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Difference Formula for Jacobi Functions and the Calderón Identity

by

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§1. Introduction

We briefly recall the continuous wavelet transform and the corresponding Calderón identity. Let $H^2(\mathbf{R})$ denote the subspace of $L^2(\mathbf{R})$ consisting of all L^2 functions f on $\mathbf{R}$ such that $\hat{f}(\lambda) = 0$ if $\lambda < 0$. We fix $\psi \in L^1(\mathbf{R})$ satisfying the so-called admissibility condition:

$$c_\psi = \int_0^\infty \frac{|\hat{\psi}(\lambda)|^2}{\lambda} d\lambda < \infty,$$

and we define the wavelet transform W_ψ on $H^2(\mathbf{R})$ by

$$(W_\psi f)(t,\xi) = \int_{-\infty}^\infty f(x) e^{-t/2} \bar{\psi}(e^{-t}x - \xi) dx \quad (t, \xi \in \mathbf{R}). \tag{1}$$

Then for any $f \in H^2(\mathbf{R})$,

$$f(x) = \frac{1}{c_\psi} \int_{-\infty}^\infty \int_{-\infty}^\infty (W_\psi f)(t,\xi) e^{-t/2} \psi(e^{-t}x - \xi) dt d\xi. \tag{2}$$

This inversion formula is equivalent to the so-called Calderón identity: For any $f \in H^2(\mathbf{R})$,

$$f = \int_0^\infty Q_t \circ Q_t^*(f) \frac{dt}{t}, \tag{3}$$

where

$$Q_t(f) = f * \psi_t, \quad \psi_t(x) = \frac{1}{t}\psi\left(\frac{x}{t}\right)$$

and Q_t^* is the adjoint operator of Q_t (cf. [M, p.16]). The formal proof of (2) (or (3)) is carried out by taking Fourier transforms at both sides and the precise one can be fond in [FJW, Theorem 1.2].

In [GMP] Grossmann-Morelt-Paul pointed out a group-theoretical interpretation of the wavelet transform. Let G_0 be the $\tilde{A}\tilde{N}$-group consisting of all matrices of the form

$$\tilde{a}_t\tilde{n}_\xi = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{t/2} & \xi e^{t/2} \\ 0 & e^{-t/2} \end{pmatrix} \quad (u, \xi \in \mathbf{R}).$$

Then $dtd\xi$ is a left invariant Haar measure on G_0 and, if we define

$$\left(T(\tilde{a}_t\tilde{n}_\xi)f\right)(x) = e^{-t/2}f(e^{-t}x - \xi), \quad f \in H^2(\mathbf{R}),$$

$(T, H^2(\mathbf{R}))$ is an irreducible unitary representation of G_0. In this scheme, we can rewrite (1) and (2) as

$$\left(W_\psi f\right)(t, \xi) = \langle f, T(a_t n_\xi)\psi\rangle$$

and

$$f(x) = \frac{1}{c_\psi}\int_{\tilde{A}}\int_{\tilde{N}}\langle f, T(\tilde{a}_t\tilde{n}_\xi)\psi\rangle \left(T(\tilde{a}_t\tilde{n}_\xi)\psi\right)(x)dtd\xi, \tag{4}$$

where $\langle\cdot,\cdot\rangle$ is the inner product of $L^2(\mathbf{R})$. This formula (4) means that T is a square-integrable representation of G_0 in the sence that the matrix coefficient $\langle f, T(a_t n_\xi)\psi\rangle$ is square-integrable on G_0, so the theory of continuous wavelet transforms is based on the one of square-integrable representations of locally compact groups. We here note that $G_0 = \tilde{A}\tilde{N}$ is a subgroup of $\tilde{G} = SL(2, \mathbf{R})$ and T is a restriction of the limit of holomorphic discrete series $\tilde{T}_{1/2}$ of $\tilde{G}$ to G_0 (see (21)):

$$\langle f, T(\tilde{a}_t\tilde{n}_\xi)\psi\rangle = \langle f, \tilde{T}_{1/2}(\tilde{a}_t\tilde{n}_\xi)\psi\rangle$$

and, as a representation of $\tilde{G}$, $\tilde{T}_{1/2}$ is not square-integrable.

As I said in the first paragraph, we can prove (4) by taking Fourier transforms at both sides, that is, by using Fourier analysis on $\tilde{N} \cong \mathbf{R}$. The aim of this paper is to replace the Fourier analysis on $\mathbf{R}$ by Fourier series on $\mathbf{T}$. Actually, this replacement is done by changing $\tilde{G} = SL(2, \mathbf{R})$ to $G = SU(1, 1)$, $(\tilde{T}_{1/2}, H^2(\mathbf{R}))$ to $(T_{1/2}, H^2(\mathbf{T}))$, and $\tilde{N}$ to the maximal compact subgroup $K \cong \mathbf{T}$ of G (see §4). As a consequence, we expect a wavelet transform associated to the KA-subset of G: For any $f \in H^2(\mathbf{T})$,

$$f(x) = c\int_K\int_A\langle f, T_{1/2}(k_\theta a_t)\psi\rangle \left(T_{1/2}(k_\theta a_t)\psi\right)(x)D(t)d\theta dt, \tag{5}$$

where $dg = D(t)d\theta dt\theta'$ is a Haar measure on $G = KAK$. If this formula (5) is true, the matrix coefficients $\langle f, T_{1/2}(k_\theta a_t)\psi\rangle$ are square-integrable on KA and they satisfy orthogonality relations. However, since $T_{1/2}$ is not square-integrable on G, we have some difficulties: How to find ψ for which $\langle f, T_{1/2}(k_\theta a_t)\psi\rangle$ is square-integrable on KA with respect to $D(t)d\theta dt$? How about their orthogonality relations? In order to construct a ψ satisfying these properties we shall condsider differences of matrix coefficients of $T_{1/2}$ (see (19), (16), and (17)). Then, these differences are square-integrable on KA and they satisfy some orthogonality relations. Hence, as we expected, we have the KA-wavelet transform (5) on $H^2(\mathbf{T})$ modulo a finite dimensional subspace.

The organization of this paper is the following. In §2 we define Jacobi polynomials (and functions) and we obtain some integrals. We define the desired differences of Jacoi functions in §3. The difference formulas and their orthogonality relations are given by Theorems 1, 2, and 3. In §4 we recall the representation theory of $SU(1,1)$ and the isomorphic group $SL(2,\mathbf{R})$. We write down matrix coefficients of the holomorphic discrete series and the limit of holomorphic discrete by using Jacobi polynomials. In §5 we generalize the Calderón identity (3) on $\mathbf{R}$. Especially, we introduce the one on $\mathbf{T}$ and we obtain some identities on $H^2(\mathbf{T})$ associated to the differences of Jacobi polynomials (see Theorem 5). Finally, in §6 we give a group-theoretical interpretation of the Calderón identities obtained in §5. The identity on $H^2(\mathbf{R})$ yields the AN-wavelets transform (4) and the one on $H^2(\mathbf{T})$ does the desired KA-wavelet transform (5) modulo a finite dimensional subspace of $H^2(\mathbf{T})$ (see Theorem 7).

This type of transforms is also studied in [AV1,2] for the case of $SO(3,1)$.

§2. Notation.

We define the classical Jacobi polynomial $G_n(\alpha,\gamma;x)$ ($n = 0,1,2,\ldots,\gamma \neq -1,-2,\ldots,-n+1$) by

$$\begin{aligned} G_n(\alpha,\gamma;x) &= {}_2F_1(-n,\alpha+n;\gamma;x) \\ &= \frac{\Gamma(\gamma)}{\Gamma(\gamma+n)} x^{1-\gamma}(1-x)^{\gamma-\alpha}\frac{d^n}{dx^n}\left(x^{\gamma+n-1}(1-x)^{\alpha+n-\gamma}\right), \quad (6) \end{aligned}$$

where ${}_2F_1$ is the hypergeometric function, and we put

$$R_n^{(\alpha,\beta)}(x) = G_n\left(\rho, \alpha+1, \frac{1-x}{2}\right) = {}_2F_1\left(-n, n+\rho; \alpha+1; \frac{1-x}{2}\right), \qquad (7)$$

where $\rho = \alpha+\beta+1$. If $\alpha, \beta > -1$, then $R_n^{(\alpha,\beta)}(x)$ is a polynomial of degree n satisfying $R_n^{(\alpha,\beta)}(1) = 1$ and

$$R_n^{(\alpha,\beta)}(-1) = \frac{\Gamma(\alpha+1)\,\Gamma(-\beta)}{\Gamma(\alpha+n+1)\,\Gamma(-n-\beta)}. \qquad (8)$$

Their orthogonality relations are given by

$$\int_{-1}^{1} R_m^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx = \delta_{mn} A_n^{(\alpha,\beta)}, \qquad (9)$$

where

$$\begin{aligned} A_n^{(\alpha,\beta)} &= \frac{2^\rho\,\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \\ &\times \frac{\Gamma(n+\rho)\,\Gamma(\alpha+1)\,\Gamma(\beta+n+1)\,\Gamma(n+1)\,\Gamma(\rho+1)}{\Gamma(2n+\rho)\,\Gamma(\alpha+n+1)\,\Gamma(\beta+1)\,\Gamma(n+\rho+1)} \end{aligned}$$

(cf. [FK2, §2]). Moreover, doing an integration by parts, from (6) we have

$$\int_{-1}^{1} x^k R_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx = 0 \ (0 \le k \le n-1) \qquad (10)$$

and if $\beta > 0$,

$$\begin{aligned} I_n^{(\alpha,\beta)} &= \int_{-1}^{1} R_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^{\beta-1} dx \\ &= 2^{\alpha+\beta}(-1)^n \frac{\Gamma(\alpha+1)\,\Gamma(n+1)\,\Gamma(\beta)}{\Gamma(n+\rho)}. \qquad (11) \end{aligned}$$

We define the Jacobi function $R_\mu^{(\alpha,\beta)}$ ($\mu \in \mathbf{C}$) by

$$\begin{aligned} R_\mu^{(\alpha,\beta)}(x) &= {}_2F_1(-\mu, \mu+\rho; \alpha+1; (1-x)/2) \qquad (12) \\ &= C_\mu^{(\alpha,\beta)} \sum_{n=0}^{\infty} D_{\mu,n}^{(\alpha,\beta)} \left(\frac{1-x}{2}\right)^n, \end{aligned}$$

where

$$
\begin{aligned}
C_\mu^{(\alpha,\beta)} &= \frac{\Gamma(\alpha+1)}{\Gamma(-\mu)\,\Gamma(\mu+\rho)}, \\
D_{\mu,n}^{(\alpha,\beta)} &= \frac{\Gamma(-\mu+n)\,\Gamma(\mu+\rho+n)}{\Gamma(\alpha+1+n)\,\Gamma(n+1)}.
\end{aligned} \tag{13}
$$

In what follows we normalize $R_\mu^{(\alpha,\beta)}$ as

$$
\phi_\mu^{(\alpha,\beta)}(r) = B_\mu^{(\alpha,\beta)}(1-r^2)^{(\beta+1)/2} r^\alpha R_\mu^{(\alpha,\beta)}(1-2r^2), \tag{14}
$$

where

$$
B_\mu^{(\alpha,\beta)} = \frac{1}{\Gamma(\alpha+1)}\sqrt{\frac{\Gamma(\alpha+\mu+1)\,\Gamma(\alpha+\beta+\mu+1)}{\Gamma(\mu+1)\,\Gamma(\mu+\beta+1)}}. \tag{15}
$$

§3. Difference formula for Jacobi functions.

We consider differences of $\phi_\mu^{(\alpha,\beta)}(r)$ (see (14)) with respect to α and β. Then we deduce their square-integrability and orthogonality relations.

We first prove the following formula.

Theorem 1 (*Difference Formula*). *Let* $Q(x) = \sqrt{x(x+1)}$.

(i) *If* $\alpha > 0, \beta > -1$, *and* $\mu(\mu-1) \geq 0$, *then*

$$
\begin{aligned}
& Q(\alpha+\mu)\phi_\mu^{(\alpha+1,\beta)}(r) - Q(\alpha+\beta+\mu)\phi_\mu^{(\alpha-1,\beta)}(r) \\
= \; & Q(\mu-1)\phi_{\mu-2}^{(\alpha+1,\beta+2)}(r) - Q(\beta+\mu+1)\phi_\mu^{(\alpha-1,\beta+2)}(r),
\end{aligned}
$$

(ii) *If* $\alpha > 0, \beta > -1$, *and* $\mu(\mu+1) \geq 0$, *then*

$$
\begin{aligned}
& Q(\beta+\mu+1)\phi_\mu^{(\alpha+1,\beta)}(r) - Q(\mu+1)\phi_{\mu+2}^{(\alpha-1,\beta)}(r) \\
= \; & Q(\alpha+\beta+\mu+2)\phi_\mu^{(\alpha+1,\beta+2)}(r) - Q(\alpha+\mu)\phi_\mu^{(\alpha-1,\beta+2)}(r)
\end{aligned}
$$

Proof. We shall prove (i). As for (ii) we can apply the exactly same process in (i) (cf. [K, Lemma 4.1]). For simplicity, we put

$$q_1 = Q(\alpha+\mu),\ q_2 = Q(\alpha+\beta+\mu),\ q_3 = Q(\mu-1),\ q_4 = Q(\beta+\mu+1).$$

We substitute the normalized Jacobi functions at both sides with (14) and their expansions of (12). We compare the coefficients of $(1-r^2)^{(\beta+1)/2}(-r)^{\alpha-1}$ r^{2n} in each side. Then, to obatin the desired equation, it suffices to show that

$$\begin{aligned} & q_1 B_\mu^{(\alpha+1,\beta)} C_\mu^{(\alpha+1,\beta)} D_{\mu,n-1}^{(\alpha+1,\beta)} - q_2 B_\mu^{(\alpha-1,\beta)} C_\mu^{(\alpha-1,\beta)} D_{\mu,n}^{(\alpha-1,\beta)} \\ = \ & q_3 B_{\mu-2}^{(\alpha+1,\beta+2)} C_{\mu-2}^{(\alpha+1,\beta+2)} \left(D_{\mu-2,n-1}^{(\alpha+1,\beta+2)} - D_{\mu-2,n-2}^{(\alpha+1,\beta+2)} \right) \\ & -q_4 B_\mu^{(\alpha-1,\beta+2)} C_\mu^{(\alpha-1,\beta+2)} \left(D_{\mu,n}^{(\alpha-1,\beta+2)} - D_{\mu,n-1}^{(\alpha-1,\beta+2)} \right). \end{aligned}$$

Furthermore, from (13) and (15) this equation is equivalent to

$$\begin{aligned} & q_1 \frac{\mu+\alpha+\beta+n}{\alpha+n} - q_2 \frac{-\mu+n-1}{n} \\ = \ & (-\mu+n-1) \left(\frac{(-\mu+n)(\mu+\alpha+\beta+n)}{\alpha+n} - (n-1) \right) \\ & +(\mu+\alpha+\beta+n) \left((\alpha+n-1) - \frac{(-\mu+n-1)(\mu+\rho+n)}{n} \right). \end{aligned}$$

The right side equals

$$\begin{aligned} & (-\mu+n-1) \left(\frac{(-\mu+n)(\mu+\alpha+\beta+n)}{\alpha+n} - \frac{(n-1)n}{n} \right) \\ & +(\mu+\alpha+\beta+n) \left(\frac{(\alpha+n-1)(\alpha+n)}{\alpha+n} - \frac{(-\mu+n-1)(\mu+\rho+n)}{n} \right) \\ = \ & \frac{\mu+\alpha+\beta+n}{\alpha+n} (q_1 + 2(\alpha+n)(-\mu+n-1)) \\ & - \frac{-\mu+n-1}{n} (q_2 + 2n(\mu+\alpha+\beta+n)) \\ = \ & q_1 \frac{\mu+\alpha+\beta+n}{\alpha+n} - q_2 \frac{-\mu+n-1}{n}. \end{aligned}$$

This completes the proof of (i).

We denote

$$\Delta_1\phi_\mu^{(\alpha,\beta)}(r) = Q(\alpha+\mu)\phi_\mu^{(\alpha+1,\beta)}(r) - Q(\alpha+\beta+\mu)\phi_\mu^{(\alpha-1,\beta)}(r) \tag{16}$$

and

$$\Delta_2\phi_\mu^{(\alpha,\beta)}(r) = Q(\beta+\mu+1)\phi_\mu^{(\alpha+1,\beta)}(r) - Q(\mu+1)\phi_{\mu+2}^{(\alpha-1,\beta)}(r). \tag{17}$$

Theorem 2 (*Square-integrability*). *Let* $\alpha, \beta > -1$ *and* $\mu = 0, 1, 2, \ldots$.

$$\begin{aligned}
&\text{(i)} && \int_0^1 \left(\phi_\mu^{(\alpha,\beta)}(r)\right)^2 \frac{4r}{(1-r^2)}dr = \frac{2}{(2\mu+\alpha+\beta+1)},\\
&\text{(ii)} && \int_0^1 \left(\Delta_1\phi_\mu^{(\alpha,\beta)}(r)\right)^2 \frac{4r}{(1-r^2)^2}dr = 2(2\mu+\beta+1),\\
&\text{(iii)} && \int_0^1 \left(\Delta_2\phi_\mu^{(\alpha,\beta)}(r)\right)^2 \frac{4r}{(1-r^2)^2}dr = 2(2\mu+\beta+3).
\end{aligned}$$

Proof. (i) is clear from (9) and (14). We shall prove (ii). As for (iii) we can apply the exactly same process in (ii) (cf. [K, Lemma 4.2]). As before, we put

$$d_3 = Q(\mu-1) \quad \text{and} \quad d_4 = Q(\beta+\mu+1).$$

The difference formula (i) in Theorem 1 yields that

$$\begin{aligned}
I &= \int_0^1 \left(\Delta_1\phi_\mu^{(\alpha,\beta)}(r)\right)^2 \frac{4r}{(1-r^2)^2}dr\\
&= 2^{-(\alpha+\beta+1)}\int_{-1}^1 \Big(d_3 B_{\mu-2}^{(\alpha+1,\beta+2)} R_{\mu-2}^{(\alpha+1,\beta+2)}(x) 2^{-1}(1-x)\\
&\quad - d_4 B_\mu^{(\alpha-1,\beta+2)} R_\mu^{(\alpha-1,\beta+2)}(x)\Big)^2 (1+x)^{\beta+1}(1-x)^{\alpha-1}dx.
\end{aligned}$$

We here note that

$$\begin{aligned}
&\left(R_{\mu-2}^{(\alpha+1,\beta+2)}(x)\right)^2 (1+x)^{\beta+1}(1-x)^{\alpha+1}\\
= \ & R_{\mu-2}^{(\alpha+1,\beta+2)}(x) \cdot \frac{R_{\mu-2}^{(\alpha+1,\beta+2)}(x) - R_{\mu-2}^{(\alpha+1,\beta+2)}(-1)}{(1+x)} \cdot (1+x)^{\beta+2}(1-x)^{\alpha+1}\\
&+R_{\mu-2}^{(\alpha+1,\beta+2)}(x) R_{\mu-2}^{(\alpha+1,\beta+2)}(-1) \cdot (1+x)^{\beta+1}(1-x)^{\alpha+1}.
\end{aligned}$$

Since

$$\frac{R_{\mu-2}^{(\alpha+1,\beta+2)}(x) - R_{\mu-2}^{(\alpha+1,\beta+2)}(-1)}{(1+x)}$$

is a polynomial of degree $< \mu - 2$ and $R_{\mu-2}^{(\alpha+1,\beta+2)}(x)$ satisfies (10) and (11), we have

$$\int_{-1}^{1} \left(R_{\mu-2}^{(\alpha+1,\beta+2)}(x)\right)^2 (1+x)^{\beta+1}(1-x)^{\alpha+1} dx = R_{\mu-2}^{(\alpha+1,\beta+2)}(-1) I_{\mu-2}^{(\alpha+1,\beta+2)}.$$

Similarly,

$$\begin{aligned} & \int_{-1}^{1} R_{\mu-2}^{(\alpha+1,\beta+2)}(x) R_{\mu}^{(\alpha-1,\beta+2)}(x)(1+x)^{\beta+1}(1-x)^{\alpha} dx \\ = \; & R_{\mu-2}^{(\alpha+1,\beta+2)}(-1) \cdot I_{\mu}^{(\alpha-1,\beta+2)} \end{aligned}$$

and

$$\int_{-1}^{1} \left(R_{\mu}^{(\alpha-1,\beta+2)}(x)\right)^2 (1+x)^{\beta+1}(1-x)^{\alpha-1} dx = R_{\mu}^{(\alpha-1,\beta+2)}(-1) I_{\mu}^{(\alpha-1,\beta+2)}.$$

Thereby,

$$\begin{aligned} I &= 2^{-(\alpha+\beta+3)} \Big(\left(d_3 B_{\mu-2}^{(\alpha+1,\beta+2)}\right)^2 R_{\mu-2}^{(\alpha+1,\beta+2)}(-1) \cdot I_{\mu-2}^{(\alpha+1,\beta+2)} \\ &\quad -8 d_3 d_4 B_{\mu-2}^{(\alpha+1,\beta+2)} B_{\mu}^{(\alpha-1,\beta+2)} R_{\mu-2}^{(\alpha+1,\beta+2)}(-1) \cdot I_{\mu}^{(\alpha-1,\beta+2)} \\ &\quad +4 \left(d_4 B_{\mu}^{(\alpha-1,\beta+2)}\right)^2 R_{\mu}^{(\alpha-1,\beta+2)}(-1) \cdot I_{\mu}^{(\alpha-1,\beta+2)} \Big) \\ &= 2^{-(\alpha+\beta+3)} d_3 B_{\mu-2}^{(\alpha+1,\beta+2)} R_{\mu-2}^{(\alpha+1,\beta+2)}(-1) \\ &\quad \times \left(d_3 B_{\mu-2}^{(\alpha+1,\beta+2)} I_{\mu-2}^{(\alpha+1,\beta+2)} - 4 d_4 B_{\mu}^{(\alpha-1,\beta+2)} I_{\mu}^{(\alpha-1,\beta+2)} \right) \\ &\quad +2^{-(\alpha+\beta+1)} d_4 B_{\mu}^{(\alpha-1,\beta+2)} I_{\mu}^{(\alpha-1,\beta+2)} \\ &\quad \times \left(-d_3 B_{\mu-2}^{(\alpha+1,\beta+2)} R_{\mu-2}^{(\alpha+1,\beta+2)}(-1) + d_4 B_{\mu}^{(\alpha-1,\beta+2)} R_{\mu}^{(\alpha-1,\beta+2)}(-1) \right) \\ &= I_1 + I_2. \end{aligned}$$

After doing a tedious calculation with (8), (11), and (15), we have

$$I_1 = 0 \quad \text{and} \quad I_2 = 2(2\mu + \beta + 1).$$

This completes the proof of (ii).

Theorem 3 (*Orthogonality*). *Let* $\alpha, \alpha', \beta > -1$ *and* $\mu, \mu' = 0, 1, 2, \ldots$.

(i) If $\mu \neq \mu'$, then

$$\int_0^1 \phi_\mu^{(\alpha,\beta)}(r)\phi_{\mu'}^{(\alpha,\beta)}(r)\frac{4r}{(1-r^2)}dr = 0$$

(ii) *If* $|\mu - \mu'| \geq 2$ *and* $\alpha + \mu = \alpha' + \mu'$, *then*

$$\int_0^1 \Delta_i\phi_\mu^{(\alpha,\beta)}(r)\Delta_i\phi_{\mu'}^{(\alpha',\beta)}(r)\frac{2r}{(1-r^2)^2}dr = 0 \quad (i = 1, 2).$$

Proof. (i) is clear from (9) and (14). We shall prove the case of $i = 1$ in (ii). As for $i = 2$ we leave it to the readers (cf. [K, Lemma 4.3]). We may suppose that $\mu > \mu'$ ($\alpha' > \alpha$), and we first form the integral in (ii) as

$$\int_{-1}^1 \left(c_1 R_\mu^{(\alpha+1,\beta)}(x)(1-x) - c_2 R_\mu^{(\alpha-1,\beta)}(x)\right) \cdot P_{\mu'}(x)$$
$$\times(1+x)^{\beta-1}(1-x)^{(\alpha+\alpha')/2-1}dx,$$

where

$$\begin{aligned} P_{\mu'}(x) &= (1+x)^{-(\beta-1)/2}(1-x)^{-(\alpha'-1)/2}\Delta_1\phi_{\mu'}^{(\alpha',\beta)}(\sqrt{(1-x)/2}) \\ &= c_1' R_{\mu'}^{(\alpha'+1,\beta)}(x)(1-x) - c_2' R_{\mu'}^{(\alpha'-1,\beta)}(x). \end{aligned}$$

The difference formula (i) in Theorem 1 yields that

$$P_{\mu'}(x)(1+x)^{-1}$$

is a polynomial with degree μ'. Since

$$\begin{aligned} & R_\mu^{(\alpha+1,\beta)}(x)(1-x) \cdot P_{\mu'}(x) \cdot (1+x)^{\beta-1}(1-x)^{(\alpha+\alpha')/2-1} \\ = \; & R_\mu^{(\alpha+1,\beta)}(x)\left(P_{\mu'}(x)(1+x)^{-1} \cdot (1-x)^{(\alpha'-\alpha)/2-1}\right)(1+x)^\beta(1-x)^{(\alpha+1)} \end{aligned}$$

and

$$\frac{\alpha'-\alpha}{2} - 1 = \frac{\mu-\mu'}{2} - 1 \geq 0, \quad \mu' + \frac{\alpha'-\alpha}{2} - 1 = \frac{\mu+\mu'}{2} - 1 < \mu$$

by the assumption on μ, μ', we obtain from (10) that

$$\int_{-1}^{1} R_\mu^{(\alpha+1,\beta)}(x)(1-x)\cdot P_{\mu'}(x)\cdot(1+x)^{\beta-1}(1-x)^{(\alpha+\alpha')/2-1}dx=0.$$

Similarly, since

$$\begin{aligned} & R_\mu^{(\alpha-1,\beta)}(x)\cdot P_{\mu'}(x)\cdot(1+x)^{\beta-1}(1-x)^{(\alpha+\alpha')/2-1} \\ = \ & R_\mu^{(\alpha-1,\beta)}(x)\left(P_{\mu'}(x)(1+x)^{-1}\cdot(1-x)^{(\alpha'-\alpha)/2}\right)(1+x)^\beta(1-x)^{(\alpha-1)} \end{aligned}$$

and

$$\mu'+\frac{\alpha'-\alpha}{2}=\frac{\mu+\mu'}{2}<\mu,$$

we have

$$\int_{-1}^{1} R_\mu^{(\alpha-1,\beta)}(x)\cdot P_{\mu'}(x)(1+x)^{\beta-1}(1-x)^{(\alpha+\alpha')/2-1}dx=0.$$

This completes the proof of (ii).

§4. $SU(1,1)$ and $SL(2,\mathbf{R})$.

We briefly recall the representation theory of $SU(1,1)$ and the isomorphic group $SL(2,\mathbf{R})$. Especially, we define the holomorphic discrete series T_h ($h\in\mathbf{Z}/2, h>1/2$) and the limit of holomorphic discrete series $T_{1/2}$, and then we give explicit forms of their matrix coefficients (see [Sa] and [Su] for general references). Then we understand that the normalization (14) of $R_\mu^{(\alpha,\beta)}$ is based on the matrix coefficients of T_h (see (19)).

Let $G=SU(1,1)$, the subgroup of $GL(2,\mathbf{C})$ consisting of all matrices of the form

$$g=\begin{pmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{pmatrix}, |\alpha|^2-|\beta|^2=1.$$

Then

$$G=KAN \quad \text{and} \quad G=KAK,$$

where

$$K=\{k_\theta=\begin{pmatrix}e^{i\theta/2} & 0\\ 0 & e^{-i\theta/2}\end{pmatrix}; 0\le\theta<4\pi\},$$

$$A = \{a_t = \begin{pmatrix} \cosh t/2 & \sinh t/2 \\ \sinh t/2 & \cosh t/2 \end{pmatrix}; t \in \mathbf{R}\},$$

$$N = \{n_\xi = \begin{pmatrix} 1 + i\xi/2 & -i\xi/2 \\ i\xi/2 & 1 - i\xi/2 \end{pmatrix}; \xi \in \mathbf{R}\}.$$

We denote the Haar measures K, A, and N by $dk_\theta = (1/4\pi)d\theta$, $da_t = dt$, and $dn_\xi = d\xi$ respectively, where $d\theta, dt$, and $d\xi$ are Lebesgue measures. Then a Haar measure dg on G is given by

$$dg = \frac{1}{2\pi} e^{2s} dk_\psi da_s dn_\xi = D(t) dk_\theta da_t dk_{\theta'}, \tag{18}$$

where $0 \le \psi < 4\pi, s, \xi \in \mathbf{R}$; $0 \le \theta, \theta' < 4\pi, t > 0$, and

$$D(t)dt = \sinh t dt = \frac{4r}{(1-r^2)^2} dr \quad (r = \tanh t/2).$$

Let $\mathcal{H}_h(D)$ $(h > 1/2)$ denote the weighted Bergman space on the unit disk D;

$$\mathcal{H}_h(D) = \{F : D \to \mathbf{C}; F \text{ is holomprphic on } D \text{ and}$$
$$\|F\|_h^2 = \Gamma(2h-1)^{-1} \int_D |F(z)|^2 (1 - |z|^2)^{2(h-1)} dz < \infty\},$$

and $\mathcal{H}_{1/2}(D)$ the H^2 Hardy space on D with norm $\|F\|_{1/2}^2 = \lim_{h \to 1/2} \|F\|_h^2$. By taking the boundary value functions on the unit circle $\mathbf{T}$, $\mathcal{H}_{1/2}(D)$ coincides with the H^2 Hardy space $H^2(\mathbf{T})$ on $\mathbf{T}$ with L^2-norm. We denote by $\langle \cdot, \cdot \rangle_h$ the inner product of $\mathcal{H}_h(D)$. Then an orthonormal basis $\{e_n^h; n = 0, 1, 2, \ldots\}$ is given by

$$e_n^h(z) = \left(\frac{\Gamma(2h+n)}{\Gamma(2h)\Gamma(n+1)} \right)^{1/2} z^n.$$

Let $h \in \mathbf{Z}/2$ and $h \ge 1/2$. For any $g \in G$, we define the operator $T_h(g)$ on $\mathcal{H}_h(D)$ by

$$(T_h(g)F)(z) = (\bar\beta + \bar\alpha)^{-2h} F\left(\frac{\alpha z + \beta}{\bar\beta \zeta + \bar\alpha} \right), \ g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}.$$

Then $(T_h, \mathcal{H}_h(D))$ is an irreducible unitary representation of G; the holpmorphic discrete sereis for $h > 1/2$ and the limit of holomorphic discrete series for $h = 1/2$. According to the above orthonormal basis, matrix coefficients $\langle T_h(g)e_n^h, e_m^h\rangle_h$ $(g \in G)$ of T_h are explicitly given as follows: For $n \geq m$,

$$\begin{aligned} & \langle T_h(g)e_n^h, e_m^h\rangle_h \\ = \ & e^{-i(m\theta+n\theta')}\langle T_h(a_t)e_n^h, e_m^h\rangle_h \\ = \ & e^{-i(m\theta+n\theta')}\frac{1}{(n-m)!}\sqrt{\frac{,\ (n+1),\ (n+2h)}{,\ (m+1),\ (m+2h)}} \\ & \times(1-r^2)^h(-r)^{n-m}{}_2F_1(-m, n+2h; n-m+1; r^2), \end{aligned}$$

where $g = k_\theta a_t k_{\theta'}$, $t > 0$, and $r = \tanh t/2$. For $m > n$, we replace n and m by m and n respectively. Hence,

$$\langle T_h(g)e_n^h, e_m^h\rangle_h = e^{-i(m\theta+n\theta')}\phi_m^{(n-m,2h-1)}(-r) \quad (n \geq m). \tag{19}$$

It is easy to see that

$$|\langle T_h(a_t)e_n^h, e_m^h\rangle_h| \leq |\phi_m^{(n-m,2h-1)}(-r)| \leq 1, \tag{20}$$

and $\langle T_h(g)e_n^h, e_m^h\rangle_h$ is square-integrable on G, equivalently, $\phi_m^{(n-m,2h-1)}(r)$ is square-integrable on $(0,1)$ with respect to $4r/(1-r^2)^2dr$, if and only if $h > 1/2$. Furthermore, (19) and (20) are also valid for $h \in \mathbf{R}$ and $h \geq 1/2$, because of the analytic continuation of the irreducible representations of the universal covering group of $SU(1,1)$.

The Cayley transform

$$\Phi : G = SU(1,1) \to \tilde{G} = SL(2,\mathbf{R}),$$

$$\Phi(g) = C^{-1}gC, \quad C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},$$

gives an isomorphism between G and $\tilde{G}$. We put

$$\tilde{A} = \Phi(A),\ \tilde{a}_t = \Phi(a_t) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$$

and

$$\tilde{N} = \Phi(N),\ \tilde{n}_\xi = \Phi(n_\xi) = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}.$$

Let $h \in \mathbf{Z}/2$ and $h \geq 1/2$. For each function f on D we define a function $E_h(f)$ on the upper half-plane $\mathbf{C}^+$ by

$$(E_h(f))(z) = \sqrt{\pi}2^{-(2h-1)}(z+i)^{-2h}f(\Psi^{-1}(z)),$$

where

$$\Psi : D \to \mathbf{C}^+, \quad \Psi(z) = C(z) = i\frac{1+z}{1-z},$$

is a complex analytic diffeomorphism of D onto $\mathbf{C}^+$. If we put $\mathcal{H}_h(\mathbf{C}^+) = E_h(\mathcal{H}_h(D))$, then $\mathcal{H}_h(\mathbf{C}^+)$ $(h > 1/2)$ is the weighted Bergman space on $\mathbf{C}^+$ defined by

$$\mathcal{H}_h(\mathbf{C}^+) = \{F : \mathbf{C}^+ \to \mathbf{C}; F \text{ is holomprphic on } \mathbf{C}^+ \text{ and}$$

$$\|F\|_h^2 = ,\ (2h-1)^{-1}\int_{\mathbf{C}^+}|F(z)|^2 y^{2(h-1)}dxdy < \infty\},$$

and $\mathcal{H}_{1/2}(\mathbf{C}^+)$ the H^2 Hardy space on $\mathbf{C}^+$ with norm $\|F\|_{1/2}^2 = \lim_{h\to 1/2}\|F\|_h^2$. By taking the boundary value functions on $\mathbf{R}$, $\mathcal{H}_{1/2}(\mathbf{C}^+)$ coincides with the H^2 Hardy space $H^2(\mathbf{R})$ on $\mathbf{R}$ with L^2-norm. Here we put

$$\tilde{T}_h(g) = E_h \circ T_h(\Phi^{-1}(g)) \circ E_h^{-1} \quad (g \in \tilde{G}).$$

Then $(\tilde{T}_h, \mathcal{H}_h(\mathbf{C}^+))$ is an irreducible unitary representation of $\tilde{G}$. Actually, the operator $\tilde{T}_h(g)$ on $\mathcal{H}_h(\mathbf{C}^+)$ is given by

$$(\tilde{T}_h(g)F)(z) = (cz+d)^{-2h}F\left(\frac{az+b}{cz+d}\right), \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{21}$$

Clearly $\{E_h(e_n^h); n = 0, 1, 2, \ldots\}$ is an orthonormal basis of $\mathcal{H}_h(\mathbf{C}^+)$ and

$$\langle \tilde{T}_h(g)E_h(e_n^h), E_h(e_m^h)\rangle_h = \langle T_h(\Phi^{-1}(g))e_n^h, e_m^h\rangle_h \quad (g \in \tilde{G}). \tag{22}$$

§5. Calderón identity.

We formulate the Calderón identity on subspaces of $L^2(\mathbf{R})$ and $L^2(\mathbf{T})$. Especially, we introduce some identities associated to the differences of Jacobi polynomials.

Let (S, dm) be a pair of a subset S of $\mathbf{R}$ and a positive measure dm on S, and W a measurable subset of $\mathbf{R}$. Let $\tilde{Q}_t$ $(t \in S)$ denote a Fourier multiplier defined by

$$\left(\tilde{Q}_t(f)\right)^{\wedge}(\lambda) = \tilde{q}_t(\lambda)\hat{f}(\lambda)$$

and suppose that for $\lambda \in \tilde{W}$,

$$|\tilde{q}_t(\lambda)| \leq C \quad \text{and} \quad \int_S |\tilde{q}_t(\lambda)|^2 dm(t) = 1, \tag{23}$$

where C does not depend on t. Let $L^2_{\tilde{W}}(\mathbf{R})$ denote the subspace of $L^2(\mathbf{R})$ consisting of all L^2 function f on $\mathbf{R}$ such that $\mathrm{supp}(f) \subset \tilde{W}$. Then for any $f \in L^2_{\tilde{W}}(\mathbf{R})$,

$$f = \int_S \tilde{Q}_t \circ \tilde{Q}_t^*(f) dm(t), \tag{24}$$

where $\tilde{Q}_t^*$ is the adjoint operator of $\tilde{Q}_t$. The formal proof follows by taking Fourier transforms at both sides;

$$\int_S (\tilde{Q}_t \circ \tilde{Q}_t^*(f))^{\wedge}(\lambda) dm(t) = \hat{f}(\lambda) \int_S |\tilde{q}_t(\lambda)|^2 dm(t) = \hat{f}(\lambda) \quad (\lambda \in \tilde{W}),$$

and the precise one follows as in [FJW, Thorem 1.2]. As an example of (24), we have (3):

Remark 4 (*Calder'on identity on* $\mathbf{R}$). We take $(S, dm) = ((0, \infty), dt/t)$ and $\tilde{W} = (0, \infty)$. We fix $\psi \in L^1(\mathbf{R})$ satisfying

$$\int_0^\infty |\hat{\psi}(\lambda)|^2 \frac{d\lambda}{\lambda} = 1 \tag{25}$$

and we define

$$\tilde{Q}_t(f) = f * \psi_t, \quad \psi_t(x) = \frac{1}{t}\psi\left(\frac{x}{t}\right) \quad (t > 0).$$

Since $\tilde{q}_t(\lambda) = \hat{\psi}_t(\lambda) = \hat{\psi}(t\lambda)$, it follows that

$$|\tilde{q}_t(\lambda)| \leq \|\psi\|_1 \quad \text{and} \quad \int_S |\tilde{q}_t(\lambda)|^2 dm(t) = \int_0^\infty |\hat{\psi}(t)|^2 \frac{dt}{t} = 1.$$

Therefore, $\tilde{q}_t$ satisfies (23), and the Calderón identity (3) on $H^2(\mathbf{R}) = L^2_{\tilde{W}}(\mathbf{R})$ follows from (24).

Next we shall consider the case of $\mathbf{T}$. Let (S, dm) be as above, and W a subset of $\mathbf{Z}$. Let Q_t denote a Fourier multiplier defined by

$$(Q_t(f))^\wedge(n) = q_t(n)\hat{f}(n)$$

and suppose that for $n \in W$,

$$|q_t(n)| \leq C \quad \text{and} \quad \int_S |q_t(n)|^2 dm(t) = 1, \tag{26}$$

where C does not depend on t. Let $L^2_W(\mathbf{T})$ denote the subspace of $L^2(\mathbf{T})$ consisting of all L^2 function f on $\mathbf{T}$ such that $\mathrm{supp}(f) \subset W$. Then for any $f \in L^2_W(\mathbf{T})$,

$$f = \int_S Q_t \circ Q_t^*(f) dm(t), \tag{27}$$

where Q_t^* is the adjoint operator of Q_t. The formal proof follows by taking Fourier series at both sides;

$$\int_S (Q_t \circ Q_t^*(f))^\wedge(n) dm(t) = \hat{f}(n) \int_S |q_t(n)|^2 dm(t) = \hat{f}(n) \quad (n \in W),$$

and the precise one follows as in [FJW, Theorem 1.2].

Now, as an example of (27), we introduce some identities associated to the differences of Jacobi polynomials (see (16) and (17)). We fix $m \in \mathbf{N}$ and $\beta > -1$. For $t > 0$ and $n \geq m$, we put

$$\begin{aligned} q^{\beta,1}_{m,t}(n) &= \frac{1}{\sqrt{2(2m+\beta+1)}} \Delta_1 \phi_m^{(n-m-1,\beta)}(t) \\ q^{\beta,2}_{m,t}(n) &= \frac{1}{\sqrt{2(2m+\beta+3)}} \Delta_2 \phi_m^{(n-m-1,\beta)}(t), \end{aligned}$$

and $Q^{\beta,i}_{m,t}$ $(i = 1, 2)$ the corresponding Fourier multipliers. We take $(S, dm) = ((0, 1), 4t/(1-t^2)^2 dt)$ and $W_m = \{n; n \geq m\}$. If we put $H^2_m(\mathbf{T}) = L^2_{W_m}(\mathbf{T})$, then $H^2_m(\mathbf{T})$ is the subspace of $L^2(\mathbf{T})$ consisting of all L^2 function f on $\mathbf{T}$

such that $\hat{f}(n) = 0$ if $n < m$. Then, (20) and Theorem 2 imply that each $q^{\beta,i}_{m,t}$ satisfies (26) and hence, (27) yields the following.

Theorem 5 (*Calderón identities on* $\mathbf{T}$). *Let* $m \in \mathbf{N}$ *and* $\beta > -1$. *Then, we have the following identities on* $H^2_m(\mathbf{T})$:

$$Id = \int_0^1 Q^{\beta,i}_{m,r} \circ (Q^{\beta,i}_{m,r})^* \frac{4r}{(1-r^2)^2} dr \quad (i = 1, 2).$$

§6. ***AN*- and *KA*-wavelets**. We give a group-theoretical interpretation of the Calderón identities ontained in §5.

AN-wavelets: Let ψ be as in Remark 4 and put

$$\psi_{\xi,t}(x) = \psi_t(x-\xi) = \frac{1}{t}\psi\left(\frac{x-\xi}{t}\right).$$

Since

$$\left(\tilde{Q}^*_t(f)\right)(x) = \langle f, \psi_{x,t}\rangle,$$

we can rewrite the Calderón identity (3) as

$$f(x) = \int_0^\infty \int_{-\infty}^\infty \langle f, \psi_{\xi,t}\rangle \psi_{\xi,t}(x) \frac{d\xi dt}{t}.$$

In the scheme of the representation $(\tilde{T}_{1/2}, H^2(\mathbf{T}))$ of $\tilde{G}$ (see §4), it follows that

$$\psi_{e^t\xi,e^t}(x) = e^{-t/2}\left(\tilde{T}_{1/2}(\tilde{a}_t\tilde{n}_\xi)\psi\right)(x)$$

and thereby, for any $f \in H^2(\mathbf{R})$,

$$f(x) = \int_{\tilde{A}}\int_{\tilde{N}} \langle f, \tilde{T}_{1/2}(\tilde{a}_t\tilde{n}_\xi)\psi\rangle_{1/2} \left(\tilde{T}_{1/2}(\tilde{a}_t\tilde{n}_\xi)\psi\right)(x) dt d\xi. \tag{28}$$

This gives a group theoretical interpretation of the Calderón identity (3), simultaneously, the continuous wavelet transform (2) (see [GMP]).

Remark 6. As we noted in §4, we can deduce the corresponding formula on $G = SU(1,1)$. Let ψ be in $L^1(\mathbf{T})$ and suppose that $E_{1/2}(\psi)$ satisfies (25).

Then for any $f \in H^2(\mathbf{T})$,

$$f = \int_A \int_N \langle f, T_{1/2}(a_t n_\xi)\psi\rangle_{1/2} T_{1/2}(a_t n_\xi)\psi dt d\xi. \tag{29}$$

KA-wavelets: We shall consider the case of $i = 2$ in Theorem 5. Let $\beta = 0$ $(h = 1/2)$ and $r = \tanh t$. Since

$$\begin{aligned}
& \Delta_2 \phi_m^{(n-m-1,0)}(r) e^{in\theta} \\
= & \left(Q(m+1)\phi_m^{(n-m,0)}(r) - Q(m+1)\phi_{m+2}^{(n-m-2,0)}(r) \right) e^{in\theta} \\
= & Q(m+1)\langle T_{1/2}(a_{-t}k_{-\theta})e_n^{1/2}, e_m^{1/2}\rangle_{1/2} \\
& -Q(m+1)\langle T_{1/2}(a_{-t}k_{-\theta})e_n^{1/2}, e_{m+2}^{1/2}\rangle_{1/2} \\
= & \sqrt{(m+1)(m+2)}\langle T_{1/2}(k_\theta a_t)e_n^{1/2}, e_m^{1/2} - e_{m+2}^{1/2}\rangle_{1/2} \\
= & \langle e_n^{1/2}, T_{1/2}(k_\theta a_t)\left(\sqrt{(m+1)(m+2)}(e_m^{1/2} - e_{m+2}^{1/2}) \right)\rangle_{1/2},
\end{aligned}$$

we see that for any $f = \sum_{n=m}^\infty a_n e^{in\theta} \in H_m^2(\mathbf{T})$,

$$\begin{aligned}
\left((Q_{m,t}^{0,2})^*(f) \right)(\theta) &= \sum_{n=m}^\infty a_n \frac{1}{\sqrt{2(2m+3)}} \Delta_2 \phi_m^{(n-m-1,0)}(r) e^{in\theta} \\
&= \langle f, T_{1/2}(k_\theta a_t)\psi_m\rangle_{1/2},
\end{aligned}$$

where

$$\psi_m = \sqrt{\frac{(m+1)(m+2)}{2(2m+3)}} \left(e_m^{1/2} - e_{m+2}^{1/2} \right).$$

Therefore, the identity in Theorem 5 implies that for any $f \in H_m^2(\mathbf{T})$,

$$f = \int_K \int_A \langle f, T_{1/2}(k_\theta a_t)\psi_m\rangle_{1/2} T_{1/2}(k_\theta a_t)\psi_m D(t) d\theta dt. \tag{30}$$

Then (30) and the orthogonality relations (ii) in Theorem 3 yield the following (see [K, Theorem 4.4]).

Theorem 7 (*KA-wavelet transform*). *Let*

$$\psi = \sum c_m \psi_m,$$

where the sum is taken over $0 \leq m \leq M, m \in 2\mathbf{N}$ *or* $0 \leq m \leq M, m \in 2\mathbf{N}+1$, *and let* $\|\psi\|_0^2 = \sum |c_m|^2$. *Then for any* f *in the* L^2*-span of* $\{e_n^{1/2}, n \geq M+1\}$,

$$f = \frac{1}{\|\psi\|_0} \int_K \int_A \langle f, T_{1/2}(k_\theta a_t)\psi\rangle_{1/2} T_{1/2}(k_\theta a_t)\psi D(t) d\theta dt.$$

Theorem 8. *Let* ψ *be as above. Then for any* $f \in H^2(\mathbf{T})$,

$$f = \frac{1}{\|\psi\|_0} \int_A \int_N \langle f, T_{1/2}(a_t n_\xi)\psi\rangle_{1/2} T_{1/2}(a_t n_\xi)\psi dt d\xi.$$

Proof. As noted in Remark 6, it suffices to show that

$$\int_0^\infty |E_{1/2}(\psi)^\wedge(\lambda)|^2 \frac{d\lambda}{\lambda} = \|\psi\|_0^2. \tag{31}$$

Proposition 9. *Let notation be as above. Then,*

$$\int_0^\infty E_{1/2}(\psi_m)^\wedge(\lambda) E_{1/2}(\psi_{m'})^\wedge(\lambda) \frac{d\lambda}{\lambda} = \begin{cases} 0 & \text{if } |m-m'| \geq 2, \\ 1 & \text{if } m = m'. \end{cases}$$

Proof. Let $\beta = 2h-1 \geq 0$ and recall that

$$E_h(e_n^h)^\wedge(\lambda) = \sqrt{2}\sqrt{\frac{\Gamma(n+1)}{\Gamma(n+1+\beta)}}(-1)^n e^{-\lambda}(2\lambda)^\beta L_n^{(\beta)}(2\lambda)$$

(cf. [Sa, p.79]). Here $L_n^{(\beta)}$ is the Laguerre polynomial defined by

$$L_n^{(\beta)}(x) = \frac{e^x x^{-\beta}}{n!} \frac{d^n}{dx^n}\left(e^{-x} x^{n+\beta}\right), \tag{32}$$

and we note that

$$L_n^{(\beta)}(0) = \frac{\Gamma(n+1+\beta)}{\Gamma(n+1)\Gamma(\beta+1)}.$$

Since $L_n^{(0)}(0) = 1$, it is easy to see that, if $|m - m'| \geq 2$, say $m \geq 2 + m'$, then

$$\frac{e^{\lambda}\lambda^{-\beta}E_{1/2}(\psi_{m'})\hat{\ }(\lambda)}{\lambda}$$

is a polynomial of degree $m' + 1 < m$. Therefore, the desired integral must be 0 by the orhogonality relations of the Laguerre polynomials.

Let $m = m'$ and $\beta > 0$. We define

$$\psi_m^h = \sqrt{\frac{(m+1)(m+2)}{2(2m+3)}}\left(e_m^h - e_{m+2}^h\right).$$

Then (32) and the similar argument used in the proof of Theorem 2 yield that

$$\begin{aligned}
& \int_0^\infty |E_h(\psi_m^h)\hat{\ }(\lambda)|^2 \frac{1}{(2\lambda)^\beta \lambda} d\lambda \\
= \ & \frac{(m+1)(m+2)}{2(2m+3)} \cdot 2 \int_0^\infty e^{-\lambda}\lambda^{\beta-1} \\
\times \ & \left(\sqrt{\frac{\Gamma(m+1)}{\Gamma(m+1+\beta)}} L_m^{(\beta)}(\lambda) - \sqrt{\frac{\Gamma(m+3)}{\Gamma(m+3+\beta)}} L_{m+2}^{(\beta)}(\lambda)\right)^2 d\lambda \\
= \ & \frac{(m+1)(m+2)}{(2m+3)} \Gamma(\beta) \\
\times \ & \left(\frac{\Gamma(m+1)}{\Gamma(m+1+\beta)} L_m^{(\beta)}(0) - 2\sqrt{\frac{\Gamma(m+1)}{\Gamma(m+1+\beta)}}\sqrt{\frac{\Gamma(m+3)}{\Gamma(m+3+\beta)}} L_m^{(\beta)}(0)\right. \\
+ \ & \left.\frac{\Gamma(m+3)}{\Gamma(m+3+\beta)} L_{m+2}^{(\beta)}(0)\right) \\
= \ & \frac{(m+1)(m+2)}{(2m+3)} \left(2 - 2\sqrt{\frac{(m+1)(m+2)}{(m+1+\beta)(m+2+\beta)}}\right) \frac{1}{\beta}.
\end{aligned}$$

Therefore, letting $\beta \to 0$ $(h \to 1/2)$, we have

$$\int_0^\infty |E_{1/2}(\psi_m)\hat{\ }(\lambda)|^2 \frac{d\lambda}{\lambda} = 1.$$

This completes the proof of Proposition 9.

(31) follows from the definition of ψ and Proposition 9.

Remark 10. (1) Noting the proofs of Theorem 2 (ii) and Proposition 9, we have an integral formula between Jacobi and Laguerre polynomials: If $\beta = 2h - 1 > 0$ and $m = 0, 1, 2, \ldots$, then

$$\begin{aligned} & \int_0^1 \phi_m^{(\alpha,\beta)}(r)\phi_{m+2}^{(\alpha,\beta)}(r)\frac{4r}{(1-r^2)^2}dr \\ = & \int_0^\infty E_h(e_m^h)^\wedge(\lambda)E_h(e_{m+2}^h)^\wedge(\lambda)\frac{1}{(2\lambda)^\beta\lambda}d\lambda. \end{aligned}$$

Actually, the both sides are equal to

$$\frac{2}{\beta}\sqrt{\frac{(m+1)(m+2)}{(m+1+\beta)(m+2+\beta)}}.$$

(2) The formulas in Theorems 7 and 8 yield that the integrals over KA and AN coincides for a suitable function f on $\mathbf{T}$. Without using the orthogonality relations of the Jacobi and Laguerre polynomials, is it possible to deduce the coincidence? If ψ has a single K-type, then the coincidence is trivial from the Iwasawa and Cartan decompositions of G, however, in our case the K-type of ψ is not single to obatin the square-integrability in Theorem 2.

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