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The Distribution of Eigenvalues of Arithmetic Infinite Graphs

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ABSTRACT. Let Γ be a principal congruence subgroups of $PGL(2, \mathbb{F}_q[t])$, then it is known that $\Gamma\backslash X$ is an infinite Ramanujan diagram, where X is the q+1-regular tress. We obtain the limit distributions of eigenvalues of the adjacency operator for $\Gamma\backslash X$. The spectral measure of X and the Sato-Tate measure appear as the limit distributions.

1. INTRODUCTION

Ramanujan graphs are defined as k-regular finite graphs whose nontrivial eigenvalues of adjacency operator have absolute values bounded by $2\sqrt{k-1}$. Such eigenvalue bound forces graphs to have high magnifications and small diameters, hence these graphs give good communication networks. They also have important applications in computer science. However, it is not easy to determine whether a given graph is Ramanujan, especially when the size of a graph is large. So it is a difficult task to give explicit constructions of an infinite family of Ramanujan graphs whose sizes increase. The first construction of an infinite family of Ramanujan graphs were given by Lubotzky, Phillips and Sarnak [LPS] and Margulis [Ma]. It is based on the arithmetic of quaternion algebras and the Ramanujan conjecture for Hecke operators acting on cusp forms of weight 2.

Morgenstern [M1] introduced the notation of Ramanujan diagrams, which is a generalization of Ramanujan graphs. To date, there are the only known method to construct an infinite family of Ramanujan diagrams which are not finite graphs. It is given by the congruence subgroups Γ of $GL(2)$ over function fields over finite fields $\mathbb{F}_q$. Using the Ramanujan conjecture proved by Drinfeld [Dr], Morgenstern [M2] showed they are Ramanujan diagrams. In particular, their nontrivial discrete spectra of the adjacency operator have absolute values bounded by $2\sqrt{q}$. In [M2], the explicit constructions in the case of principal congruence subgroups $\Gamma(A)\,(A \in \mathbb{F}_q[t])$ are given. However, their concrete eigenvalues are difficult to compute.

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In this paper, we describe the limit distribution of eigenvalues of the adjacency operators for the Ramanujan diagrams attached to these principal congruence subgroups $\Gamma(A)$. It does not look so clear whether these diagrams have as many eigenvalues as their distributions converge to a measure which is not zero almost everywhere (see [E2] and [E3]). For our investigation, we will use the trace formulas described in [Na]. The results about the limit distribution of eigenvalues in the case of finite Ramanujan graphs are described, for example, in [Te] and [L2], which contain several conjectures.

We give our main theorems (Theorem 5.1 and Theorem 5.2) in Section 5. In Section 2 we denote some results on Bruhat-Tits and quotient graphs. In Section 3 we describe the trace formulas in an useful form for our aim and in Section 4 we investigate the scattering determinant for $\Gamma(A)$ which is necessary later.

2. Preliminaries

In this section we prepare some notations and basic facts about the Bruhat-Tits trees attached to function fields.

Let $\mathbb{F}_q$ be the finite field with q elements, $\mathbb{F}_q[t]$ the ring of polynomials in t over $\mathbb{F}_q$, and $k = \mathbb{F}_q(t)$ its quotient field. Let k_∞ be the completion of k with respect to the norm $|\cdot|_\infty$ at $1/t$, and r_∞ the ring of local integers. Then $k_\infty = \mathbb{F}_q((t^{-1}))$ is the fields of Laurent series in uniformizer t^{-1} over $\mathbb{F}_q$, and $r_\infty = \mathbb{F}_q[[t^{-1}]]$ is the ring of Taylor series in t^{-1} over $\mathbb{F}_q$. If an element a in k_∞ is written as $\sum_{i=n}^{\infty} a_i t^{-i}$ $(a_n \neq 0)$, then $|a|_\infty = q^{-n}$.

Throughout this paper we put $G = PGL(2, k_\infty)$ and $K = PGL(2, r_\infty)$. Note that K is a maximal compact subgroup of G. As described in [Se1, II.1.1], we can endow $X := G/K$ with the structure of the $q+1$-regular tree, also denoted by X or X_{q+1} for short. The tree X has a natural distance d, namely, if u and v are adjacent in X we let $d(u,v) = 1$. When a graph Y is given, we write $V(Y)$ (resp. $E(Y)$) for the set of vertices (resp. edges) of Y. As a complete set of representatives of $V(X) = G/K$ we can take the following set of matrices:

$$\left\{ \begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix} \in G \;\middle|\; n \in \mathbb{Z}, x \in k_\infty, x \bmod t^n r_\infty \right\}. \tag{1}$$

From the way of construction of the tree X, the neighbors of a vertex gK ($g \in G$) are $q+1$ cosets gs_iK $(i = 1, \cdots, q+1)$, where

$$\{s_1, \cdots, s_{q+1}\} = \left\{ \begin{pmatrix} t^{-1} & \alpha \\ 0 & 1 \end{pmatrix} \;\middle|\; \alpha \in \mathbb{F}_q \right\} \cup \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}.$$

The group G acts on the tree X as a group of automorphisms. This action induces an action of G on the boundary ∂X of X, which is just the usual fractional linear transformation of G on $\mathbb{P}^1(k_\infty)$. See [Se1] or [Lu] for definitions of terms with X. We assume from now on that Γ denotes a discrete subgroup of G which acts without inversions on X. Then the quotient set $\Gamma\backslash X$ naturally has the structure of graphs. If Γ is a lattice (i.e., a discrete subgroup of finite covolume) in G, Lubotzky [Lu, Theorem 6.1.] shows that the quotient graph $\Gamma\backslash X$ is the union of a finite graph together with finitely many infinite rays. For example, when $\Gamma(1) := PGL(2, \mathbb{F}_q[t])$ the quotient graph $\Gamma(1)\backslash X$ is isomorphic to a half-line tree [Sel, II.1.6].

In this paper we consider $\mathbb{C}$-valued functions defined on the set of vertices $V(X)$. If a function f on $V(X)$ satisfies $f(\gamma g) = f(g)$ for all $\gamma \in \Gamma$ and $g \in V(X)$, f is called an automorphic function for Γ, namely, just functions on the quotient graph $\Gamma\backslash X$. Now let denote the stabilizer of $v \in V(\Gamma\backslash X)$ (resp. $e \in E(\Gamma\backslash X)$) in Γ by Γ_v (resp. Γ_e). The quotient graph $\Gamma\backslash X$ can be made into a measure space by a Haar measure m induced from G, which we normalize so that the volume of K is 1. It is an atomic measure on $\Gamma\backslash X$ that assigns to a vertex $v \in V(\Gamma\backslash X)$ the measure

$$m(v) = |\Gamma_v|^{-1}, \tag{2}$$

(see [Sel, II.1.5]). For later use we put

$$m(e) = |\Gamma_e|^{-1}$$

for $e \in E(\Gamma\backslash X)$.

Next we define a natural operator for X, which is called the adjacency operator, by

$$(Tf)(v) = \sum_{d(v,u)=1} f(u) \qquad (f\colon V(X) \to \mathbb{C}). \tag{3}$$

It induces an operator for $\Gamma\backslash X$ and this induced operator can be represented as

$$(Tf)(v) = \sum_{e=(v,u)\in E(\Gamma\backslash X)} \frac{m(e)}{m(v)} f(u) \qquad (f\colon V(\Gamma\backslash X) \to \mathbb{C}),$$

where $e = (v, u)$ denotes the edge from v to u. It is known that T is self-adjoint and $\|T\| \leq q + 1$. Generalizing the operator T, we define the operator T_m $(m = 0, 1, 2, \cdots)$, which average functions on $V(X)$ at distance m:

$$(T_m f)(v) = \sum_{d(v,u)=m} f(u) \qquad (f\colon V(X) \to \mathbb{C}).$$

Note that $T_0 = I =$ identity and $T_1 = T$. Then we have the recursive relations:

$$T_1^2 = T_2 + (q+1)T_0$$
$$T_1 T_m = T_{m+1} + qT_{m-1} \quad (m \geq 2).$$

We can show that these relations yield the following identity:

$$\sum_{m=0}^{\infty} T_m u^m = \frac{1-u^2}{1-Tu+qu^2}, \tag{4}$$

where u is an indeterminate.

3. Trace formulas

From this section we let q be an odd prime power and Γ a principal congruence subgroup of G:

$$\Gamma(A) = \{\gamma \in PGL(2, \mathbb{F}_q[t]) \mid \gamma \equiv I \pmod{A}\} \quad (A \in \mathbb{F}_q[t]).$$

We assume $\deg A = a \geq 1$. The quotient graph $\Gamma\backslash X$ is an infinite graph, so that there will be continuous spectra of T. Let σ be the number of inequivalent cusps of $\Gamma\backslash X$. The continuous spectra are furnished by the Eisenstein series $E_i(g,s)$ at each cusp κ_i $(i = 1, \cdots, \sigma)$ (see Section 4 for the definition) and parametrized by the interval $[-2\sqrt{q}, 2\sqrt{q}]$. The Eisenstein series $E_i(g,s)$ is invariant under Γ, so it can be expanded as a Fourier series at each cusp κ_j. In the case of principal congruence groups, Li [L1] obtains an explicit form of the Fourier series in terms of the L-functions associated to characters χ on $\mathbb{F}_q[t]$ mod A.

The constant terms of the Fourier series of $E_i(g,s)$ at cusps κ_j define the $\sigma\times\sigma$-matrix $\Phi(s)$ which is called the scattering matrix for Γ. Then it is known [L1, Theorem 7] that $\Phi(s)$ satisfies the functional equation $\Phi(s) = \Phi(1-s)$. We put $\varphi(s) = \det \Phi(s)$ and call it the scattering determinant for Γ. By the above computation [L1] we find that $\varphi(s)$ is a rational function in q^{2s}, so we put

$$\varphi(s) = c\frac{(q^{2s}-qa_1)(q^{2s}-qa_2)\cdots(q^{2s}-qa_m)}{(q^{2s}-qb_1)(q^{2s}-qb_2)\cdots(q^{2s}-qb_n)}, \tag{5}$$

where c, a_i, b_j are constants and we assume that the right hand side is written to be irreducible. It is understood that the determinant $\varphi(s)$ plays an important role in the theory. It controls the Eisenstein series in the sense that if $\varphi(s)$ is analytic at some point, then so are all the Eisenstein series. Moreover $\varphi(s)$ appears in the Selberg trace formula described below as the contribution of the continuous spectra of T. In Section 4 we will observe more detailed expression of $\varphi(s)$ for Γ.

Next we set some notations to describe the trace formula for Γ. Let $\mathfrak{P}_\Gamma$ denote the set of primitive hyperbolic conjugacy classes of Γ. For a hyperbolic conjugacy class $\{P\}$ of Γ, we put $N(\{P\}) = N(P) = \sup\{|\lambda_i|_\infty^2 \mid \lambda_i$ is an eigenvalue of the matrix $P\}$ and $\deg\{P\} = \deg P = \log_q N(P)$. Let D be the set of the discrete spectra of the adjacency operator T for $\Gamma\backslash X$, which is a finite set. Denote the number of elements of D by $|D|$. Then the trace formula for Γ is found explicitly in the following:

Theorem 3.1. [Na] *For a discrete spectrum $\lambda_j \in D$ of T on $L^2(V(\Gamma\backslash X), m)$, we set $\lambda_j = q^{s_j} + q^{1-s_j}$ and $s_j = 1/2 + ir_j$. Assume that the sequence $c(n) \in \mathbb{C}\,(n \in \mathbb{Z})$ satisfies $c(n) = c(-n)$ and $\sum_{n\in\mathbb{Z}} q^{\frac{|n|}{2}}|c(n)| < \infty$. Then we have the following formula:*

$$
\begin{aligned}
(6)\qquad \sum_{j=1}^{|D|} h(r_j) &= \mathrm{vol}(\Gamma\backslash X)k(0) \\
&+ \sum_{\{P\}\in\mathfrak{P}_\Gamma} \sum_{l=1}^{\infty} \frac{\deg P}{q^{\frac{l \deg P}{2}}} c(l \deg P) \\
&+ \left(\sigma - \mathrm{Tr}\,\Phi\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}c(0) + \sum_{m=1}^{\infty} c(2m)\right) \\
&+ \frac{1}{4\pi}\int_{-\frac{\pi}{\log q}}^{\frac{\pi}{\log q}} h(r)\frac{\varphi'}{\varphi}\left(\frac{1}{2} + ir\right) dr \\
&- \sigma\left(a + \frac{1}{q-1}\right)c(0).
\end{aligned}
$$

Here the functions $h(\cdot)$ and $k(\cdot)$ are determined by $c(\cdot)$ via the Selberg transform.

The Selberg zeta function for Γ is defined by

$$
Z_\Gamma(s) = \prod_{\{P\}\in\mathfrak{P}_\Gamma} \left(1 - N(P)^{-s}\right)^{-1}.
$$

Here we put

$$
(8)\qquad N_m = \sum_{\substack{\{P\}\in\mathfrak{P}_\Gamma \\ \deg P | m}} \deg P \qquad (m \geq 1).
$$

Then the Selberg zeta function $Z_\Gamma(s)$ can also be written as

$$
(9)\qquad Z_\Gamma(u) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} u^m\right),
$$

where we put $u = q^{-s}$.

Using the trace formula (6), we see that the Selberg zeta function $Z_\Gamma(u)$ can be expressed as the determinant of T:

Theorem 3.2. [Na]

$$(10)\qquad Z_\Gamma(u) = (1-u^2)^{-\chi}(1-qu^2)^{\rho} \times \prod_{\lambda\in D}(1-\lambda u+qu^2)^{-1}\cdot \prod_{|b_j|<1}(1-qb_ju^2)^{-1}\cdot \prod_{|b_j|>1}(1-qb_j^{-1}u^2),$$

where $\chi = \mathrm{vol}(\Gamma\backslash X)\frac{q-1}{2}, \rho = \frac{1}{2}\left(\sigma - \mathrm{Tr}\,\Phi(\frac{1}{2})\right)$ *and* b_j*'s are as in* (5).

Now taking the logarithmic derivative of (9) and (10) in u, we obtain

$$(11)\qquad \sum_{m=1}^{\infty} N_m u^m = \frac{(q-1)\mathrm{vol}(\Gamma\backslash X)u^2}{1-u^2} - \frac{\left(\sigma - \mathrm{Tr}\,\Phi(\frac{1}{2})\right)qu^2}{1-qu^2} - \sum_{\lambda\in D}\frac{-\lambda u+2qu^2}{1-\lambda u+qu^2} + \sum_{|b_j|<1}\frac{2qb_ju^2}{1-qb_ju^2} - \sum_{|b_j|>1}\frac{2qb_j^{-1}u^2}{1-qb_j^{-1}u^2}.$$

Here we have

$$\begin{aligned} &-\sum_{\lambda\in D}\frac{-\lambda u+2qu^2}{1-\lambda u+qu^2} \\ &= -\sum_{\lambda\in D}\frac{1-\lambda u+qu^2-(1-u^2)+(q-1)u^2}{1-\lambda u+qu^2} \\ (12)\qquad &= -|D| + \sum_{\lambda\in D}\frac{1-u^2}{1-\lambda u+qu^2} - (q-1)\frac{u^2}{1-u^2}\sum_{\lambda\in D}\frac{1-u^2}{1-\lambda u+qu^2}. \end{aligned}$$

Let $\mathrm{Tr}\,T_m$ denote the the sum of the discrete spectra of T_m on $L^2(V(\Gamma\backslash X), m)$. The relation (4) yields

$$(13)\qquad \sum_{m=0}^{\infty}\mathrm{Tr}\,T_m\,u^m = \sum_{\lambda\in D}\frac{1-u^2}{1-\lambda u+qu^2},$$

from which we have

$$\begin{aligned}(14)\qquad \sum_{m=1}^{\infty}\sum_{1\le k<\frac{m}{2}}\mathrm{Tr}\,T_{m-2k}u^m &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mathrm{Tr}\,T_n u^{n+2k} \\ &= \frac{u^2}{1-u^2}\sum_{\lambda\in D}\frac{1-u^2}{1-\lambda u+qu^2} - \frac{u^2}{1-u^2}|D|.\end{aligned}$$

Hence by (13) and (14), the equation (12) is equal to

$$\begin{aligned} &\sum_{m=1}^{\infty}\mathrm{Tr}\,T_m u^m - (q-1)\left(\sum_{m=1}^{\infty}\sum_{1\le k<\frac{m}{2}}\mathrm{Tr}\,T_{m-2k}u^m + \frac{u^2}{1-u^2}|D|\right) \\ &= \sum_{m=1}^{\infty}\left(\mathrm{Tr}\,T_m - (q-1)\sum_{1\le k<\frac{m}{2}}\mathrm{Tr}\,T_{m-2k}\right)u^m - \frac{(q-1)u^2}{1-u^2}|D|. \end{aligned}$$

Thus we find that (11) can be computed as

$$\sum_{m=1}^{\infty} N_m u^m = \sum_{m=1}^{\infty} (q-1)\mathrm{vol}(\Gamma\backslash X) u^{2m} - \sum_{m=1}^{\infty}\left(\sigma - \operatorname{Tr}\Phi\left(\frac{1}{2}\right)\right) q^m u^{2m}$$
$$+ \sum_{m=1}^{\infty}\left(\operatorname{Tr} T_m - (q-1)\sum_{1\le k<\frac{m}{2}} \operatorname{Tr} T_{m-2k}\right) u^m - \sum_{m=1}^{\infty}(q-1)|D|u^{2m}$$
$$+ \sum_{|b_j|<1}\sum_{m=1}^{\infty} 2q^m b_j^m u^{2m} - \sum_{|b_j|>1}\sum_{m=1}^{\infty} 2q^m b_j^{-m} u^{2m}.$$

Therefore we obtain the following formulas connecting each N_m with the traces of T_m's. By induction we also have Proposition 3.1 (ii).

Proposition 3.1. *For $m \ge 1$ let N_m be as in* (8) *and $\operatorname{Tr} T_m$ the sum of the discrete spectra of T_m for $\Gamma\backslash X$. Then*

(i) *if m is odd,* $N_m = \operatorname{Tr} T_m - (q-1)\sum_{1\le k<\frac{m}{2}} \operatorname{Tr} T_{m-2k}$;

if m is even,
$$N_m = \operatorname{Tr} T_m - (q-1)\sum_{1\le k<\frac{m}{2}} \operatorname{Tr} T_{m-2k}$$
$$+ (q-1)\big(\mathrm{vol}(\Gamma\backslash X) - |D|\big)$$
$$- \left(\sigma - \operatorname{Tr}\Phi\left(\frac{1}{2}\right)\right) q^{\frac{m}{2}}$$
$$+ 2q^{\frac{m}{2}} \sum_{|b_j|<1} b_j^{\frac{m}{2}} - 2q^{\frac{m}{2}} \sum_{|b_j|>1} b_j^{-\frac{m}{2}}.$$

(ii) *if m is odd,* $\operatorname{Tr} T_m = N_m + (q-1)\sum_{1\le k<\frac{m}{2}} q^{k-1} N_{m-2k}$;

if m is even,
$$\operatorname{Tr} T_m = N_m + (q-1)\sum_{1\le k<\frac{m}{2}} q^{k-1} N_{m-2k}$$
$$- \big(\mathrm{vol}(\Gamma\backslash X) - |D|\big)(q-1)q^{\frac{m}{2}-1}$$
$$+ \left(\sigma - \operatorname{Tr}\Phi\left(\frac{1}{2}\right)\right)\left(\frac{m}{2}q - \frac{m}{2} + 1\right)q^{\frac{m}{2}-1}$$
$$- 2(q-1)q^{\frac{m}{2}-1}\sum_{|b_j|<1}\sum_{1\le n<\frac{m}{2}} b_j^n - 2q^{\frac{m}{2}}\sum_{|b_j|<1} b_j^{\frac{m}{2}}$$
$$+ 2(q-1)q^{\frac{m}{2}-1}\sum_{|b_j|>1}\sum_{1\le n<\frac{m}{2}} b_j^{-n} + 2q^{\frac{m}{2}}\sum_{|b_j|>1} b_j^{-\frac{m}{2}}.$$

Remark. For $\Gamma = \Gamma(A) \subset PGL(2, \mathbb{F}_q[t])$ it follows by the definition of $\deg P$ ($\{P\} \in \mathfrak{P}_\Gamma$) that $\deg P$ is even, so $N_m = 0$ if m is odd. See [Na] for the geometric meaning of $\deg P$. For a function f on $V(X)$, let $\tilde{f}$ be its alternating function on $V(X)$ defined by

$$\tilde{f}(v) := \begin{cases} f(v) & \text{if } d(v, v_0) \text{ is even} \\ -f(v) & \text{if } d(v, v_0) \text{ is odd,} \end{cases} \tag{15}$$

where v_0 is the vertex which corresponds to $\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \in G/K$. Then we find that if let m be odd and $T_m f = \lambda f$ for a function f on $V(X)$ then we get $T_m \tilde{f} = -\lambda \tilde{f}$. Noting that if $f \in L^2(V(\Gamma\backslash X), m)$ then $\tilde{f} \in L^2(V(\Gamma\backslash X), m)$, we have $\mathrm{Tr}\, T_m = 0$ for odd m.

4. Scattering determinants

In this section we investigate more details about the scattering determinant $\varphi(s)$ for $\Gamma(A)$. In the following, let $a \equiv b$ mean $a \equiv b \,(\mathrm{mod}\, A)$, $M(\mathbb{F}_q[t])$ be the set of monic polynomials in $\mathbb{F}_q[t]$ and $P(\mathbb{F}_q[t])$ be the set of monic and prime polynomials in $\mathbb{F}_q[t]$. We denote by $(a, \cdots, b)$ the element in $M(\mathbb{F}_q[t])$ of highest order which divides all $a, \cdots, b \in \mathbb{F}_q[t]$.

We recall that $\Gamma(1)$ has one inequivalent cusp. By analogy of the case of $PSL(2,\mathbb{Z})$, we find that a set $\mathcal{C}$ of $\Gamma(A)$-inequivalent cusps $\kappa = -\frac{\delta}{\gamma}\,(\gamma, \delta \in \mathbb{F}_q[t], (\gamma, \delta) = 1)$ can be given by the equivalence classes of pairs $< \gamma, \delta >$ such that $(\gamma, \delta, A) = 1$ under the following equivalence relation: An element $< \gamma, \delta >$ is equivalent to $< \gamma', \delta' >$ if and only if $\gamma \equiv \alpha\gamma' (\mathrm{mod}\, A)$ and $\delta \equiv \alpha\delta' (\mathrm{mod}\, A)$ for some $\alpha \in \mathbb{F}_q^{\times}$. The number of inequivalent cusps of $\Gamma(A)\,(\deg A \geq 1)$ is given by

$$\sigma = \sigma(A) = \frac{1}{q-1} |A|_\infty^2 \prod_{\substack{B \in P(\mathbb{F}_q[t]) \\ B|A}} \left(1 - \frac{1}{|B|_\infty^2}\right). \tag{16}$$

Let Γ_{κ_i} be the stabilizer in Γ of a cusp κ_i and we choose $\alpha_i, \beta_i \in \mathbb{F}_q[t]$ such that the matrix

$$\rho_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \in SL(2, \mathbb{F}_q[t]) \tag{17}$$

sends $\kappa_i = -\frac{\delta_i}{\gamma_i}$ to ∞. Then the Eisenstein series $E_i(g, s)\,(g \in G/K, \mathrm{Re}(s) > 1)$ attached to κ_i is defined by

$$E_i(g, s) = E_{\gamma_i, \delta_i}(g, s) = \sum_{\tau \in \Gamma_{\kappa_i} \backslash \Gamma} \psi_s(\rho_i \tau g),$$

where $\psi_s(g) = |\det g|_\infty^s h((0,1)g)^{-2s}$and $h(\cdot)$ is a function which sends row vectors (x, y) to $\mathbb{C}$ by $h((x, y)) = \sup\{|x|_\infty, |y|_\infty\}$. For $c, d \in \mathbb{F}_q[t]$ with $c \equiv \gamma_i, d \equiv \delta_i$ and $(c, d) = 1$ we have $\alpha_i d - \beta_i c \equiv 1$ by (17), so that there exist

$a, b \in \mathbb{F}_q[t]$ such that $a \equiv \alpha_i, b \equiv \beta_i$ and $ad - bc = 1$ (see the proof of [Sc, p. 74, lemma]). By taking account of this fact and using these a and b, $E_i(g,s)$ can be written as

$$E_i(g,s) = \sum_{\substack{c\equiv\gamma_i, d\equiv\delta_i \\ (c,d)=1}} \frac{|\det g|_\infty^s}{h\left((0,1)\begin{pmatrix} a & b \\ c & d\end{pmatrix} g\right)^{2s}} = \sum_{\substack{c\equiv\gamma_i, d\equiv\delta_i \\ (c,d)=1}} \frac{q^{ns}}{h((ct^n, cx+d))^{2s}},$$

for $g = \begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix} \in G/K$ as in (1).

We now put

$$F_i(g,s) = \sum_{c\equiv\gamma_i, d\equiv\delta_i} \frac{|\det g|_\infty^s}{h\left((0,1)\begin{pmatrix} a & b \\ c & d\end{pmatrix} g\right)^{2s}} = \sum_{c\equiv\gamma_i, d\equiv\delta_i} \frac{q^{ns}}{h((ct^n, cx+d))^{2s}}.$$

Then

$$\begin{aligned} F_i(g,s) &= \sum_{\substack{Y\in M(\mathbb{F}_q[t]) \\ (Y,A)=1}} \sum_{\substack{(c,d)=Y \\ c\equiv\gamma_i, d\equiv\delta_i}} \frac{q^{ns}}{h\left((ct^n, cx+d)\right)^{2s}} \\ &= \sum_{\substack{Y\in M(\mathbb{F}_q[t]) \\ (Y,A)=1}} \frac{1}{|Y|_\infty^{2s}} \sum_{\substack{(c,d)=1 \\ c\equiv Y^{-1}\gamma_i, d\equiv Y^{-1}\delta_i}} \frac{q^{ns}}{h\left((ct^n, cx+d)\right)^{2s}} \\ &= \sum_{U\in(\mathbb{F}_q[t]/A\mathbb{F}_q[t])^\times} \sum_{\substack{Y\in M(\mathbb{F}_q[t]) \\ Y\equiv U}} \frac{1}{|Y|_\infty^{2s}} \sum_{\substack{(c,d)=1 \\ c\equiv U^{-1}\gamma_i, d\equiv U^{-1}\delta_i}} \frac{q^{ns}}{h\left((ct^n, cx+d)\right)^{2s}}, \end{aligned}$$

where Y^{-1} denotes the inverse of Y in $(\mathbb{F}_q[t]/A\mathbb{F}_q[t])^\times$.

Let $U_1, U_2, \cdots, U_r$ be the set of representatives of $A^\times := (\mathbb{F}_q[t]/A\mathbb{F}_q[t])^\times / \mathbb{F}_q^\times$, where

$$r = \frac{\phi(A)}{q-1} = \frac{1}{q-1}|A|_\infty \prod_{\substack{B\in P(\mathbb{F}_q[t]) \\ B|A}} \left(1 - \frac{1}{|B|_\infty}\right). \tag{21}$$

Now we define the function $\zeta(s,U) = \zeta(s,U;A)$ for $U \in \mathbb{F}_q[t]$ by

$$\zeta(s,U) = \begin{cases} \displaystyle\sum_{\substack{X\in\mathbb{F}_q[t] \\ X\equiv U}} \frac{1}{|X|_\infty^s} = \sum_{\alpha\in\mathbb{F}_q^\times} \sum_{\substack{Y\in M(\mathbb{F}_q[t]) \\ Y\equiv\alpha U}} \frac{1}{|Y|_\infty^s} & \text{if } U \not\equiv 0 \\ \displaystyle\sum_{\substack{X\in\mathbb{F}_q[t] \\ X\equiv 0, X\neq 0}} \frac{1}{|X|_\infty^s} & \text{if } U \equiv 0. \end{cases} \tag{22}$$

Then we have

$$F_i(g,s) = \sum_{\nu=1}^{r} \sum_{\alpha\in\mathbb{F}_q^\times} \sum_{\substack{Y\in M(\mathbb{F}_q[t]) \\ Y\equiv\alpha U_\nu}} \frac{1}{|Y|_\infty^{2s}} \sum_{\substack{(c,d)=1 \\ c\equiv\alpha^{-1}U_\nu^{-1}\gamma_i \\ d\equiv\alpha^{-1}U_\nu^{-1}\delta_i}} \frac{q^{ns}}{h\left((ct^n, cx+d)\right)^{2s}}$$

$$= \sum_{\nu=1}^{r} \zeta(2s, U_\nu) E_{U_\nu^{-1}\gamma_i, U_\nu^{-1}\delta_i}(g,s). \tag{23}$$

Here we see that $A^\times$ acts on $\mathcal{C}$ by setting $U_\nu \cdot < \gamma, \delta > := < U_\nu^{-1}\gamma, U_\nu^{-1}\delta >$ for $< \gamma, \delta > \in \mathcal{C}$ and $U_\nu \in A^\times$. By taking account of this and changing indexes of cusps κ_i appropriately, the relations (23) can be written as follows:

Lemma 4.1.

$$\begin{pmatrix} F_1 \\ \vdots \\ F_\sigma \end{pmatrix} = \begin{pmatrix} \mathrm{J} & & \\ & \ddots & \\ & & \mathrm{J} \end{pmatrix} \begin{pmatrix} E_1 \\ \vdots \\ E_\sigma \end{pmatrix}, \tag{24}$$

where each block J *is the* $r \times r$*-matrix* $\mathrm{J} = \left(\zeta(2s, U_i^{-1}U_j)\right)_{i,j}$ *and* r *is as in* (21).

Next we turn to compute the constant term of the Fourier expansion of $F_i(g,s)$ at κ_j, namely, the Fourier expansion of $F_i(\rho_j^{-1}g, s)$ at ∞. We have that for $g = \begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix} \in G/K$,

$$\begin{aligned} F_i(\rho_j^{-1}g,s) &= \sum_{c\equiv\gamma_i, d\equiv\delta_i} \frac{|\det \rho_j^{-1}g|_\infty^s}{h\left((0,1)\begin{pmatrix} a & b \\ c & d\end{pmatrix}\rho_j^{-1}g\right)^{2s}} \\ &= \sum_{c\equiv\gamma_i, d\equiv\delta_i} \frac{q^{ns}}{h\left((c\delta_j - d\gamma_j)t^n, (c\delta_j - d\gamma_j)x + (-c\beta_j + d\alpha_j)\right)^{2s}} \\ &= \sum_{\substack{c\equiv\gamma_i\delta_j - \delta_i\gamma_j \\ d\equiv -\gamma_i\beta_j + \delta_i\alpha_j}} \frac{q^{ns}}{h\left((ct^n, cx+d)\right)^{2s}}, \end{aligned}$$

so we put

$$F(g,s) = \sum_{c\equiv\gamma, d\equiv\delta} \frac{q^{ns}}{h\left((ct^n, cx+d)\right)^{2s}} \tag{25}$$

and will treat $F(g,s)$. The constant term of the Fourier expansion at ∞ of $F(g,s)$ is defined by

$$a_0(F; t^n, s) = \int_{A\mathbb{F}_q[t]\backslash k_\infty} F\left(\begin{pmatrix} t^n & x' \\ 0 & 1\end{pmatrix}, s\right) dx',$$

where the invariant measure dx' on $A\mathbb{F}_q[t]\backslash k_\infty$ is normalized so that the total volume of $A\mathbb{F}_q[t]\backslash k_\infty$ is equal to 1. If we normalize the Haar measure dx on k_∞ so that $\mathrm{vol}(r_\infty) = 1$, then

$$\begin{aligned}
a_0(F; t^n, s) &= \frac{1}{\mathrm{vol}(t^{a-1}r_\infty)} \int_{t^{a-1}r_\infty} F\left(\begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix}, s\right) dx \\
&= \frac{1}{q^{a-1}} \int_{t^{a-1}r_\infty} \sum_{c\equiv\gamma, d\equiv\delta} \frac{q^{ns}}{h\left((ct^n, cx+d)\right)^{2s}} dx.
\end{aligned}$$

First we consider the case of $\gamma \equiv 0$, in which case the term with $c = 0$ appears and $\delta \not\equiv 0$, then $a_0(F; t^n, s)$ is equal to

$$\begin{aligned}
(27) \qquad & \frac{1}{q^{a-1}} \int_{t^{a-1}r_\infty} \sum_{d\equiv\delta} \frac{q^{ns}}{|d|_\infty^{2s}} dx \\
&+ \frac{1}{q^{a-1}} \int_{t^{a-1}r_\infty} \sum_{\substack{c\equiv 0, c\neq 0 \\ d\equiv\delta}} \frac{q^{ns}}{h\left((ct^n, cx+d)\right)^{2s}} dx.
\end{aligned}$$

The first term of (27) is equal to

$$\frac{1}{q^{a-1}} \sum_{d\equiv\delta} \frac{q^{ns}}{|d|_\infty^{2s}} \int_{t^{a-1}r_\infty} dx = \zeta(2s, \delta) q^{ns}.$$

The second term of (27) is equal to

$$(28) \qquad \frac{1}{q^{a-1}} \sum_{\substack{c\equiv 0, c\neq 0 \\ \ell\in\mathbb{F}_q[t]}} \int_{A\mathbb{F}_q[t]\backslash k_\infty} \frac{q^{ns}}{h\left((ct^n, cx+\ell A+\delta)\right)} dx.$$

We now set $\ell = \ell' c + \ell'' (0 \le \deg \ell'' < \deg c)$, so $cx + \ell A + \delta = c(x+\ell' A) + \ell'' A + \delta$. Summing over $\ell \in \mathbb{F}_q[t]$ and changing variables we find that (28) is

$$\begin{aligned}
& \frac{1}{q^{a-1}} \sum_{\substack{c\equiv 0, c\neq 0 \\ 0\le \deg \ell'' < \deg c}} \int_{k_\infty} \frac{q^{ns}}{h\left((ct^n, cx+\ell'' A+\delta)\right)^{2s}} dx \\
&= \frac{1}{q^{a-1}} \sum_{\substack{c\equiv 0, c\neq 0 \\ 0\le \deg \ell'' < \deg c}} \int_{k_\infty} \frac{q^{ns}}{|c|_\infty^{2s} h\left(\left(t^n, x + \frac{\ell'' A+\delta}{c}\right)\right)^{2s}} dx \\
&= \frac{1}{q^{a-1}} \sum_{c\equiv 0, c\neq 0} \frac{|c|_\infty}{|c|_\infty^{2s}} \int_{k_\infty} \frac{q^{ns}}{h\left((t^n, x)\right)^{2s}} dx \\
&= \frac{1}{q^{a-1}} \zeta(2s-1, 0)\, q^{-ns} \int_{k_\infty} \frac{1}{h\left(\left(1, \frac{x}{t^n}\right)\right)^{2s}} dx \\
&= \frac{1}{q^{a-1}} \zeta(2s-1, 0) \frac{1-q^{-2s}}{1-q^{1-2s}}\, q^{n(1-s)},
\end{aligned}$$

by the following formula derived from the definition:

$$\int_{k_\infty} \frac{1}{h((1,x))^{2s}} dx = \frac{1-q^{-2s}}{1-q^{1-2s}}.$$

Next we consider the case of $\gamma \not\equiv 0$ in (25). In this case, the constant term $a_0(F; t^n, s)$ is computed in the same way as the computation of the second term of (27) and we get

$$a_0(F; t^n, s) = \frac{1}{q^{a-1}} \zeta(2s-1, \gamma) \frac{1-q^{-2s}}{1-q^{1-2s}} q^{n(1-s)}.$$

To sum up, we find that the coefficient of $q^{n(1-s)}$ in the constant term of the Fourier expansion of $F_i(\rho_j^{-1} g, s)$ at ∞ is

$$\frac{1}{q^{a-1}} \zeta(2s-1, \gamma_i\delta_j - \delta_i\gamma_j) \frac{1-q^{-2s}}{1-q^{1-2s}}. \tag{29}$$

Here we recall the definition of the scattering matrix $\Phi(s)$ for Γ. The constant term of the Fourier expansion of $E_i(g, s)$ at a cusp κ_j is of the form:

$$\delta_{ij} q^{ns} + \varphi_{ij}(s) q^{n(1-s)}, \tag{30}$$

where δ_{ij} is Kronecker's δ (see [L1]). Then we define the matrix $\Phi(s)$ by $\Phi(s) = \left(\varphi_{ij}(s)\right)_{i,j=1,\cdots,\sigma}$. By (24) we can write the above conclusion (29) as follows:

Proposition 4.1. *Let* $\mathbf{J}$ *be the* $r \times r$*-matrix as in Lemma* 4.1 *and* $\Phi(s)$ *the scattering matrix for* $\Gamma(A)$. *Then*

(31)
$$\begin{pmatrix} \mathbf{J} & & \\ & \ddots & \\ & & \mathbf{J} \end{pmatrix} \Phi(s) = \frac{1}{q^{a-1}} \frac{1-q^{-2s}}{1-q^{1-2s}} \Big(\zeta(2s-1, \gamma_i\delta_j - \delta_i\gamma_j) \Big)_{i,j=1,\cdots,\sigma}.$$

Let $\hat{A}^\times$ be the set of characters of $A^\times$ and $L(s, \chi)$ the L-function associated to a character $\chi \in \hat{A}^\times$:

$$L(s, \chi) = \sum_{B \in M(\mathbb{F}_q[t])} \frac{\chi(B)}{|B|_\infty^s}.$$

Corollary 4.1. *Let* $\varphi(s)$ *be the scattering determinant* $\det \Phi(s)$ *for* $\Gamma(A)$. *Then*

$$\varphi(s) = \left(\frac{1}{q^{a-1}} \frac{1-q^{-2s}}{1-q^{1-2s}} \right)^\sigma \frac{\det\big(\zeta(2s-1, \gamma_i\delta_j - \delta_i\gamma_j)\big)_{i,j}}{\prod_{\chi \in \hat{A}^\times} L(2s, \chi)^h}, \tag{32}$$

where

$$h = \frac{\sigma}{r} = |A|_\infty \prod_{\substack{B \in P(\mathbb{F}_q[t]) \\ B|A}} \left(1 + \frac{1}{|B|_\infty}\right).$$

In particular, the poles of $\varphi(s)$ are contained within the zeros of the function

$$(q^{2s} - q^2) \prod_{\chi \in \hat{A}^\times} L(2s, \chi)^h.$$

Here we count zeros and poles with multiplicity.

Proof. First we claim that the determinant of the matrix J is computed as

$$\det \mathrm{J} = \prod_{\chi \in \hat{A}^\times} L(2s, \chi). \tag{33}$$

To obtain (33) we begin with

$$\begin{aligned}
\sum_{j=1}^{r} \zeta(s, U_i^{-1} U_j) \chi(U_j) &= \sum_{j=1}^{r} \sum_{\substack{R \in \mathbb{F}_q[t] \\ R \equiv U_i^{-1} U_j (\mathrm{mod}\, A)}} \frac{\chi(U_j)}{|R|_\infty^s} \\
&= \chi(U_i) \sum_{j=1}^{r} \sum_{\substack{R \in \mathbb{F}_q[t] \\ R \equiv U_i^{-1} U_j (\mathrm{mod}\, A)}} \frac{\chi(R)}{|R|_\infty^s} \\
&= \chi(U_i) \sum_{R \in M(\mathbb{F}_q[t])} \frac{\chi(R)}{|R|_\infty^s} \\
&= \chi(U_i) L(s, \chi),
\end{aligned}$$

where we consider $\chi \in \hat{A}^\times$ as a Dirichlet character $\mathrm{mod}\, A$ on $\mathbb{F}_q[t]$. Hence we have

$$\Big(\zeta(s, U_i^{-1} U_j)\Big)_{i,j=1,\cdots,r} \begin{pmatrix} \chi(U_1) \\ \vdots \\ \chi(U_r) \end{pmatrix} = L(s, \chi) \begin{pmatrix} \chi(U_1) \\ \vdots \\ \chi(U_r) \end{pmatrix},$$

from which we get (33). Taking the determinant of (31) we obtain (32).

Next we note that the function $\zeta(s, U)$ for $U \in \mathbb{F}_q[t]/A\mathbb{F}_q[t]$ in (22) is computed as

$$\zeta(s, U) = \begin{cases} \dfrac{1}{q^{us}} + \dfrac{(q-1)q^s}{q^{as}(q^s - q)} & \text{if } U \not\equiv 0 \\ \dfrac{(q-1)q^s}{q^{as}(q^s - q)} & \text{if } U \equiv 0, \end{cases} \tag{34}$$

where $u = \min\{\deg X \,|\, X \in \mathbb{F}_q[t], X \equiv U (\mathrm{mod}\, A)\}$. So

$$\lim_{q^s \to q} (q^s - q) \zeta(s, U) = \frac{q(q-1)}{q^a},$$

which is independent of U. By this we see that the poles of

$$\det\Big(\zeta(2s-1,\gamma_i\delta_j-\delta_i\gamma_j)\Big)_{i,j=1,\cdots,\sigma}$$

may come from the zeros of $q^{2s}-q^2$ with multiplicity one. Observing that $\varphi(s)$ is holomorphic on $\mathrm{Re}(s)=1/2$ by [L1, p. 241, lemma2], we get this Corollary. □

To obtain Lemma 4.2 below, we require the next claim:

Claim 4.1. *Let $\varphi(s)$ be written as in* (5). *Then $a_i\neq 0\,(i=1,\cdots,m)$.*

Proof. In the right hand side of (32) we let $\mathrm{Re}(s)\to\infty$. First by (34) we find that if $U\equiv\alpha$ for some $\alpha\in\mathbb{F}_q^\times$ then $\zeta(s,U)\to 1$ and otherwise $\zeta(s,U)\to 0$ as $\mathrm{Re}(s)\to\infty$. So every element in the $\sigma\times\sigma$-matrix $\Big(\zeta(2s-1,\gamma_i\delta_j-\delta_i\gamma_j)\Big)_{i,j}$ converges and $\left|\det\Big(\zeta(2s-1,\gamma_i\delta_j-\delta_i\gamma_j)\Big)_{i,j}\right|<\infty$ as $\mathrm{Re}(s)\to\infty$. This implies

$$|\varphi(s)|<\infty \qquad \text{as } \mathrm{Re}(s)\to\infty, \tag{35}$$

since $L(s,\chi)\to 1$.

Next we note that if $\varphi(s)$ in (5) has factors q^{2s} then these factors exist either in the numerator or in the denominator since (5) is written to be irreducible, and that by the functional equation $\varphi(s)\varphi(1-s)=1$ the set of factors $\{q^{2s}-qa_i|a_i\neq 0\}$ in the numerator in (5) and the set $\{q^{2s}-qb_j|b_j\neq 0\}$ in the denominator are one-to-one correspondent. Hence if we assume that $\varphi(s)$ in (5) has a factor with $a_i=0$, then it follows that $\varphi(s)\to\infty$ as $\mathrm{Re}(s)\to\infty$. But this contradicts (35). Thus we have the assertion. □

Lemma 4.2. *Let $\varphi(s)$ be as in* (5), *then in fact we have*

$$\varphi(s)=c\frac{(q^{2s}-1)(q^{2s}-qa_2)\cdots(q^{2s}-qa_m)}{(q^{2s}-q^2)(q^{2s}-qb_2)\cdots(q^{2s}-qb_n)}, \tag{36}$$

where $|a_i|>1$ for $i=2,\cdots,m$ and $|b_j|<1$ for $j=2,\cdots,n$.

Proof. By Corollary 4.1, [L1] and the theory of Eisenstein series, we have that on $\mathrm{Re}(s)\geq 1/2$ the function $\varphi(s)$ has a simple pole at $s=1+n\pi i/\log q\,(n\in\mathbb{Z})$ (i.e., $\varphi(s)$ as in (5) has a factor $q^{2s}-q^2$ in the denominator) and is holomorphic at the other s. Moreover by the fact that $L(2s,\chi)$ does not vanish on $\mathrm{Re}(s)>1/2$ for each character χ mod A, we can write $\varphi(s)$ as in (36) with $|b_j|<1\,(j=2,\cdots,n)$.

By the functional equation $\varphi(s)\varphi(1-s)=1$ we see that a factor $q^{2s}-qa_i$ in the numerator, where $a_i\neq 0$ by Lemma 4.1, corresponds to the factor $q^{2s}-qb_i$ with $b_i=1/a_i$ in the denominator. So we have the assertion as for the numerator. □

The estimate in the next lemma has an important role for later use. It should be noted that the volume of $\Gamma(A)\backslash X$ are computed via the construction of $\Gamma(A)\backslash X$ in [M2] as follows:

$$\text{(37)} \qquad \operatorname{vol}(\Gamma(A)\backslash X) = \frac{2\,|A|_\infty^3}{(q+1)(q-1)^2} \prod_{\substack{B\in P(\mathbb{F}_q[t]) \\ B|A}} \left(1 - \frac{1}{|B|_\infty^2}\right).$$

Lemma 4.3. *Under the notation in Lemma 4.2 we have*

$$\sum_{|b_j|<1} 1 = o\big(\operatorname{vol}(\Gamma(A)\backslash X)\big)$$

as $degA \to \infty$.

Proof. On account of Corollary 4.1 we divide the set of b_j $(j = 2, \cdots, n)$ as in (36) into two types: (i) the contribution of $\prod_{\chi\in\hat{A}^\times} L(2s,\chi)^h$ and (ii) that of $\det\Big(\zeta(2s-1, \gamma_i\delta_j - \delta_i\gamma_j)\Big)_{i,j}$, which might possibly exist and are correspond to b_j's which are 0.

First we estimate the sum over b_j of (i). As is well known, the L-series $L(s,\chi)$ is equal to the L-series $L(s,\psi)$ of a primitive character ψ modulo the conductor of χ, multiplied by some factors, the number of which is less than that of divisors of A. This and Theorem 6 of [We, p. 134] imply that for any $\varepsilon > 0$

$$\text{(38)} \qquad \sum_{b_j:(\mathrm{i})} 1 \ll_\varepsilon (a + q^{a\varepsilon}) \cdot h \cdot r = (a + q^{a\varepsilon}) \cdot \sigma(A).$$

By (16) and (37) this gives us

$$\text{(39)} \qquad \sum_{b_j:(\mathrm{i})} 1 = o\big(\operatorname{vol}(\Gamma(A)\backslash X)\big).$$

Next as for the sum over b_j of (ii), by (34) we have

$$\sum_{b_j:(\mathrm{ii})} 1 \le a\cdot\sigma(A) = o\big(\operatorname{vol}(\Gamma(A)\backslash X)\big).$$

Hence we obtain the assertion. □

5. The distribution of eigenvalues

As mentioned in Introduction, by virtue of the Ramanujan conjecture proved by Drinfeld [Dr], Morgenstern [M2] shows that any nontrivial discrete spectra λ, (i.e. the eigenvalues except $\pm(q+1)$) of T on $L^2(V(\Gamma\backslash X), m)$ satisfies $|\lambda| \le 2\sqrt{q}$. We put a normalized operator $T' = T/2\sqrt{q}$. Let D' be the set of the nontrivial discrete spectra of T'. Every element λ' of D' satisfies $|\lambda'| \le 2$. Then we consider certain limit distribution of the eigenvalues of T' for $\Gamma\backslash X$.

First we prepare two probability measures on $\Omega = [-2, 2]$ and their basic facts which will be used later. One is the Sato-Tate measure or Wigner semi-circle:

$$\mu_\infty(x) = \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}dx.$$

The other is defined for a real number $q (> 1)$ by

$$\mu_q(x) = \frac{q+1}{(q^{1/2} + q^{-1/2})^2 - x^2}\,\mu_\infty(x).$$

The measure $\mu_q(x)$ is found to be the spectral measure of the adjacency operator for the $q+1$-regular tree X. The Chebychev polynomials of the second kind $X_m(x)\,(m = 0, 1, 2, \cdots)$, which are defined by

$$X_m(x) = \frac{\sin((m+1)\theta)}{\sin\theta} \tag{40}$$

when $x = 2\cos\theta\,(0 \le \theta \le \pi)$, are known to be orthogonal with respect to $\mu_\infty(x)$. We have

$$\int_\Omega X_m(x)d\mu_\infty(x) = \begin{cases} 1 & (m = 0) \\ 0 & (m > 0) \end{cases} \tag{41}$$

and these polynomials have the following identity:

$$\sum_{m=0}^{\infty} X_m(x)u^m = \frac{1}{1 - xu + u^2}, \tag{42}$$

where u is an indeterminate. Next let us define the polynomials $X_{m,q}(x)\,(m = 0, 1, 2, \cdots)$ by the relation:

$$X_{m,q}(x) := X_m(x) - q^{-1}X_{m-2}(x), \tag{43}$$

where we set $X_m(x) := 0$ for $m < 0$. Then we can check that

$$\int_\Omega X_{m,q}(x)d\mu_q(x) = \begin{cases} 1 & (m = 0) \\ 0 & (m > 0) \end{cases} \tag{44}$$

and

$$\sum_{m=0}^{\infty} X_{m,q}(x)u^m = \frac{1 - u^2/q}{1 - xu + u^2}. \tag{45}$$

Now we normalize the operator $T'_m = T_m/q^{\frac{m}{2}}$, then by (4)

$$\sum_{m=0}^{\infty} T'_m u^m = \frac{1 - u^2/q}{1 - T'u + u^2}. \tag{46}$$

Hence the relations (45) and (46) yield

$$T'_m = X_{m,q}(T'), \quad T_m = q^{\frac{m}{2}} X_{m,q}(T/q^{\frac{1}{2}}). \tag{47}$$

To prove our main theorems, we need the following lemmas. The next lemma describes the number of eigenvalues of T on $L^2(V(\Gamma\backslash X), m)$. See also the work in [HLW].

Lemma 5.1. *As before, let D be the set of the discrete spectra of T for $\Gamma\backslash X$ and* $\mathrm{vol}(\Gamma\backslash X)$ *the volume of* $\Gamma\backslash X$ *which is normalized so that the Haar measure of K is* 1. *Then*

$$\begin{aligned}|D| = \mathrm{vol}(\Gamma\backslash X) &+ \frac{1}{2}\left(\sigma - \mathrm{Tr}\,\Phi\left(\frac{1}{2}\right)\right)\\ &+ 1 - \sum_{|b_i|<1} 1 - \sigma\left(a + \frac{1}{q-1}\right),\end{aligned} \tag{48}$$

where b_j is as in Lemma 4.2. *In particular we have*

$$|D| \sim \mathrm{vol}(\Gamma\backslash X) \qquad \text{as } \deg A \to \infty. \tag{49}$$

Proof. To the trace formula (6) we apply the test function such that $c(0) = 1$ and $c(n) = 0$ for $n \neq 0$, from which $h(r) = 1$ ($r \in \mathbb{C}$) and $k(0) = 1$. Then we obtain

$$\begin{aligned}|D| = \mathrm{vol}(\Gamma\backslash X) &+ \frac{1}{2}\left(\sigma - \mathrm{Tr}\,\Phi\left(\frac{1}{2}\right)\right)\\ &+ \frac{1}{4\pi}\int_{-\frac{\pi}{\log q}}^{\frac{\pi}{\log q}} \frac{\varphi'}{\varphi}\left(\frac{1}{2} + ir\right) dr - \sigma\left(a + \frac{1}{q-1}\right).\end{aligned}$$

When $\varphi(s)$ is written as in (5), it follows by computations that

$$\begin{aligned}\frac{1}{4\pi}\int_{-\frac{\pi}{\log q}}^{\frac{\pi}{\log q}} \frac{\varphi'}{\varphi}\left(\frac{1}{2} + ir\right) dr &= \sum_{|a_i|<1} 1 - \sum_{|b_i|<1} 1\\ &= 1 - \sum_{|b_i|<1} 1\end{aligned}$$

since Lemma 4.2. Thus we get (48).

Next by the result in [L1] we find

$$\mathrm{Tr}\,\Phi\left(\frac{1}{2}\right) = -|A|_\infty \prod_{\substack{B\in P(\mathbb{F}_q[t])\\ B|A}} \left(1 + \frac{1}{|B|_\infty}\right) \tag{50}$$

(see [Na]). By this, (16) and Lemma 4.3, we get (49). □

Lemma 5.2. *Let* $\Gamma(A)\,(A \in \mathbb{F}_q[t])$ *be a principal congruence subgroup of* $PGL(2, \mathbb{F}_q[t])$. *We assume* $\deg A = a \geq 1$, *then*

$$\min\{\deg P \mid P \in \Gamma(A) \textit{ is hyperbolic}\} \geq 2a.$$

Proof. Let P is a hyperbolic element of $\Gamma(A)$. By simple argument it follows that $N(P) = |\mathrm{Tr}\, P|_\infty^2$. For $P \in \Gamma(A)$ we have $\mathrm{Tr}\, P \equiv 2\,(\mathrm{mod} A)$, and hence $|\mathrm{Tr}\, P|_\infty \geq q^a$ since P is hyperbolic. Thus $\deg P = \log_q N(P) \geq 2a$. □

Combing the above results, we can establish the following theorems about the limit distribution of eigenvalues of T' for infinite Ramanujan diagrams $\Gamma(A)\backslash X$.

Theorem 5.1. *Let q be fixed. Then for any sequence of polynomials $\{A_i\}$ ($i = 1, 2, \cdots; A_i \in \mathbb{F}_q[t]$) such that $\deg A_i \to \infty$ as $i \to \infty$, the nontrivial discrete spectra D_i' of $T' = T/\sqrt{q}$ for $\Gamma(A_i)\backslash X_{q+1}$ are equidistributed with respect to the measure $\mu_q(x)$ on $\Omega = [-2, 2]$. That is, let $C(\Omega)$ be the set of continuous functions on Ω, then for any $f(x) \in C(\Omega)$ the following holds:*

$$\lim_{i\to\infty} \frac{1}{|D_i'|} \sum_{\lambda \in D_i'} f(\lambda) = \int_\Omega f(x)\,d\mu_q(x). \tag{51}$$

Proof. The space spanned by the set of polynomials $\{X_{m,q}(x)\}$ ($m = 0, 1, 2, \cdots$) is dense in $C(\Omega)$, so it suffices to check that $f(x) = X_{m,q}(x)$ satisfies (51) for each m.

When $m = 0$, (51) holds since $X_{m,q}(x) = 1$ and (44). Next let $m \geq 1$ be fixed. For $\Gamma(A)\backslash X$, taking $f(x) = X_{m,q}(x)$ gives us that

$$\begin{aligned}\sum_{\lambda \in D'} f(\lambda) &= \mathrm{Tr}\, X_{m,q}(T') - X_{m,q}\left(\frac{q+1}{\sqrt{q}}\right) - X_{m,q}\left(-\frac{q+1}{\sqrt{q}}\right) \\ &= q^{-\frac{m}{2}}\mathrm{Tr}\, T_m - X_{m,q}\left(\frac{q+1}{\sqrt{q}}\right) - X_{m,q}\left(-\frac{q+1}{\sqrt{q}}\right)\end{aligned}$$

by (47). Now it follows from Lemma 5.2 that for each n, N_n tends to 0 as $\deg A \to \infty$. Using this, Proposition 3.1 (ii), (16), (36), (37), Lemma 4.3, (49) and (50), we obtain that $\frac{1}{|D_i'|}\sum_{\lambda \in D_i'} f(\lambda) \to 0$ as $i \to \infty$. This and (44) complete the proof. □

Theorem 5.2. *For any sequence of couples $\{q_i, A_i\}$ ($i = 1, 2, \cdots; A_i \in \mathbb{F}_{q_i}[t]$) such that $q_i \to \infty$ and $\deg A_i \to \infty$ as $i \to \infty$, the nontrivial discrete spectra D_i' of $T' = T/\sqrt{q_i}$ for $\Gamma(A_i)\backslash X_{q_i+1}$ are equidistributed with respect to the measure $\mu_\infty(x)$ on $\Omega = [-2, 2]$. That is, for any $f(x) \in C(\Omega)$ the following holds:*

$$\lim_{i\to\infty} \frac{1}{|D_i'|} \sum_{\lambda \in D_i'} f(\lambda) = \int_\Omega f(x)\,d\mu_\infty(x). \tag{52}$$

Proof. We use the same method as in the proof of Theorem 5.1. Since the space spanned by the set of polynomials $\{X_m(x)\}$ is dense in $C(\Omega)$, it suffices to check that for each m, $f(x) = X_m(x)$ satisfies (52).

When $m = 0$ we have (52) since $X_m(x) = 1$. Next for fixed $m \geq 1$ we have

$$\sum_{\lambda \in D'} f(\lambda) = \mathrm{Tr}\, X_m(T') - X_m\left(\frac{q+1}{\sqrt{q}}\right) - X_m\left(-\frac{q+1}{\sqrt{q}}\right). \tag{53}$$

Now using Proposition 3.1 (ii), (43) and (47), we obtain by induction that

$$\begin{aligned}
&\text{if } m \text{ is odd,} \quad q^{\frac{m}{2}} \mathrm{Tr}\, X_m(T') = \sum_{0 \leq k < \frac{m}{2}} q^k N_{m-2k}\,; \\
&\text{if } m \text{ is even,} \quad q^{\frac{m}{2}} \mathrm{Tr}\, X_m(T') = \sum_{0 \leq k < \frac{m}{2}} q^k N_{m-2k} \\
&\qquad + |D| - (\mathrm{vol}(\Gamma\backslash X) - |D|)(q^{\frac{m}{2}} - 1) \\
&\qquad + \left(\sigma - \mathrm{Tr}\,\Phi\left(\frac{1}{2}\right)\right)\frac{m}{2} q^{\frac{m}{2}} \\
&\qquad - 2q^{\frac{m}{2}} \sum_{|b_j|<1} \sum_{1 \leq n \leq \frac{m}{2}} b_j^n \\
&\qquad + 2q^{\frac{m}{2}} \sum_{|b_j|>1} \sum_{1 \leq n \leq \frac{m}{2}} b_j^{-n}.
\end{aligned} \tag{54}$$

For $\Gamma(A)\backslash X$ it follows from Lemma 5.2 that for each n, N_n tends to 0 as $\deg A \to \infty$, which holds independently of q. According to this, (16), (36), (37), the proof of Lemma 4.3, Lemma 5.1, (50), (53) and (54), we have that $\frac{1}{|D_i'|}\sum_{\lambda \in D_i'} f(\lambda) \to 0$ as $i \to \infty$. By (41) we have the assertion. □

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