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Transcendence of the Values of Certain Series with Hadamard's Gaps

by

Taka-aki Tanaka

Taka-aki Tanaka
Department of Mathematics
Keio University

Department of Mathematics
Faculty of Science and Technology
Keio University

3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522 Japan

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Abstract

Transcendence of the number $\sum_{k=0}^{\infty} \alpha^{r_k}$, where α is an algebraic number with $0 < |\alpha| < 1$ and $\{r_k\}_{k\geq 0}$ is a sequence of positive integers such that $\lim_{k\to\infty} r_{k+1}/r_k = d \in \boldsymbol{N} \setminus \{1\}$, is proved by Mahler's method. This result implies the transcendence of the number $\sum_{k=0}^{\infty} \alpha^{kd^k}$.

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1 Introduction.

Let $f(z) = \sum_{k=0}^{\infty} z^{e_k}$ be a power series in the complex variable z with a strictly increasing sequence $\{e_k\}_{k\geq 0}$ of exponents. From the Hadamard's gap theorem, if $\liminf_{k\to\infty} e_{k+1}/e_k > 1$, then $f(z)$ has the unit circle $|z| = 1$ as a natural boundary. The transcendence of the value $f(\alpha)$ of such a series at a nonzero algebraic number α inside the unit circle has been investigated by various authors. In 1844, Liouville proved the transcendency of $\sum_{k=0}^{\infty} 2^{-k!}$, the first example of a transcendental number. For the case of $\limsup_{k\to\infty} e_{k+1}/e_k = \infty$, there were some results on the transcendence of $f(\alpha)$, which are included in the result of Cijsouw and Tijdeman [1]. On the other hand, only special sequences $\{e_k\}_{k\geq 0}$ have been treated in the remaining case of $\limsup_{k\to\infty} e_{k+1}/e_k < \infty$. Let d be an integer greater than 1. In 1929, Mahler [3] proved that, if $e_k = d^k$, $f(\alpha)$ is transcendental. Mahler's method was generalized by Loxton and van der Poorten [2], who proved the transcendence of $f(\alpha)$ when $\{e_{k+1}/e_k\}_{k\geq 0}$ is a sequence of integers greater than 1. However, for the case that $\lim_{k\to\infty} e_{k+1}/e_k = d$ and $\{e_{k+1}/e_k\}_{k\geq 0}$ is not necessarily a sequence of integers, for example $e_k = kd^k$, no transcendence result had been known. In this paper we prove the transcendence of $f(\alpha)$ under these conditions.

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Theorem 1. *Let $\{r_k\}_{k\geq 0}$ be a sequence of positive integers such that $\lim_{k\to\infty} r_{k+1}/r_k = d$, where d is an integer greater than 1. Suppose that there exists a positive number M such that $r_{k+1} \geq dr_k - M$ for all $k \geq 0$. Let*

$$f(z) = \sum_{k=0}^{\infty} z^{r_k}$$

and let α be an algebraic number with $0 < |\alpha| < 1$. Then the number $f(\alpha)$ is transcendental.

Example. Let α be an algebraic number with $0 < |\alpha| < 1$ and d an integer greater than 1. Then the numbers

$$(1) \qquad \sum_{k=0}^{\infty} \alpha^{kd^k}, \quad \sum_{k=0}^{\infty} \alpha^{2kd^k+(-d)^k}, \quad \sum_{k=0}^{\infty} \alpha^{[\omega d^k+\eta]}, \quad \text{and} \quad \sum_{k=1}^{\infty} \alpha^{k\cdot\binom{2k}{k}}$$

are transcendental, where $\omega > 0$, $\eta \geq 0$, $[x]$ denotes the largest integer not exceeding a real number x, and $\binom{m}{n}$ is the binomial coefficient.

Applying Mahler's method, we proved in [5] the transcendence of the number $\sum_{k=0}^{\infty} \alpha^{a_k}$ generated by a linear recurrence $\{a_k\}_{k\geq 0}$ of nonnegative integers with $a_k = g\rho^k + o(\rho^k)$, where $g > 0$ and $\rho > 1$, under some additional conditions. However, the transcendence of the first two numbers in (1) cannot be deduced from our result in [5] although the sequences of their exponents are linear recurrences.

Theorem 1 can be deduced from Theorem 2 below. We prepare the notation for stating the theorem. For any algebraic number α, we denote by $\overline{|\alpha|}$ the maximum of the absolute values of the conjugates of α and by $\mathrm{den}(\alpha)$ the smallest positive integer such that $\mathrm{den}(\alpha)\cdot\alpha$ is an algebraic integer. It is easily seen that $\overline{|\alpha+\beta|} \leq \overline{|\alpha|} + \overline{|\beta|}$ and $\overline{|\alpha\beta|} \leq \overline{|\alpha|}\,\overline{|\beta|}$ for any algebraic numbers α and β. Furthermore, for any algebraic number α, we define

$$\|\alpha\| = \max\{\overline{|\alpha|}, \mathrm{den}(\alpha)\}.$$

Then for any $\alpha \neq 0$ we have the inequalities

$$(2) \qquad \log|\alpha| \geq -2[\boldsymbol{Q}(\alpha) : \boldsymbol{Q}] \log\|\alpha\|$$

and

$$(3) \qquad \log\left\|\alpha^{-1}\right\| \leq 2[\boldsymbol{Q}(\alpha) : \boldsymbol{Q}] \log\|\alpha\|$$

(cf. [4, Lemma 2.10.2]).

Let K be an algebraic number field. We denote by $K[[z]]$ the ring of formal power series in the variable z with coefficients in K. Let

$$f_k(z) = \sum_{l=0}^{\infty} \sigma_l^{(k)} z^l \in K[[z]] \qquad (k \geq 0)$$

and let $\alpha \in K$ with $0 < |\alpha| < 1$. In what follows, $c_1, c_2, \ldots$ denote positive constants independent of k and depending only on $f_k(z)$ $(k \geq 0)$ and α, and if they may depend also on parameters x as well as y, they will be denoted by $c_1(x), c_2(x, y), \ldots$. Let $\{r_k\}_{k \geq 0}$ be a sequence of positive integers with the following properties:

(I) $r_k \to \infty$ as k tends to infinity;

(II) $f_k(\alpha^{r_k}) = a_k f_0(\alpha) + b_k$ $(k \geq 1)$, where $a_k, b_k \in K$ and

$$\log \|a_k\|, \log \|b_k\| \leq c_1 r_k;$$

(III) for any $\varepsilon > 0$ and for any $l \geq 0$, there exists a constant $c_2(\varepsilon, l) > 0$ such that

$$\log \left\| \sigma_l^{(k)} \right\| \leq \varepsilon r_k (1 + l)$$

for all $k \geq c_2(\varepsilon, l)$;

(IV) for any $\varepsilon > 0$ there exists a constant $c_3(\varepsilon) > 0$ such that

$$\log |\sigma_l^{(k)}| \leq \varepsilon r_k (1 + l)$$

for all $k \geq c_3(\varepsilon)$ and for any $l \geq 0$.

Let $s_0, s_1, \ldots$ be variables and put $F(z; s) = \sum_{l=0}^{\infty} s_l z^l$. Then $F(z; \sigma^{(k)}) = f_k(z)$ $(k \geq 0)$. We assume that

(V) if $P_0(z; s), \ldots, P_p(z; s)$ are polynomials in z and $\{s_l\}_{l \geq 0}$ with degrees at most p in z and coefficients in K and if we put

$$E(z; s) = \sum_{j=0}^{p} P_j(z; s) F(z; s)^j = \sum_{l=0}^{\infty} R_l(s) z^l,$$

then there exists a positive integer $I(p)$, independent of k and depending only on $F(z; s)$ and p, with the following property. If k is sufficiently large and $P_0(z; \sigma^{(k)}), \ldots, P_p(z; \sigma^{(k)})$ are not all zero, then there is an l such that $l \leq I(p)$ and $R_l(\sigma^{(k)}) \neq 0$.

Theorem 2. *If the properties* (I) – (V) *are satisfied, then the number* $f_0(\alpha)$ *is transcendental.*

REMARK. If the constant $c_2(\varepsilon, l)$ in the property (III) does not depend on l, then the property (IV) is satisfied by the property (III). This is the very case that Loxton and van der Poorten [2] dealt with.

2 Proof of the theorems.

Proof of Theorem 1. We may assume that $r_0 = 1$, replacing $r_0, r_1, r_2, \ldots$ by $1, r_0, r_1, \ldots$ if necessary. Define

$$f_k(z) = \sum_{h=0}^{\infty} \alpha^{r_{h+k} - r_k d^h} z^{d^h} \qquad (k \geq 0).$$

Then

$$\sigma_l^{(k)} = \begin{cases} \alpha^{r_{h+k} - r_k d^h} & (l = d^h) \\ 0 & (\text{otherwise}) \end{cases} \tag{4}$$

and $f_0(\alpha) = \sum_{h=0}^{\infty} \alpha^{r_h} = f(\alpha)$, which is transcendental by Theorem 2 if the properties (I) – (V) are satisfied.

The sequence $\{r_k\}_{k \geq 0}$ obviously has the property (I). Let $K = \boldsymbol{Q}(\alpha)$. Then $f_k(z) \in K[[z]]$ $(k \geq 0)$ and

$$f_k(\alpha^{r_k}) = \sum_{h=0}^{\infty} \alpha^{r_{h+k}} = f_0(\alpha) - \sum_{h=0}^{k-1} \alpha^{r_h}.$$

Since $r_{k+1} > r_k$ for all sufficiently large k by the assumption, there is a constant $C \geq 1$ such that $\max_{0 \leq h \leq k-1} r_h \leq C r_k$ for all $k \geq 1$. Hence

$$\log \left\| - \sum_{h=0}^{k-1} \alpha^{r_h} \right\| \leq \log k + \left(\max_{0 \leq h \leq k-1} r_h \right) \log \|\alpha\| \leq c_1 r_k,$$

and the property (II) is satisfied.

Using (3), we have

$$\log \left\| \alpha^{r_{h+k} - r_k d^h} \right\| \leq 2[K : \boldsymbol{Q}] |r_{h+k} - r_k d^h| \log \|\alpha\| . \tag{5}$$

By (4), (5), and $\|0\| = 1$, in order to prove that the property (III) is satisfied, it suffices to show that for any $\varepsilon > 0$ and for any $h \geq 0$, there exists a constant $c_2(\varepsilon, h) > 0$ such that

$$|r_{h+k} - r_k d^h| \leq \varepsilon r_k d^h$$

for all $k \geq c_2(\varepsilon, h)$. If $h = 0$, this inequality holds for all $k \geq 0$. Since $\lim_{k\to\infty} r_{k+1}/r_k = d$, for any $\varepsilon > 0$ and for any $h \geq 1$, there exists a constant $c_2(\varepsilon, h) > 0$ such that

$$1 - \frac{\varepsilon}{(1+\varepsilon)h} < \frac{r_{k+1}}{dr_k} < 1 + \frac{\varepsilon}{(1+\varepsilon)h}$$

for all $k \geq c_2(\varepsilon, h)$. Then

$$\frac{|r_{h+k} - r_k d^h|}{r_k d^h} = \left| \frac{r_{k+h}}{dr_{k+h-1}} \cdots \frac{r_{k+1}}{dr_k} - 1 \right| \leq \sum_{m=1}^{h} h^m \left(\frac{\varepsilon}{(1+\varepsilon)h} \right)^m \leq \frac{\frac{\varepsilon}{1+\varepsilon}}{1 - \frac{\varepsilon}{1+\varepsilon}} = \varepsilon.$$

Next we prove that the property (IV) is satisfied. Since

$$\begin{aligned} r_{h+k} - r_k d^h &= (r_{k+h} - dr_{k+h-1}) + d(r_{k+h-1} - dr_{k+h-2}) + \cdots + d^{h-1}(r_{k+1} - dr_k) \\ &\geq -M(1 + d + \cdots + d^{h-1}) \end{aligned}$$

by the assumption in the theorem,

$$\log |\sigma_{d^h}^{(k)}| = (r_{h+k} - r_k d^h) \log |\alpha| \leq \frac{-M(d^h - 1)}{d-1} \log |\alpha| < -M(1 + d^h) \log |\alpha|.$$

Then for any $\varepsilon > 0$ there exists a constant $c_3(\varepsilon) > 0$ such that $\varepsilon r_k \geq -M \log |\alpha|$ for all $k \geq c_3(\varepsilon)$, and the property (IV) is fulfilled.

Finally we show that the property (V) is satisfied by the same way as in the proof of Theorem 2.10.1 in [4]. Choose a positive integer $\lambda(p)$, depending on p, such that

$$\max_{0 \leq j \leq p} \deg_z P_j(z; s) < d^{\lambda(p)}.$$

Suppose that $P_0(z; \sigma^{(k)}), \ldots, P_p(z; \sigma^{(k)})$ are not all zero and put

$$p' = p'(k) = \max\{j \mid P_j(z; \sigma^{(k)}) \neq 0\}, \qquad a = a(k) = \deg_z P_{p'}(z; \sigma^{(k)}).$$

Then

$$E(z; \sigma^{(k)}) = \sum_{j=0}^{p'} P_j(z; \sigma^{(k)}) f_k(z)^j = \sum_{l=0}^{\infty} R_l(\sigma^{(k)}) z^l.$$

We prove that $R_l(\sigma^{(k)}) \neq 0$ for some l. This can be done by choosing

$$l = a + \sum_{m=1}^{p'} d^{\lambda(p)+m}$$

and considering the d-adic expansion of the positive integer l in place of the $\{d_1, d_2, \ldots\}$-adic expansion in the proof of Theorem 2.10.1 in [4]. Since $a(k) < d^{\lambda(p)}$ and $p'(k) \leq p$ for any k, we can take $I(p) = d^{\lambda(p)+p+1}$ and the property (V) is fulfilled. Then by Theorem 2, $f(\alpha)$ is transcendental, and the proof of the theorem is completed.

We prove Theorem 2 by the method of Loxton and van der Poorten [2] and Nishioka [4].

Proof of Theorem 2. We assume on the contrary that $f_0(\alpha)$ is algebraic. We may suppose $f_0(\alpha) \in K$.

Proposition 1 (Loxton and van der Poorten [2], see also Nishioka [4, Proposition 2.9.2])**.** *Let m be a nonnegative integer. There exists an infinite subset $\Lambda(m)$ of the set $\boldsymbol{N}$ of positive integers such that for any polynomial $P(s_0, \ldots, s_m) \in K[s_0, \ldots, s_m]$ the following two properties are equivalent*:

(i) $P(\sigma_0^{(k)}, \ldots, \sigma_m^{(k)}) = 0$ *for infinitely many* $k \in \Lambda(m)$.

(ii) $P(\sigma_0^{(k)}, \ldots, \sigma_m^{(k)}) = 0$ *for all* $k \in \Lambda(m)$.

Let m be a nonnegative integer and put

$$V(m) = \{P(s_0, \ldots, s_m) \in K[s_0, \ldots, s_m] \mid P(\sigma_0^{(k)}, \ldots, \sigma_m^{(k)}) = 0 \text{ for all } k \in \Lambda(m)\}.$$

Then $V(m)$ is a prime ideal of $K[s_0, \ldots, s_m]$ by Proposition 1.

Proposition 2 (Loxton and van der Poorten [2], see also Nishioka [4, Proposition 2.9.3])**.** *For any positive integer p, there exist $p+1$ polynomials $P_0(z; s_0, \ldots, s_{p^2}), \ldots, P_p(z; s_0, \ldots, s_{p^2}) \in K[z, s_0, \ldots, s_{p^2}]$ with degrees at most p in z such that the function*

$$E_p(z; s) = \sum_{j=0}^{p} P_j(z; s_0, \ldots, s_{p^2}) F(z; s)^j = \sum_{l=0}^{\infty} R_l(s) z^l$$

has the following two properties:

(i) $R_l(s) = R_l(s_0, \ldots, s_{p^2}) \in V(p^2)$ *for all l with* $l \leq p^2$;

(ii) *there exists a positive integer* $I(p)$, *independent of* k *and depending only on* $F(z;s)$ *and* p, *such that* $\mathrm{ord}_{z=0}E_p(z;\sigma^{(k)}) \leq I(p)$ *for all sufficiently large* $k \in \Lambda(p^2)$.

Proposition 3. *For any positive integer* p *and any positive number* ε, *if* $k \geq c_4(\varepsilon, p)$, *then*

$$\log \left\| E_p(\alpha^{r_k}; \sigma^{(k)}) \right\| \leq \varepsilon r_k c_5(p) + c_6 r_k p.$$

Proof. By the property (III), $\left\|\sigma_l^{(k)}\right\| \leq e^{\varepsilon r_k(1+l)}$ for all $k \geq c_2(\varepsilon, l)$. Let $P_j(z; s_0, \ldots, s_{p^2}) = \sum_{l=0}^{p} Q_{jl}(s_0, \ldots, s_{p^2}) z^l$. Since $Q_{jl}(s_0, \ldots, s_{p^2}) \in K[s_0, \ldots, s_{p^2}]$, we have

$$\left\| Q_{jl}(\sigma_0^{(k)}, \ldots, \sigma_{p^2}^{(k)}) \right\| \leq c_7(p) e^{\varepsilon r_k c_8(p)}$$

for all $k \geq \max_{0 \leq l \leq p^2} c_2(\varepsilon, l)$. Since

$$\begin{aligned}
E_p(\alpha^{r_k}; \sigma^{(k)}) &= \sum_{j=0}^{p} P_j(\alpha^{r_k}; \sigma_0^{(k)}, \ldots, \sigma_{p^2}^{(k)}) F(\alpha^{r_k}; \sigma^{(k)})^j \\
&= \sum_{j=0}^{p} P_j(\alpha^{r_k}; \sigma_0^{(k)}, \ldots, \sigma_{p^2}^{(k)}) f_k(\alpha^{r_k})^j \\
&= \sum_{j=0}^{p} P_j(\alpha^{r_k}; \sigma_0^{(k)}, \ldots, \sigma_{p^2}^{(k)}) (a_k f_0(\alpha) + b_k)^j \\
&= \sum_{j=0}^{p} \left(\sum_{l=0}^{p} Q_{jl}(\sigma_0^{(k)}, \ldots, \sigma_{p^2}^{(k)}) \alpha^{r_k l} \right) (a_k f_0(\alpha) + b_k)^j,
\end{aligned}$$

noting that $\|\alpha^{r_k}\| \leq c_9^{r_k}$, we obtain

$$\left\| E_p(\alpha^{r_k}; \sigma^{(k)}) \right\| \leq c_{10}(p) e^{\varepsilon r_k c_{11}(p)} c_9^{r_k p} \left(e^{2c_1 r_k} (\|f_0(\alpha)\| + 1) \right)^p$$

for $k \geq \max_{0 \leq l \leq p^2} c_2(\varepsilon, l)$, which implies the proposition.

Proposition 4. *For any positive integer* p *and any positive number* ε, *there exist infinitely many* $k \in \Lambda(p^2)$ *such that* $E_p(\alpha^{r_k}; \sigma^{(k)}) \neq 0$ *and*

$$\log |E_p(\alpha^{r_k}; \sigma^{(k)})| \leq -c_7 r_k p^2 + \varepsilon r_k c_8(p).$$

Proof. In what follows, we always assume that $k \in \Lambda(p^2)$. By the property (i) of Proposition 2,

$$E_p(\alpha^{r_k}; \sigma^{(k)}) = \sum_{l > p^2} R_l(\sigma^{(k)}) \alpha^{r_k l}.$$

Let

$$n_k = \min\{l \mid R_l(\sigma^{(k)}) \neq 0\} \qquad (k \geq 0).$$

By the property (ii) of Proposition 2, there is an l such that $l \leq I(p)$ and $R_l(\sigma^{(k)}) \neq 0$ for all sufficiently large k. Hence there exists an integer N such that $n_k = N$ for infinitely many k. If $n_k = N$,

$$|E_p(\alpha^{r_k}; \sigma^{(k)}) - R_N(\sigma^{(k)})\alpha^{r_k N}| \leq \sum_{l=N+1}^{\infty} |R_l(\sigma^{(k)})\alpha^{r_k l}|. \tag{6}$$

Let

$$P_j(z; s_0, \ldots, s_{p^2}) = \sum_{l=0}^{p} Q_{jl}(s_0, \ldots, s_{p^2}) z^l, \qquad F(z;s)^j = \sum_{l=0}^{\infty} G_{jl}(s) z^l.$$

Then by the property (IV),

$$|Q_{jl}(\sigma_0^{(k)}, \ldots, \sigma_{p^2}^{(k)})| \leq c_9(p) e^{\varepsilon r_k c_{10}(p)}$$

and

$$|G_{jl}(\sigma^{(k)})| = \left| \sum_{l_1+\cdots+l_j=l} \sigma_{l_1}^{(k)} \cdots \sigma_{l_j}^{(k)} \right| \leq (l+1)^j e^{\varepsilon r_k (j+l)}$$

for $k \geq c_3(\varepsilon)$. Therefore

$$|R_l(\sigma^{(k)})| \leq c_{11}(p) e^{\varepsilon r_k c_{12}(p)} (l+1)^p e^{\varepsilon r_k (p+l)} \tag{7}$$

for $k \geq c_3(\varepsilon)$. On the other hand, noting that $N \leq I(p)$, we obtain

$$\left\| R_N(\sigma^{(k)}) \right\| \leq c_{13}(p) e^{\varepsilon r_k c_{14}(p)} \tag{8}$$

for $k \geq c_{15}(\varepsilon, p)$. By (7)

$$\begin{aligned} \log |R_l(\sigma^{(k)})\alpha^{r_k l}| &\leq \log c_{11}(p) + \varepsilon r_k c_{12}(p) + p \log(l+1) + \varepsilon r_k (p+l) + r_k l \log|\alpha| \\ &\leq \varepsilon r_k c_{16}(p) + (1 - c_{17}\varepsilon) r_k l \log|\alpha| \end{aligned}$$

if $k \geq c_{18}(\varepsilon, p)$. Choose ε so small that $1 - c_{17}\varepsilon > 0$. Then for $k \geq c_{18}(\varepsilon, p)$,

$$\sum_{l=N+1}^{\infty} |R_l(\sigma^{(k)})\alpha^{r_k l}| \leq e^{\varepsilon r_k c_{16}(p)} c_{19} e^{(1-c_{17}\varepsilon) r_k (N+1) \log|\alpha|}. \tag{9}$$

By (2), (8), and (9), if $k \geq c_{20}(\varepsilon, p)$ and $n_k = N$, then

$$\begin{aligned}
&\log \sum_{l=N+1}^{\infty} |R_l(\sigma^{(k)})\alpha^{r_k l}|/|R_N(\sigma^{(k)})\alpha^{r_k N}| \\
\leq\ & \varepsilon r_k c_{16}(p) + \log c_{19} + (1 - c_{17}\varepsilon) r_k (N+1) \log|\alpha| \\
& +2[K:\boldsymbol{Q}] \log c_{13}(p) + 2[K:\boldsymbol{Q}]\varepsilon r_k c_{14}(p) - r_k N \log|\alpha| \\
=\ & \log c_{19} + 2[K:\boldsymbol{Q}] \log c_{13}(p) \\
& + r_k \Big(\varepsilon \Big(c_{16}(p) + 2[K:\boldsymbol{Q}] c_{14}(p) - c_{17}(N+1)\log|\alpha| \Big) + \log|\alpha| \Big).
\end{aligned}$$

Noting that $N \leq I(p)$, we have

$$\varepsilon \Big(c_{16}(p) + 2[K:\boldsymbol{Q}] c_{14}(p) - c_{17}(N+1)\log|\alpha| \Big) + \log|\alpha| < 0$$

if $\varepsilon < c_{21}(p)$. Hence we have

$$\sum_{l=N+1}^{\infty} |R_l(\sigma^{(k)})\alpha^{r_k l}|/|R_N(\sigma^{(k)})\alpha^{r_k N}| \to 0 \qquad \text{as} \quad k \to \infty\ (n_k = N).$$

Therefore by (6)

$$E_p(\alpha^{r_k}; \sigma^{(k)})/R_N(\sigma^{(k)})\alpha^{r_k N} \to 1 \qquad \text{as} \quad k \to \infty\ (n_k = N).$$

Noting that $N > p^2$ and using (7), we obtain the assertions of the proposition.

Now we complete the proof of the theorem by choosing $p > 2[K:\boldsymbol{Q}]c_6/c_7$. By Proposition 3, 4, and (2), for infinitely many $k \in \Lambda(p^2)$, we have

$$\begin{aligned}
-c_7 r_k p^2 + \varepsilon r_k c_8(p) &\geq \log|E_p(\alpha^{r_k}; \sigma^{(k)})| \\
&\geq -2[K:\boldsymbol{Q}] \log \left\| E_p(\alpha^{r_k}; \sigma^{(k)}) \right\| \\
&\geq -2[K:\boldsymbol{Q}](\varepsilon r_k c_5(p) + c_6 r_k p).
\end{aligned}$$

Dividing both sides by r_k, we get

$$-c_7 p^2 + \varepsilon c_8(p) \geq -2[K:\boldsymbol{Q}](\varepsilon c_5(p) + c_6 p).$$

Letting ε tend to 0, we obtain

$$-c_7 p^2 \geq -2[K:\boldsymbol{Q}] c_6 p,$$

which contradicts the choice of p, and the proof of the theorem is completed.

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Taka-aki Tanaka
Department of Mathematics
Keio University
Hiyoshi 3–14–1, Kohoku-ku
Yokohama 223–8522
Japan

E-mail: takaaki@math.keio.ac.jp