]$ mod A.

The constant terms of the Fourier series of $E_i(g,s)$ at cusps κ_j define the $(\sigma\times\sigma)$-matrix $\Phi(s)$ which is called the scattering matrix of Γ. Then $\Phi(s)$ satisfies the functional equation $\Phi(s) = \Phi(1-s)$ [L1, Theorem 7]. We put $\varphi(s) = \det\Phi(s)$ and call it the scattering determinant of Γ. By the above computation [L1] we find that $\varphi(s)$ is a rational function in q^{2s}, so we put

$$\varphi(s) = c\frac{(q^{2s}-qa_1)(q^{2s}-qa_2)\cdots(q^{2s}-qa_m)}{(q^{2s}-qb_1)(q^{2s}-qb_2)\cdots(q^{2s}-qb_n)}, \tag{3}$$

where $c, a_i\,(i=1,\cdots,m), b_j\,(j=1,\cdots,n)$ are constants and we assume that the right hand side is written to be irreducible.

It is understood that the determinant $\varphi(s)$, in particular the set of its poles, plays an important role in the theory. We note that by the functional equation $\varphi(s)=\varphi(1-s)$ the set of poles of $\varphi(s)$ is one-to-one correspond to the set of its zeros. The function $\varphi(s)$ controls the Eisenstein series in the sense that if $\varphi(s)$ is analytic at some point, then so are all the Eisenstein series. Moreover $\varphi(s)$ appears in the Selberg trace formula described below as the contribution of the continuous spectra of T.

In [N2] we give more detailed information about the poles of $\varphi(s)$ for Γ by a slightly different computation from [L1] above.

Lemma 3.1. *Let* $A^\times=(\mathbb{F}_q[t]/A\mathbb{F}_q[t])^\times/\mathbb{F}_q^\times$ *and* $\hat{A}^\times$ *be the set of characters of* $A^\times$. *Let* $L(s,\chi)$ *be the Dirichlet L-function associated with a character* $\chi\in\hat{A}^\times$. *Then the poles of* $\varphi(s)$ *for* $\Gamma(A)$ *are contained within the zeros of the function*

$$(q^{2s}-q^2)\prod_{\chi\in\hat{A}^\times}L(2s,\chi)^h,$$

where $h=\sigma/r, r=\phi(A)/(q-1)$ *and* $\phi(A)$ *is the Euler totient function. Here we count zeros and poles with multiplicity.*

Next we set some notations to describe the trace formula of $\Gamma=\Gamma(A)$. For hyperbolic elements P of Γ we put $N(P)=\sup\{|\lambda_i|_\infty^2 \mid \lambda_i$ is an eigenvalue of the matrix $P\}$ and $\deg P=\log_q N(P)$. Recalling that any hyperbolic element P of Γ is written in the form $P=P_0^k\,(k\geq 1)$ with a primitive hyperbolic element P_0, we define the Mangoldt function $\Lambda(P)$ on hyperbolic elements of Γ by $\Lambda(P)=\deg P_0=\log_q N(P_0)$. Let $\mathfrak{P}_\Gamma$ (resp. $\mathfrak{P}'_\Gamma$) denote the set of primitive hyperbolic conjugacy classes (resp. hyperbolic conjugacy classes) of Γ. Let D be the set of discrete spectra of the adjacency operator T on $L^2(\Gamma\backslash\mathcal{T},w)$, which is a finite set. Denote the number of elements of D by $|D|$. Then the trace formula of Γ is given explicitly in the following [N1]:

Theorem 3.1. *For a discrete spectrum* $\lambda_j\in D$ *of* T *on* $L^2(\Gamma\backslash\mathcal{T},w)$, *we set* $\lambda_j=q^{s_j}+q^{1-s_j}$ *and* $s_j=1/2+ir_j$. *Assume that the sequence* $c(n)\in\mathbb{C}\,(n\in\mathbb{Z})$

satisfies $c(n) = c(-n)$ *and* $\sum_{n\in\mathbb{Z}} q^{\frac{|n|}{2}}|c(n)| < \infty$. *Then if* $\Gamma = \Gamma(1)$, *we have*

$$(4)\qquad h\left(\frac{i}{2}\right) + h\left(\frac{i}{2} + \frac{\pi}{\log q}\right) = \mathrm{vol}(\Gamma\backslash\mathcal{T})k(0)$$
$$+\left(-\frac{q}{q^2-1}k(0) + \frac{q}{2(q-1)}c(0)\right) + \sum_{\{P_0\}\in\mathfrak{P}_\Gamma}\sum_{l=1}^{\infty}\frac{\deg P_0}{q^{\frac{l\deg P_0}{2}}}c(l\deg P_0)$$
$$+2\sum_{m=1}^{\infty}(q^{-m}+1)c(2m) + \frac{q-2}{(q-1)^2}k(0) + \frac{3q-4}{2(q-1)}c(0),$$

and if $\Gamma = \Gamma(A)\,(a = \deg A \geq 1)$,

$$(5)\qquad \sum_{j=1}^{|D|} h(r_j) = \mathrm{vol}(\Gamma\backslash\mathcal{T})k(0) + \sum_{\{P_0\}\in\mathfrak{P}_\Gamma}\sum_{l=1}^{\infty}\frac{\deg P_0}{q^{\frac{l\deg P_0}{2}}}c(l\deg P_0)$$
$$+\left(\sigma - \mathrm{Tr}\,\Phi\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}c(0) + \sum_{m=1}^{\infty}c(2m)\right)$$
$$+\frac{1}{4\pi}\int_{-\frac{\pi}{\log q}}^{\frac{\pi}{\log q}} h(r)\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)dr - \sigma\left(a + \frac{1}{q-1}\right)c(0).$$

Here the functions $h(\cdot)$ *and* $k(\cdot)$ *are determined by* $c(\cdot)$ *via the Selberg transform.*

In the case of $\Gamma = \Gamma(1)$, Efrat [E3] computes the spectral decomposition of $L^2(\Gamma\backslash\mathcal{T}, w)$. The set of discrete spectra of T for $\Gamma(1)$ consists of two trivial eigenvalues $\pm(q+1)$. We also remark that the study of trace formulas was done by Akagawa [Ak] in the case of the full modular group $\Gamma = \Gamma(1)$, but his direct method is not applicable for other lattices Γ. Modifying his computation and using the structure of a $(q+1)$-regular tree $\mathcal{T}$, we obtain an explicit trace formula for $\Gamma(A)$ more naturally and plainly. Our computation seems to be similar to that of the case of $PSL(2,\mathbb{R})$ [He] and applicable for generic Γ's.

The Ihara-Selberg zeta function of Γ is defined by

$$\zeta_\Gamma(s) = \prod_{\{P_0\}\in\mathfrak{P}_\Gamma}\left(1 - N(P)^{-s}\right)^{-1}.$$

Here we put

$$(7)\qquad N_m = \sum_{\substack{\{P\}\in\mathfrak{P}'_\Gamma \\ \deg P = m}} \Lambda(P) = \sum_{\substack{\{P_0\}\in\mathfrak{P}_\Gamma \\ \deg P_0 | m}} \deg P_0, \qquad m \geq 1,$$

then the zeta function $\zeta_\Gamma(s)$ can also be written as

$$(8)\qquad \zeta_\Gamma(u) = \exp\left(\sum_{m=1}^{\infty}\frac{N_m}{m}u^m\right),$$

where we put $u = q^{-s}$.

Later we will write N_m in two different forms and these formulas are keys to the proofs of the results in Section 3 and Section 4.

We apply Theorem 3.1 to the test function $c(n)$:

$$c(n) = \begin{cases} -(\log q)\, q^{-|n|(s-\frac{1}{2})} & n \neq 0 \\ 0 & n = 0, \end{cases}$$

where $s \in \mathbb{C}$ is fixed with $\mathrm{Re}(s) > 1$, then this gives the following determinant expression with T of the Ihara-Selberg zeta function $\zeta_\Gamma(u)$ [N1]:

Theorem 3.2. *For $\Gamma = \Gamma(1)$ we have*

$$\zeta_\Gamma(u) = \frac{1 - qu^2}{1 - q^2u^2}, \tag{9}$$

and for $\Gamma = \Gamma(A)$ $(\deg A \geq 1)$

$$\begin{aligned} \zeta_\Gamma(u) = (1-u^2)^{-\chi}(1-qu^2)^{\rho} \prod_{\lambda \in D} (1 - \lambda u + qu^2)^{-1} \\ \times \prod_{|b_j|<1} (1 - qb_j u^2)^{-1} \cdot \prod_{|b_j|>1} (1 - qb_j^{-1} u^2), \end{aligned} \tag{10}$$

where $\chi = \mathrm{vol}(\Gamma\backslash\mathcal{T})\frac{q-1}{2}, \rho = \frac{1}{2}\left(\sigma - \mathrm{Tr}\,\Phi(\frac{1}{2})\right)$ and b_j's are as in (3).

We here should mention the related result of Scheja [Scl]. By combinatorial argument, he obtained a determinant expression of the Ihara-Selberg zeta function $\zeta_{\Gamma(A)}(s)$ which is different from Theorem 3.2. This expression is given by certain deformation $\Delta^*(u)$ of the operator $\Delta(u) = 1 - Tu + qu^2$ on $V(\Gamma\backslash\mathcal{T})$ in Theorem 3.2, where we let $\Delta^*(u)$ act on the finite part of $\Gamma\backslash\mathcal{T}$ (i.e. removing the ends).

4. Asymptotic Distribution of Eigenvalues

As described in Section 2, Morgenstern [M2] shows that any nontrivial discrete spectra λ (i.e. $\lambda \neq \pm(q+1)$) of T on $L^2(\Gamma(A)\backslash\mathcal{T}, w)$ satisfies $|\lambda| \leq 2\sqrt{q}$, and moreover gives an explicit structure of $\Gamma(A)\backslash\mathcal{T}$. However, it is difficult to compute these eigenvalues concretely. In this section we will find certain asymptotic distribution of them.

Now we set a normalized operator $T' = T/\sqrt{q}$ and let D' be the set of nontrivial discrete spectra of T', then every element λ' of D' satisfies $|\lambda'| \leq 2$. We prepare two probability measures on $\Omega = [-2, 2]$. One is the Sato-Tate measure or Wigner semi-circle:

$$\mu_\infty(x) = \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}dx.$$

The other is defined for a real number $q\,(>1)$ by

$$\mu_q(x) = \frac{q+1}{(q^{1/2}+q^{-1/2})^2 - x^2}\,\mu_\infty(x).$$

The measure $\mu_q(x)$ is known to be the spectral measure of T on a $(q+1)$-regular tree $\mathcal{T}$. These measures can also be found in number theory (for example, see [KS1] [Sa1] [Se2]).

The Chebychev polynomials of the second kind $X_m(x)\,(m = 0,1,2,\cdots)$, which are defined by

$$X_m(x) = \frac{\sin((m+1)\theta)}{\sin\theta}$$

when $x = 2\cos\theta\,(0 \le \theta \le \pi)$, are known to be orthogonal with respect to $\mu_\infty(x)$. We have

$$\int_\Omega X_m(x)d\mu_\infty(x) = \begin{cases} 1 & (m=0) \\ 0 & (m>0) \end{cases}$$

and these polynomials have the following identity:

$$\sum_{m=0}^{\infty} X_m(x)u^m = \frac{1}{1-xu+u^2},$$

where u is an indeterminate. Next let us define the polynomials $X_{m,q}(x)\,(m = 0,1,2,\cdots)$ by the relation:

$$X_{m,q}(x) := X_m(x) - q^{-1}X_{m-2}(x),$$

where we set $X_m(x) := 0$ for $m < 0$. Then we can check that

$$\int_\Omega X_{m,q}(x)d\mu_q(x) = \begin{cases} 1 & (m=0) \\ 0 & (m>0) \end{cases} \tag{12}$$

and

$$\sum_{m=0}^{\infty} X_{m,q}(x)u^m = \frac{1-u^2/q}{1-xu+u^2}. \tag{13}$$

On the other hand, generalizing T, we define the operator $T_m\,(m = 0,1,2,\cdots)$, which average functions on $V(\mathcal{T})$ at distance m:

$$(T_m f)(v) = \sum_{d(v,u)=m} f(u), \qquad f\colon V(\mathcal{T}) \to \mathbb{C}$$

Note that $T_0 = I =$ identity and $T_1 = T$. Then we have the recursive relations:

$$T_1^2 = T_2 + (q+1)T_0,$$
$$T_1T_m = T_{m+1} + qT_{m-1}, \qquad m \ge 2,$$

from which we can show the following identity:

$$\sum_{m=0}^{\infty} T_m u^m = \frac{1-u^2}{1-Tu+qu^2}, \tag{14}$$

$$\sum_{m=0}^{\infty} T'_m u^m = \frac{1-u^2/q}{1-T'u+u^2}, \tag{15}$$

where we put $T'_m = T_m/q^{\frac{m}{2}}$. Hence the relations (13) and (15) yield

$$\begin{aligned} T'_m &= X_{m,q}(T'), \\ T_m &= q^{\frac{m}{2}} X_{m,q}(T/q^{\frac{1}{2}}). \end{aligned} \tag{16}$$

Under the above preparation, we can establish the following result [N2].

Theorem 4.1. *(i) Let q be fixed. Then for any sequence of polynomials $\{A_i\}$ $(i = 1,2,\cdots; A_i \in \mathbb{F}_q[t])$ such that $\deg A_i \to \infty$ as $i \to \infty$, the nontrivial discrete spectra D'_i of $T' = T/\sqrt{q}$ for $\Gamma(A_i)\backslash\mathcal{T}_{q+1}$ are equidistributed with respect to the measure $\mu_q(x)$ on $\Omega = [-2,2]$. That is, let $C(\Omega)$ be the space of continuous functions on Ω, then for any $f(x) \in C(\Omega)$ the following holds:*

$$\lim_{i\to\infty} \frac{1}{|D'_i|} \sum_{\lambda \in D'_i} f(\lambda) = \int_\Omega f(x)\,d\mu_q(x). \tag{17}$$

(ii) For any sequence of couples $\{q_i, A_i\}$ $(i = 1,2,\cdots; A_i \in \mathbb{F}_{q_i}[t])$ such that $q_i \to \infty$ and $\deg A_i \to \infty$ as $i \to \infty$, the nontrivial discrete spectra D'_i of $T' = T/\sqrt{q_i}$ for $\Gamma(A_i)\backslash\mathcal{T}_{q_i+1}$ are equidistributed with respect to the measure $\mu_\infty(x)$ on $\Omega = [-2,2]$. That is, for any $f(x) \in C(\Omega)$ the following holds:

$$\lim_{i\to\infty} \frac{1}{|D'_i|} \sum_{\lambda \in D'_i} f(\lambda) = \int_\Omega f(x)\,d\mu_\infty(x). \tag{18}$$

Sketch of the proof. We will give the proof of (i). By the same idea we can also prove (ii). First note that the space spanned by the set of polynomials $\{X_{m,q}(x)\}$ $(m = 0,1,2,\cdots)$ is dense in $C(\Omega)$, so it suffices to check that $f(x) = X_{m,q}(x)$ satisfies (17) for each m.

If we set $f(x) = X_{m,q}(x)$ and denote by $\mathrm{Tr}\,T_m$ the sum of the nontrivial eigenvalues of T_m for $\Gamma(A)$, then $\sum_{\lambda\in D'} f(\lambda) = q^{-\frac{m}{2}}\mathrm{Tr}\,T_m$ by (16). Taking the logarithmic derivative of (10) and (8) in u, we obtain the formula connecting $\mathrm{Tr}\,T_m$ with N_m's by using (14). This formula contains the terms of the contribution of the scattering determinant $\varphi(s)$. By Lemma 3.1 it is possible to estimate these terms.

As we let $\deg A \to \infty$ for principal congruence subgroups $\Gamma(A)$, we see that $N_m \to 0$ for each m, and by the trace formula (5) and Lemma 3.1, we obtain

$|D'| \sim \mathrm{vol}(\Gamma(A)\backslash\mathcal{T})$. Combing these facts, we can obtain that for each $m \neq 0$, $\frac{1}{|D_i'|}\sum_{\lambda\in D_i'} f(\lambda) \to 0$ as $i \to \infty$. This and (12) complete the proof. □

5. Prime Geodesic Theorems

In this section we exhibit the distribution of the primitive hyperbolic conjugacy classes of $\Gamma(A)$ and its application to a number-theoretic problem about the fundamental units of orders of real quadratic function fields.

First we recall a number-theoretic interpretation of the primitive hyperbolic conjugacy classes $\mathfrak{P}_\Gamma$ of $\Gamma = \Gamma(1) = PGL(2, \mathbb{F}_q[t])$ (cf. [Ak]), which is an analogue of the case of $PSL(2,\mathbb{Z})$. As before let q be an odd prime power and we denote by $\mathcal{D}$ the subset of $\mathbb{F}_q[t]$ consisting of monic and square-free polynomials of even degree and by $\mathcal{L}$ the subset of $\mathbb{F}_q[t]$ consisting of monic polynomials. The mapping $d \mapsto k(\sqrt{d})$ establishes a one-to-one correspondence between $\mathcal{D}$ and the set of real quadratic function fields. Let $\omega = x + y\sqrt{d}\,(x, y \in k)$ be a quadratic irrational function (i.e. not an element of k) and satisfy the quadratic equation:

$$C\omega^2 - B\omega + A = 0, \qquad A, B, C \in \mathbb{F}_q[t].$$

If we require g.c.d $(A, B, C) = 1$ and that the coefficient of the highest power in t of $2Cy$ is 1, then the polynomials A, B, C are uniquely determined, so we write $\omega = \{A, B, C\}$. Then the discriminant $D(\omega)$ of ω is defined by $D(\omega) = B^2 - 4AC = 4C^2y^2d$. If two quadratic irrational functions ω_1, ω_2 are equivalent by the following equivalence relation: $\omega_2 = \frac{a\omega_1+b}{c\omega_1+d}$ with a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, then they have the same discriminant and we say that they are Γ-equivalent.

For $d \in \mathcal{D}$ and $l \in \mathcal{L}$ we put $\mathcal{O}_{l\sqrt{d}} = \mathbb{F}_q[t] + \mathbb{F}_q[t]l\sqrt{d}$, which is an order in $k(\sqrt{d})$. Let $h_{l\sqrt{d}}$ be the narrow class number of $\mathcal{O}_{l\sqrt{d}}$ in $k(\sqrt{d})$ and $e_{l\sqrt{d}} = t_0 + u_0 l\sqrt{d}$ be a fundamental unit of $\mathcal{O}_{l\sqrt{d}}$ with $|e_{l\sqrt{d}}|_\infty > 1$. For every real quadratic irrational function $\omega = \{A, B, C\}$ of discriminant dl^2 we put

$$P_0(\omega) = \begin{pmatrix} (t_0 + Bu_0)/2 & -Au_0 \\ Cu_0 & (t_0 - Bu_0)/2 \end{pmatrix},$$

then $N(P_0(\omega)) = |e_{l\sqrt{d}}|_\infty^2$. For each $d \in \mathcal{D}$ and $l \in \mathcal{L}$, we denote by $R(dl^2)$ a complete set of representative of Γ-equivalence classes of the real quadratic irrational functions of discriminant dl^2. We note that the number of elements of $R(dl^2)$ is equal to $h_{l\sqrt{d}}$. Then a complete set of representatives of primitive hyperbolic conjugacy classes of Γ is given by the following set:

$$\bigcup_{d\in\mathcal{D}} \bigcup_{l\in\mathcal{L}} \bigcup_{\omega\in R(dl^2)} \{P_0(\omega)\}.$$

Before describing our result, we review the *Prime Geodesic Theorem* in the case of $\Gamma \subset PSL(2,\mathbb{R})$, where we concentrate on congruence subgroups Γ of $PSL(2,\mathbb{Z})$. Recall that for Γ the Selberg zeta function $Z_\Gamma(s)$ is defined by

$$Z_\Gamma(s) = \prod_{\{P_0\}\in\mathfrak{P}_\Gamma} \prod_{n=0}^{\infty} \left(1 - N(P_0)^{-(s+n)}\right),$$

and these primitive conjugacy classes $\{P_0\} \in \mathfrak{P}_\Gamma$ correspond to oriented prime closed geodesics on $\Gamma\backslash\mathbb{H}$ ($\mathbb{H}$is the upper half plane) whose lengths are $\log N(P_0)$. In the Prime Geodesic Theorem (PGT), which is about counting closed geodesics and is the geometric counterpart of the Prime Number Theorem (PNT), the Selberg zeta function plays a similar role to the Riemann zeta function $\zeta(s)$ in the PNT. However, there are also differences between them.

The Riemann Hypothesis of the Riemann zeta function implies (and is implies by) that the remainder term in the PNT is $O(x^{\frac{1}{2}}\log x)$. In the case of PGT, if let $\lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \cdots$ be the eigenvalues of the Laplacian Δ on $L^2(\Gamma\backslash\mathbb{H})$, we have that for the full modular group $\Gamma(1) = PSL(2,\mathbb{Z})$ it holds $\lambda_1 > \frac{1}{4}$, so that via the trace formula the Selberg zeta function $Z_{\Gamma(1)}(s)$ satisfies the *Riemann Hypothesis*, i.e. $Z_{\Gamma(1)}(s)$ has no zeros in $\mathrm{Re}(s) > \frac{1}{2}, s \ne 1$. Nevertheless too many zeros of $Z_{\Gamma(1)}(s)$ on the line $\mathrm{Re}(s) = \frac{1}{2}$, unlike $\zeta(s)$, makes it difficult to estimate the remainder term of $\pi_{\Gamma(1)}(x)$, where we put

$$\pi_\Gamma(x) = |\{\text{distinct}\,\{P_0\} \in \mathfrak{P}_\Gamma \mid N(P_0) \le x\}|. \tag{19}$$

The best known result about estimating this term is the following Prime Geodesic Theorem:

Theorem 5.1. [LRS] *Let Γ be a congruence subgroup of $PSL(2,\mathbb{Z})$. Then it holds that*

$$\pi_\Gamma(x) = \mathrm{Li}(x) + O_\varepsilon(x^{\frac{7}{10}+\varepsilon}), \qquad \varepsilon > 0, \tag{20}$$

where $\mathrm{Li}(x) = \int_e^x \frac{dt}{\log t} = \int_1^{\log x} \frac{e^t}{t} dt$.

In the course of *Arithmetic Quantum Chaos*, Luo and Sarnak [LS] established the mean Lindelöf Hypothesis in the λ_j-aspect of the Rankin-Selberg L-functions attached to L^2-eigenfunctions ϕ_j (Maass cusp forms) of Δ for Γ, and then using Kuznetsov's trace formula and Weil's bound for Kloosterman sums, they deduced the result (20) for the full modular group $PSL(2,\mathbb{Z})$ first. For any congruence subgroup Γ, Luo, Rudnick and Sarnak [LRS] yielded a new lower bound on λ_1 and gave the remainder term in (20) by coincidence.

For a number-theoretic interpretation of Theorem 5.1, see [Sa1]. We point out that it is naturally expected as in the case of the PNT that

$$\pi_\Gamma(x) = \mathrm{Li}(x) + O_\varepsilon(x^{\frac{1}{2}+\varepsilon}) \tag{21}$$

for any $\varepsilon > 0$, but the abundance of the eigenvalues puts this bound completely out of reach.

We have arrived at describing our result. If we put

$$\mathrm{Li}_q(x) = 2 \sum_{\substack{1 \le m \le \log_q x \\ m:\mathrm{even}}} \frac{q^m}{m},$$

which is an analog of $\mathrm{Li}(x)$ in Theorem 5.1, then in the case of function fields we have the following result, which should be compared with (21):

Theorem 5.2. *Let Γ be a principal congruence subgroup $\Gamma(A)$ $(A \in \mathbb{F}_q[t])$ of $PGL(2, \mathbb{F}_q[t])$ and $\pi_\Gamma(x)$ be defined as in (19). Then we have*

$$\pi_\Gamma(x) = \mathrm{Li}_q(x) + O\left(\frac{x^{\frac{1}{2}}}{\log x}\right). \tag{22}$$

By a number-theoretic interpretation of $\mathfrak{P}_{\Gamma(1)}$ described first, this theorem deduces the asymptotic formula of certain distribution of fundamental units and class numbers in real quadratic fields over k:

Corollary 5.1.

$$\sum_{|\epsilon_{t\sqrt{d}}|_\infty^2 \le x} h_{t\sqrt{d}} = \mathrm{Li}_q(x) + O\left(\frac{x^{\frac{1}{2}}}{\log x}\right). \tag{23}$$

We remark that the remainder term of size $O(x^{\frac{1}{2}}/\log x)$ in Theorem 5.2 is just what we expect from the theory of primes (cf. [In] for example). We will give a sketch of the proof of Theorem 5.2. Here we use the ordinary method in analytic number theory. For other investigation, see the forthcoming paper.

Our proof is based on the two facts. One is that the graph $\Gamma\backslash\mathcal{T}$ is a Ramanujan diagram (Theorem 2.1), which concludes that the Ihara-Selberg zeta function $\zeta_\Gamma(s)$ satisfies the *Riemann Hypothesis*, i.e. $\zeta_\Gamma(s)^{-1}$ has no zeros in $\frac{1}{2} < \mathrm{Re}(s) < 1$. The other is that (an integral power of) $\zeta_\Gamma(s)$ is a rational function (Theorem 3.2), i.e. the zeros of $\zeta_\Gamma(s)$ is essentially a finite set. This implies that the role of the Ihara-Selberg zeta function $\zeta_\Gamma(s)$ in the proof of Theorem 5.2 is more similar to that of $\zeta(s)$ in the PNT rather than that of the Selberg zeta function $Z_\Gamma(s)$, which has too many zeros as described above, in the PGT. However, we note that the number of those essential zeros of $\zeta_\Gamma(s)$ seems to be much more than that of a congruence zeta function, like the relation between the Selberg zeta function and the Riemann zeta function (cf. [N2]).

Proof. First we prepare some functions. Recall that we denote by $\mathfrak{P}_\Gamma$ (resp. $\mathfrak{P}'_\Gamma$) the primitive hyperbolic conjugacy classes (resp. the hyperbolic conjugacy classes) of Γ and define the Mangoldt function on hyperbolic elements P of

Γ by $\Lambda(P) = \log_q N(P_0) = \deg P_0$ when $P = P_0^k$ with a primitive hyperbolic element P_0 and $k \geq 1$. We define Chebyshev's ψ-function and ϑ-function for Γ by

$$(24) \qquad \psi(x) = \sum_{\substack{\{P\}\in\mathfrak{P}'_\Gamma \\ N(P)\leq x}} \Lambda(P) = \sum_{\substack{\{P\}\in\mathfrak{P}'_\Gamma \\ \deg P\leq \log_q x}} \deg P_0 = \sum_{1\leq m\leq \log_q x} N_m,$$

$$(25) \qquad \vartheta(x) = \sum_{\substack{\{P_0\}\in\mathfrak{P}_\Gamma \\ N(P_0)\leq x}} \deg P_0,$$

respectively.

Next it follows from definition that for odd m, $N_m = 0$. By (8) we have

$$u\frac{\zeta'_\Gamma(u)}{\zeta_\Gamma(u)} = \sum_{m=1}^{\infty} N_m u^m, \qquad |u| < q^{-1},$$

which gives

$$N_m = \frac{1}{2\pi i}\int_{|u|=\ell_1} \frac{\zeta'_\Gamma(u)}{\zeta_\Gamma(u)} u^{-m}\, du,$$

where $0 < \ell_1 < q^{-1}$. Now we shift the path of integration to the circle $|u| = \ell_2\,(1 < \ell_2)$. Then by using Cauchy's theorem of residues, Theorem 3.2 and Lemma 3.1 and noting that $\int_{|u|=\ell_2} \frac{\zeta'_\Gamma(u)}{\zeta_\Gamma(u)} u^{-m} du \to 0$ as $\ell_2 \to \infty$ since $\frac{\zeta'_\Gamma(u)}{\zeta_\Gamma(u)} = O(1/u)$ for $|u| \geq \ell\,(\ell > 1)$, we obtain the following formula: Let m is even. Then if $\Gamma = \Gamma(1)$ we have

$$N_m = 2(q^m - q^{\frac{m}{2}}),$$

and if $\Gamma = \Gamma(A)\,(\deg A \geq 1)$,

$$N_m = 2q^m + q^{\frac{m}{2}} \sum_r (q^{imr} + q^{-imr}) + (q-1)\mathrm{vol}(\Gamma\backslash\mathcal{T}) - \left(\sigma - \mathrm{Tr}\,\Phi\left(\frac{1}{2}\right)\right) q^{\frac{m}{2}} + 2q^{\frac{m}{2}} \sum_{|b_j|<1} b_j^{\frac{m}{2}},$$

where the sum $\sum_r$ is taken over nontrivial eigenvalues $\lambda = q^s + q^{1-s}\,(s = \frac{1}{2} + ir)$. Hence for even m,

$$(26) \qquad N_m = 2q^m + O(q^{\frac{m}{2}})$$

since $\Gamma\backslash\mathcal{T}$ is a Ramanujan diagram.

Substituting this result into (24), we have

$$\psi(x) = 2 \sum_{\substack{1 \le m \le \log_q x \\ m:\text{even}}} q^m + O\left(\sum_{\substack{1 \le m \le \log_q x \\ m:\text{even}}} q^{\frac{m}{2}} \right)$$

$$= 2 \sum_{\substack{1 \le m \le \log_q x \\ m:\text{even}}} q^m + O(x^{\frac{1}{2}}). \tag{27}$$

On the other hand, it follows from (26) that $\vartheta(x) = O(x \log^2 x)$. Since it holds

$$\psi(x) = \vartheta(x) + \vartheta(x^{\frac{1}{2}}) + \sum_{n=3}^{\infty} \vartheta(x^{\frac{1}{n}}) \tag{28}$$

by (24) and (25), we have $\psi(x) = \vartheta(x) + O(x^{\frac{1}{2}} \log^2 x)$. This and (27) yield

$$\vartheta(x) = 2 \sum_{\substack{1 \le m \le \log_q x \\ m:\text{even}}} q^m + O(x^{\frac{1}{2}} \log^2 x),$$

from which we have $\vartheta(x) = O(x)$. Using (28) again, we obtain

$$\psi(x) = \vartheta(x) + O(x^{\frac{1}{2}}). \tag{29}$$

Therefore by (27) it follows that

$$\vartheta(x) = 2 \sum_{\substack{1 \le m \le \log_q x \\ m:\text{even}}} q^m + O(x^{\frac{1}{2}}). \tag{30}$$

Now by definition we have

$$\pi_\Gamma(x) = \sum_{\substack{\{P_0\} \in \mathfrak{P}_\Gamma \\ N(P_0) \le x}} 1 = \int_1^x \frac{1}{\log_q t} d\vartheta(t)$$

$$= \frac{\vartheta(x)}{\log_q x} + \frac{1}{\log q} \int_q^x \frac{\vartheta(t)}{t \log_q^2 t} dt. \tag{31}$$

If we put

$$C(u) = 2 \sum_{\substack{1 \le m \le u \\ m:\text{even}}} q^m,$$

then it holds by an integration by parts that

$$2 \sum_{\substack{1 \le m \le \log_q x \\ m:\text{even}}} \frac{q^m}{m} = \frac{C(\log_q x)}{\log_q x} + \int_1^{\log_q x} \frac{C(u)}{u^2} du$$

$$= \frac{C(\log_q x)}{\log_q x} + \frac{1}{\log q} \int_q^x \frac{C(\log_q t)}{t \log_q^2 t} dt. \tag{32}$$

Combining (30), (31) and (32), we finally obtain that

$$\begin{aligned}\pi_\Gamma(x) &= \frac{C(\log_q x)}{\log_q x} + O\left(\frac{x^{\frac{1}{2}}}{\log_q x}\right) + \frac{1}{\log q}\int_q^x \frac{C(\log_q t)}{t\log_q^2 t}dt \\ &+ O\left(\int_q^x \frac{t^{\frac{1}{2}}}{t\log_q^2 t}dt\right) \\ &= 2\sum_{\substack{1\le m\le \log_q x \\ m:\text{even}}} \frac{q^m}{m} + O\left(\frac{x^{\frac{1}{2}}}{\log x}\right) + O\left(\frac{x^{\frac{1}{2}}}{\log^2 x}\right) \\ &= \mathrm{Li}_q(x) + O\left(\frac{x^{\frac{1}{2}}}{\log x}\right).\end{aligned}$$

□

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