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Prime Geodesic Theorem Via the Explicit Formula of Ψ for Hyperbolic 3-Manifolds

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PRIME GEODESIC THEOREM VIA THE EXPLICIT FORMULA OF Ψ FOR HYPERBOLIC 3-MANIFOLDS

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Running Title. Prime Geodesic Theorem for Hyperbolic 3-manifolds

Abstract. We obtain a lower bound for the error term of the prime geodesic theorem for hyperbolic 3-manifolds. Our result is $\Omega_{\pm}\left(\frac{x(\log\log x)^{\frac{1}{3}}}{\log x}\right)$. We also generalize Sarnak's upper bound $O(x^{\frac{5}{3}+\varepsilon})$ to compact manifolds.

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1. Introduction

For a $(d+1)$-dimensional hyperbolic manifold with Γ being the fundamental group, the prime geodesic theorem is

$$\pi_\Gamma(x) = \mathrm{li}(x^d) + \sum_{n=1}^{M} \mathrm{li}(x^{s_n}) + (\text{error}), \tag{1.1}$$

where $\pi_\Gamma(x)$ is the number of prime geodesics P whose length $l(P)$ satisfies that $N(P) := e^{l(P)} \le x$, and $s_1, \ldots s_M$ are the zeros of the Selberg zeta function $Z(s)$ in the interval $(0,2)$. The chief concern of this paper is to give estimates of the error term in (1.1).

Hejhal [4] obtained a lower bound in two-dimensional cases i.e. $d = 1$, by using the explicit formula for $\Psi_1(x) := \int_1^x \Psi_\Gamma(t)dt$. Here we put $\Psi_\Gamma(x) = \sum_{\substack{\{P\} \\ N(P)\le x}} \Lambda_\Gamma(P)$, where $\Lambda_\Gamma(P)$ is defined by

$$\frac{Z'}{Z}(s) = \sum_{\substack{\{P\} \\ N(P)\le x}} \frac{\Lambda_\Gamma(P)}{N(P)^s}$$

and is an analogue of the von-Mangoldt function in the theory of the Riemann zeta function. His result is as follows:

Theorem 1.1. *When $\Gamma \subset PSL(2, \mathbf{R})$ is a cocompact subgroup or a cofinite subgroup satisfying that* $\sum_{\gamma_n > 0} \frac{x^{\beta_n - \frac{1}{2}}}{\gamma_n{}^2} = O\left(\frac{1}{1+(\log x)^2}\right)$,

$$\pi_\Gamma(x) = \mathrm{li}(x) + \sum_{n=1}^{M} \mathrm{li}(x^{s_n}) + \Omega_\pm\left(\frac{x^{\frac{1}{2}}(\log\log x)^{\frac{1}{2}}}{\log x}\right) \qquad \text{as } x \to \infty,$$

where $\beta_n + i\gamma_n$ are poles of the scattering determinant.

In this paper, we generalize it to three-dimensional cases. In Sections 3 and 4, we prove the following main theorem of this paper:

Theorem 1.2. *When $\Gamma \subset PSL(2, \mathbf{C})$ is a cocompact subgroup or a cofinite subgroup satisfying that* $\sum_{\gamma_n > 0} \frac{x^{\beta_n - 1}}{\gamma_n{}^2} = O\left(\frac{1}{1+(\log x)^3}\right)$,

$$\pi_\Gamma(x) = \mathrm{li}(x^2) + \sum_{n=1}^{M} \mathrm{li}(x^{s_n}) + \Omega_\pm\left(\frac{x(\log\log x)^{\frac{1}{3}}}{\log x}\right) \qquad \text{as } x \to \infty,$$

where $\beta_n + i\gamma_n$ are poles of the scattering determinant.

Cocompact cases are dealt with in Section 3 (Theorem 3.21), and its generalization to cofinite cases under the assumption in Theorem 1.2 is given in Section 4 (Theorem 4.10). Since the order of $Z(s)$ is three, abundance of the zeros of $Z(s)$ gives rise to a difficulty concerning the estimate of $\Psi_1(x)$. We overcame it by considering the explicit formula for $\Psi_2(x) := \int_1^x \Psi_1(t)dt$. In cofinite cases, we can omit the contribution of the continuous spectra under the assumption in Theorem 1.2. We will see in Example 4.11 that any Bianchi group satisfies this assumption. The conjectural exponent of x in the error term in (1.1) is $\frac{d}{2}$. Theorems 1.1 and 1.2 give sharp estimates in that sense.

On the other hand, upper estimates of the error term in (1.1) have been studied by many people in the case of $d = 1$. For higher dimensional cases, the only known result is Sarnak's error term $O(x^{\frac{5}{3}+\varepsilon})$ in [10] for $\Gamma = PSL(2, O_K)$ with K being an imaginary quadratic field ($\neq \mathbf{Q}(i), \mathbf{Q}(\sqrt{-3})$) of class number one. (Some conditional results are obtained in [7].) We generalize Sarnak's estimate to cocompact groups in section 5 (Theorem 5.4) and to general Bianchi groups in Section 6 (Theorem 6.1) as follows.

Theorem 1.3. *When $\Gamma \subset PSL(2, \mathbf{C})$ is a cocompact subgroup or $\Gamma = PSL(2, O_K)$ with K an imaginary quadratic field,*

$$\pi_\Gamma(x) = \mathrm{li}(x^2) + \sum_{n=1}^{M} \mathrm{li}(x^{s_n}) + O(x^{\frac{5}{3}+\varepsilon})$$

as $x \to \infty$.

The proof uses the explicit formula for $\Psi_\Gamma(x)$.

2. Preliminaries

Throughout this paper we put G to be $PSL(2,\mathbf{C})$ and Γ to be a cofinite subgroup of G. Let j be an element in the quaternion field which satisfies $j^2 = -1$, $ij = -ji$, and let $\mathbf{H}$ be the three-dimensional hyperbolic space:

$$\mathbf{H} := \{v = z + yj \mid z = x_1 + x_2 i \in \mathbf{C}, y > 0\}$$

with the Riemannian metric

$$dv^2 = \frac{dx_1{}^2 + dx_2{}^2 + dy^2}{y^2}.$$

It induces a hyperbolic distance $d(v,v')$ given by

$$\cosh d(v,v') = \frac{|z-z'|^2 + y^2 + y'^2}{2yy'},$$

where $v = z + yj$ and $v' = z' + y'j$. The volume measure is given by

$$\frac{dx_1 dx_2 dy}{y^3}.$$

The group $PSL(2,\mathbf{C})$ acts on $\mathbf{H}$ transitively by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(v) := (av+b)(cv+d)^{-1} = \frac{(az+b)\overline{(cz+d)} + a\bar{c}y^2 + yj}{|cz+d|^2 + |c|^2y^2}.$$

The Laplacian for $\mathbf{H}$ is defined by

$$\Delta := -y^2\left(\frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial}{\partial y}.$$

We denote the eigenvalues of Δ by $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_M \le 1 < \lambda_{M+1} \ldots$.

Next we give the classification of conjugacy classes.

Definition 2.1. *An element $P \in \Gamma - \{1\}$ is called*

parabolic	*iff*	$\lvert\mathrm{tr}(P)\rvert = 2$	*and*	$\mathrm{tr}(P) \in \mathbf{R}$,
hyperbolic	*iff*	$\lvert\mathrm{tr}(P)\rvert > 2$	*and*	$\mathrm{tr}(P) \in \mathbf{R}$,
elliptic	*iff*	$\lvert\mathrm{tr}(P)\rvert < 2$	*and*	$\mathrm{tr}(P) \in \mathbf{R}$,

and loxodromic in all other cases. An element of $PSL(2,\mathbf{C})$ is called parabolic, elliptic, hyperbolic, loxodromic if its preimages in $SL(2,\mathbf{C})$ have this property. A conjugacy class $\{P\}$ in Γ is called hyperbolic, elliptic, parabolic if each P in the class has this property.

The norm of a hyperbolic or loxodromic element P is defined by $N(P) = |a(P)|^2$, if $a(P) \in \mathbf{C}$ is the eigenvalue of $P \in G$ such that $|a(P)| > 1$.

Definition 2.2. *An element $P \in \Gamma - \{1\}$ is called primitive iff it is not an essential power of any other element. A conjugacy class $\{P\}$ in Γ is called primitive if each P in the class has this property.*

For every hyperbolic matrix $P \in \Gamma$ there exist exactly one primitive hyperbolic element $P_0 \in \Gamma$ and exactly one $n \in \mathbf{N}$ such that $P = {P_0}^n$. We define that $\pi_\Gamma(\mathrm{x})$ is the number of P_0 which is primitive hyperbolic or loxodromic and satisfies $N(P_0) \leq x$.

Definition 2.3. *For $\mathrm{Re}(s) > 2$, the Selberg zeta function for Γ is defined by*

$$Z(s) := \prod_{\{P_0\}} \prod_{(k,l)} (1 - a(P_0)^{-2k} \overline{a(P_0)}^{-2l} N(P_0)^{-s}),$$

where the product on $\{P_0\}$ is taken over all hyperbolic or loxodromic conjugacy classes of Γ, and (k,l) runs through all the pairs of positive integers satisfying the following congruence relation: $k \equiv l \pmod{m(P_0)}$ with $m(P)$ the order of the torsion of the centralizer of P.

For the Selberg zeta function, Elstrodt, Grunewald and Mennicke proved the following Lemma.

Lemma 2.4. [2, p. 208, Lemma 4.2] *For $\mathrm{Re}(s) > 2$, we have*

$$\frac{Z'}{Z}(s) = \sum_{\{P\}} \frac{N(P) \log N(P_0)}{m(P)|a(P) - a(P)^{-1}|^2} N(P)^{-s},$$

where P_0 is a primitive element associated with P, and $\{P\}$ runs through the hyperbolic or loxodromic conjugacy classes of Γ.

Recall that the von-Mangoldt function $\Lambda(n)$ appears in the logarithmic derivative of the Riemann zeta function:

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}. \tag{2.1}$$

Comparing Lemma 2.4 and (2.1), the following definition is natural.

Definition 2.5. *For a hyperbolic or loxodromic element P of Γ, we define*

$$\Lambda_\Gamma(P) := \frac{N(P) \log N(P_0)}{m(P)|a(P) - a(P)^{-1}|^2},$$

and

$$\Psi_\Gamma(x) := \sum_{\substack{\{P\} \\ N(P) \leq x}} \Lambda_\Gamma(P),$$

where P_0 is a primitive element associated with P, and $\{P\}$ runs through hyperbolic or loxodromic classes of Γ.

Then we have

$$\frac{Z'}{Z}(s) = \sum_{\{P\}} \Lambda_\Gamma(P) N(P)^{-s}. \tag{2.2}$$

3. Ω-RESULT FOR COCOMPACT GROUPS

First, we introduce some properties of $Z(s)$ for cocompact Γ.

A determinant expression of Selberg zeta functions was discovered by Sarnak[9] and Voros[12] for compact Riemann surfaces. Koyama generalized it to 3-dimensional Bianchi groups [6, Theorem 4.4]. He expressed $Z(s)$ multiplied with some gamma factors in terms of the determinant of the Laplacian.

In our cases, since Γ is cocompact, we can omit in his formula the contribution from the parabolic classes and the continuous spectra. We introduce the spectral zeta function generalized by a variables s:

$$\zeta(w, s, \Delta) := \sum_{n=0}^{\infty} \frac{1}{(\lambda_n - s(2-s))^w} \quad \left(\mathrm{Re}(w) > \frac{3}{2}\right).$$

Then we immediately have the following theorem.

Theorem 3.1. *Let*

$$\hat{Z}(s) := Z_I(s) Z_E(s) Z(s)$$

with

$$Z_I(s) = \exp\left(-\frac{\mathrm{vol}(\Gamma\backslash\mathbf{H})}{6}(s-1)^3\right)$$

$$Z_E(s) = \exp\left(\sum_{\{R\}} \frac{\log N(P_0)}{2m(R)} \sum_{m=0}^{\nu_R - 1} \left(1 - \cos\frac{2m\pi}{\nu_R}\right)^{-1} s\right),$$

where $\{R\}$ runs through all the primitive elliptic conjugacy classes of Γ, and ν_R is the order of R. We denote by $m(R)$ the order of the maximal finite subgroup of the centralizer of R.

Then

$$\hat{Z}(s) = e^{c + c' s(2-s)} \mathrm{det}_D(\Delta - s(2-s)),$$

where det_D is the determinant of the Laplacian composed of the discrete spectra :

$$\mathrm{det}_D(\Delta - s(2-s)) := \exp\left(-\frac{\partial}{\partial\omega}\bigg|_{\omega=0} \zeta(\omega, s, \Delta)\right).$$

It is the zeta-regularization of a divergent product $\prod_{n=0}^{\infty}(\lambda_n - s(2-s))$.

For this Selberg zeta function, Elstrodt, Grunewald and Mennicke showed the following functional equation.

Lemma 3.2. [2, p. 209, Corollary 4.4] *The zeta function $Z(s)$ satisfies the functional equation*

$$Z(2-s) = \exp\left(-\frac{\mathrm{vol}(\Gamma\backslash\mathbf{H})}{3\pi}(s-1)^3 + 2E(s-1)\right) Z(s),$$

where $E := \sum_{\{R\}} \frac{\log N(P_0)}{m(R)|(\mathrm{tr}R)^2-4|}$, *and the summation of $\{R\}$ is taken over all elliptic conjugacy classes of Γ.*

Here we have the following property for $Z(s)$ from the trace formula. From Theorem 3.1, the zeros of $Z(s)$ are expressed as $s_n = 1+i\sqrt{\lambda_n - 1}$ and $\tilde{s}_n = 1 - i\sqrt{\lambda_n - 1}$. Let $t_n := \sqrt{\lambda_n - 1}$.

Proposition 3.3. *We have,*

$$\frac{Z'}{Z}(s) = \frac{1}{s-2} + \sum_{|s-s_n|<1} \frac{1}{s-s_n} + \sum_{|s-\tilde{s}_n|<1} \frac{1}{s-\tilde{s}_n} + O(|s|^2+1).$$

where $s_n = 1+it_n$ and $\tilde{s}_n = 1-it_n$ are the zeros of $Z(s)$ on $\mathrm{Re}(s) = 1$.

Proof. From Theorem 3.1, we get

$$\frac{Z'}{Z}(s) + \frac{Z_I'}{Z_I}(s) + \frac{Z_E'}{Z_E}(s) = c'(2s-2) + \frac{d}{ds}\left(\log(\det{}_D(\Delta - s(2-s)))\right), \tag{3.1}$$

together with

$$\frac{Z_I'}{Z_I}(s) = O(|s|^2+1), \tag{3.2}$$

and

$$\frac{Z_E'}{Z_E}(s) = O(1). \tag{3.3}$$

About the right hand side of (3.1), by Hadamard's theory, we have

$$\det{}_D(\Delta - s(2-s)) = e^{P(s)} \prod_{n=0}^{\infty} \left(1-\frac{s}{s_n}\right)\left(1-\frac{s}{\tilde{s}_n}\right) e^{\frac{s}{s_n}+\frac{s}{\tilde{s}_n}}, \tag{3.4}$$

where $P(s)$ is an integral function and the product is absolutely convergent for all s. Elstrodt, Grunewald and Mennicke [2, p. 215 Lemma 5.8] show the order of (3.4) is three. Then we have

$$(3.5)\quad \frac{d}{ds}(\log(\det_D(\Delta - s(2-{}^{\cdot}s))))$$
$$= O(|s|^2) + \frac{1}{s-2} + \sum_{n=0}^{\infty}\left(\frac{1}{s-s_n} + \frac{1}{s_n}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{s-\tilde{s}_n} + \frac{1}{\tilde{s}_n}\right).$$

Gathering together (3.1), (3.2), (3.3) and (3.5), we have Proposition 3.3. □

Now we need information about the distribution of the imaginary parts of the complex zeros of $Z(s)$ for cocompact Γ.

Proposition 3.4. [2, p. 215, Theorem 5.6] *Suppose that $T > 0$, $T \neq t_n$, then for all $n \geq M+1$, the counting function $N(T) := \#\{n \mid n \geq M+1,\ t_n < T\}$ satisfies*

$$N(T) = \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H})}{6\pi^2}T^3 + O(T^2) \quad \text{as} \quad T \to \infty,$$

where $\mathrm{vol}(\Gamma\backslash\mathbf{H})$ *is the volume of the fundamental domain* $\Gamma\backslash\mathbf{H}$.

From Propositions 3.3 and 3.4, we obtain the following estimates:

Lemma 3.5. *For $\varepsilon > 0$, we get*

$$(3.6)\qquad \frac{Z'}{Z}(1+\varepsilon+it) \ll \frac{|t|^2}{\varepsilon}, \quad (|t| \geq 2)$$

$$(3.7)\qquad \frac{Z'}{Z}(2+\varepsilon+it) \ll \frac{1}{\varepsilon},$$

$$(3.8)\qquad \frac{Z'}{Z}(-\varepsilon+it) \ll |t|^2 + 1,$$

$$(3.9)\qquad \left|\frac{Z'}{Z}(s)\right| \ll |t|^{2\max(0,2-\sigma)}\log|t| \ \ \left(s = \sigma+it,\ \sigma > 1+\frac{1}{\log|t|}, |t| \geq 2\right).$$

Moreover, for any T there exists τ in $[T, T+1]$ such that

$$(3.10)\qquad \int_0^2 \left|\frac{Z'}{Z}(\sigma+i\tau)\right| d\sigma \ll T^2\log T.$$

Proof. By Propositions 3.3 and 3.4, we get

$$\frac{Z'}{Z}(1+\varepsilon+it) \ll \frac{1}{\varepsilon+it-1} + \frac{t^2}{\varepsilon} + O(|t|^2).$$

This implies (3.6). Here ε is any number with $0 < \varepsilon \leq \frac{1}{2}$.

Since Definition 2.3 converges for $\mathrm{Re}(s) > 2$, we have from Proposition 3.3 that

$$\frac{Z'}{Z}(s) = \frac{1}{s-2} + O(1) \quad \text{as } s \to 2,$$

for $\mathrm{Re}(s) > 2$. It leads us to (3.7).

For proving (3.8), we again appeal to Proposition 3.3. Putting $s = -\varepsilon + it$ gives the conclusion.

Next we deduce (3.9) from (3.6) and (3.7) together with the Phragmen-Lindelöf principle:

$$\left|\frac{Z'}{Z}(s)\right| \ll |t|^{2\max(0,2-\sigma)} \log|t|$$

for $s = \sigma + it$, and $|t| \geq 2$, $\sigma > 1 + \frac{1}{\log|t|}$.

To see (3.10), we integrate the left hand side of (3.10) over τ in

$$\mathcal{T} := \{\tau \mid T < \tau < T+1,\ |\tau - t_n| \geq T^{-3}\}. \tag{3.11}$$

By Propositions 3.3, 3.4 and (3.11), we estimate

$$\begin{aligned}\int_{\mathcal{T}} \int_0^2 \left|\frac{Z'}{Z}(\sigma + i\tau)\right| d\sigma d\tau &\ll \int_0^2 \sum_{|s_n - T| \leq 2} \log T d\sigma \\ &\ll T^2 \log T.\end{aligned}$$

On the other hand, $|\mathcal{T}| = 1 + O(T^{-2}) > \frac{1}{2}$ for sufficiently large T. This proves (3.10) for sufficiently large T. For small T the assertion is trivial. □

Now, we prove the following theorem about $\Psi_\Gamma(x)$.

Theorem 3.6. *Let $\Psi_1(x) := \int_1^x \Psi_\Gamma(t)dt$. Then we have*

$$\begin{aligned}\Psi_1(x) = \alpha x + \beta x \log x + \alpha_1 + \sum_{n=0}^{M} \frac{x^{s_n+1}}{s_n(s_n+1)} + \sum_{n=0}^{M} \frac{x^{\tilde{s}_n+1}}{\tilde{s}_n(\tilde{s}_n+1)} \\ + \sum_{t_n \geq 0} \frac{x^{s_n+1}}{s_n(s_n+1)} + \sum_{t_n > 0} \frac{x^{\tilde{s}_n+1}}{\tilde{s}_n(\tilde{s}_n+1)}\end{aligned}$$

with some constants α, β and α_1, where $s_n = 1 + it_n$ and $\tilde{s}_n = 1 - it_n$ are the zeros of $Z(s)$.

For the proof of Theorem 3.6, we define $\Psi_2(x)$, and express it with $N(P)$ and $\Lambda_\Gamma(P)$.

Lemma 3.7. *Let $\Psi_2(x) = \int_1^x \Psi_1(t)dt$. We have*

$$\Psi_1(x) = \sum_{N(P) \leq x} (x - N(P))\Lambda(P)$$

$$2\Psi_2(x) = \sum_{N(P)\leq x} (x - N(P))^2 \Lambda(P).$$

The following theorem is used for the proof of Lemma 3.7.

Theorem 3.8. [5, Theorem A] *Let $\lambda_1, \lambda_2, \dots,$ be a real sequence which increases (in the wide sense) and has the limit infinity, and let*

$$C(x) = \sum_{\lambda_n \leq x} c_n,$$

where the c_n may be real or complex, and the notation indicates a summation over the (finite) set of positive integers n for which $\lambda_n \leq x$. Then, if $X \geq \lambda_1$ and $\phi(x)$ has a continuous derivative, we have

$$\sum_{\lambda_n \leq X} c_n \phi(\lambda_n) = -\int_{\lambda_1}^{X} C(x)\phi'(x)dx + C(X)\phi(X).$$

If, further, $C(X)\phi(X) \to 0$ as $X \to \infty$, then

$$\sum_{n=1}^{\infty} c_n \phi(\lambda_n) = -\int_{\lambda_1}^{\infty} C(x)\phi'(x)dx,$$

provided that either side is convergent.

Proof of Lemma 3.7. $\Psi_1(x)$ is obtained by substituting $n = P$, $\lambda_n = N(n)$, $\phi(x) = x - N(n)$ and $c_n = \Lambda(n)$ in Theorem 3.8. Similarly $\Psi_2(x)$ is also obtained by putting $n = P$, $\lambda_n = N(n)$, $\phi(x) = (x - N(n))^2$ and $c_n = \Lambda(n)$ in Theorem 3.8. □

We express $\Psi_2(x)$ with $Z(s)$ by using the following fact.

Theorem 3.9. [5, p. 31, Theorem B] *If k is a positive integer, $c > 0$, $y > 0$, then*

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{y^s ds}{s(s+1)\dots(s+k)} = \begin{cases} 0, & (y \leq 1), \\ \frac{1}{k!}(1 - \frac{1}{y})^k, & (y \geq 1). \end{cases}$$

From Lemma 3.7, we have

$$2\frac{\Psi_2(x)}{x^2} = \sum_{N(P)\leq x} \left(1 - \frac{N(P)}{x}\right)^2 \Lambda(P).$$

This is the case of $k = 2$ in Theorem 3.9, so we can express

$$\Psi_2(x) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s+2}}{s(s+1)(s+2)} \sum_{N(P)\leq x} \frac{\Lambda(P)}{N(P)^s} ds \quad \text{for} \quad c > 2.$$

From (2.2), we have

$$\Psi_2(x) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s+2}}{s(s+1)(s+2)} \frac{Z'}{Z}(s) ds. \tag{3.12}$$

Now we begin the proof of Theorem 3.6.

Proof of Theorem 3.6. Suppose $T \geq 1000$, and let $A := N + \frac{1}{2}$ where N is a positive integer. We have by Cauchy's theorem

$$
\begin{aligned}
(3.13)\quad & \frac{1}{2\pi i}\int_{3-iT}^{3+iT} \frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)ds \\
&= \frac{1}{2\pi i}\left(\int_{-A-iT}^{-A+iT} + \int_{-A+iT}^{3+iT} - \int_{-A-iT}^{3-iT}\right)\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)ds \\
&\qquad + \sum_{z\in R(A,T)} \mathrm{Res}_{s=z}\left(\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)\right),
\end{aligned}
$$

where

$$R(A,T) := \{z \in \mathbf{C} \mid -A \leq \mathrm{Re}(z) \leq 3, -T \leq \mathrm{Im}(z) \leq T\}.$$

We will estimate each integral in the right hand side of (3.13), which will be denoted by I_1, I_2 and I_3.

We first estimate

$$I_1 \ll \int_{-A}^{-A+i} + \int_{-A+i}^{-A+iT}.$$

We use Lemma 3.2 since $A \geq \frac{3}{2}$. Then we have

$$
\begin{aligned}
&\int_{-A+i}^{-A+iT} \frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)ds \\
&\qquad = \int_{-A+i}^{-A+iT} \frac{x^{s+2}}{s(s+1)(s+2)}\left(-\frac{Z'}{Z}(2-s) + \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H})}{\pi}(s-1)^2 - 2E\right) ds.
\end{aligned}
$$

Since $-\frac{Z'}{Z}(2-s) - 2E = O(1)$, by denoting $s = \sigma + it$, we have

$$
\begin{aligned}
&\int_{-A+i}^{-A+iT} \frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)ds \\
&\qquad = O\left(\int_1^T \frac{x^{2-A}}{t^3}dt\right) + \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H})}{\pi}\int_{-A+i}^{-A+iT}\frac{x^{s+2}}{s+2}ds = O(x^{2-A}).
\end{aligned}
$$

Since we have

$$\left|\int_{-A}^{-A+i} \frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)ds\right| \leq 1 \times \left|\frac{x^{2-A}}{A^3} \times O(A^2)\right| = O(x^{2-A}),$$

we obtain

$$I_1 = O(x^{2-A}).$$

Next, for I_2, we divide it into the following three parts,

$$\int_{-A+iT}^{3+iT} \frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)ds = \int_{-A+iT}^{-1+iT} + \int_{-1+iT}^{1+iT} + \int_{1+iT}^{3+iT}.$$

We put them to be J_1, J_2 and J_3, respectively.

By Lemma 3.2,

$$J_1 \leq \int_{-A+iT}^{-1+iT} \left| \frac{x^{s+2}}{s(s+1)(s+2)} \right| \left\{ \left| \frac{Z'}{Z}(2-s) \right| + \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H})}{\pi} |(s-1)^2| + |2E| \right\} |ds|.$$

Since $|\frac{Z'}{Z}(2-s)| \leq |\frac{Z'}{Z}(3)|$, we have

$$\begin{aligned} J_1 &\leq \left(\frac{Z'}{Z}(3)\frac{1}{T^3} + \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H})}{\pi T} + \frac{2E}{T^3} \right) \int_{-A}^{0} x^{\sigma+2} d\sigma. \\ &= O\left(\frac{x^2(1-x^{-A})}{T\log x} \right). \end{aligned}$$

For J_2, we use Lemma 3.2 again. We have

$$J_2 \leq \int_{-1+iT}^{1+iT} \frac{x^{\sigma+2}}{T^3} \left\{ \left| \frac{Z'}{Z}(2-s) \right| + \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H})}{\pi} |(s-1)^2| + |2E| \right\} |ds|.$$

Since

$$\int_{-1+iT}^{1+iT} \left| \frac{Z'}{Z}(2-s) \right| |ds| = -\int_{3-iT}^{1-iT} \left| \frac{Z'}{Z}(s) \right| |ds|,$$

we obtain

$$J_2 = O\left(\int_{1-iT}^{3-iT} \frac{x^{\sigma+2}}{T^3} \left| \frac{Z'}{Z}(s) \right| |ds| \right) + O\left(\frac{x^2(x-1)}{T\log x} \right).$$

It is obvious that

$$J_3 \leq \int_{1+iT}^{3+iT} \frac{x^{\sigma+2}}{T^3} \left| \frac{Z'}{Z}(s) \right| |ds|.$$

From (3.12), we have

$$\Psi_2(x) = \frac{1}{2\pi i} \int_{3-iT}^{3+iT} \frac{x^{s+2}}{s(s+1)(s+2)} \frac{Z'}{Z}(s) ds + O\left(\frac{x^5}{T^2} \right).$$

Equation (3.13) now becomes

(3.14)

$$\begin{aligned} \Psi_2(x) + O\left(\frac{x^5}{T^2} \right) &= O(x^{2-A}) + O\left(\frac{x^3}{T\log x} \right) + O\left(\frac{1}{T^3} \int_{1+iT}^{3+iT} x^{\sigma+2} \left| \frac{Z'}{Z}(s) \right| |ds| \right) \\ &\quad + \sum_{z \in R(A,T)} \mathrm{Res}_{s=z} \left(\frac{x^{s+2}}{s(s+1)(s+2)} \frac{Z'}{Z}(s) \right). \end{aligned}$$

It is easily seen that

$$\mathrm{Res}_{s=s_n} \left(\frac{x^{s+2}}{s(s+1)(s+2)} \frac{Z'}{Z}(s) \right) = \frac{\mu(n) x^{s_n+2}}{s_n(s_n+1)(s_n+2)} \quad (n \geq 1),$$

$$\mathrm{Res}_{s=\tilde{s}_n}\left(\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)\right)=\frac{\mu(n)x^{\tilde{s}_n+2}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)}\quad(n\geq 0),$$

where $\mu(n)$ is the multiplicity of s_n.

For calculating the residue at $s=0$, we have

$$\frac{Z'}{Z}(s)=\frac{\mathrm{vol}(\Gamma\backslash\mathbf{H})}{2}-a+2c'+\frac{1}{s}-\frac{1}{2}+\left(-\mathrm{vol}(\Gamma\backslash\mathbf{H})-2c'-\frac{1}{4}\right)s+\dots,$$

where

$$a:=\sum_R\frac{\log N(P_0)}{2m(R)}\sum_{m=0}^{\nu_R-1}\left(1-\cos\frac{2m\pi}{\nu_R}\right)^{-1},$$

by Theorem 3.1, and

$$\begin{aligned}x^{s+2}&=x^2\left(1+s\log x+\frac{(s\log x)^2}{2!}+\dots\right),\\ \frac{1}{s+1}&=1-s+s^2-\dots,\\ \frac{1}{s+2}&=\frac{1}{2}\left(1-\frac{s}{2}+\frac{s^2}{4}\dots\right).\end{aligned}$$

From these calculations, we can express

$$\mathrm{Res}_{s=0}\left(\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)\right)=\alpha_0x^2+\beta_0x^2\log x,$$

where $\alpha_0:=\frac{1}{2}\left(\frac{\mathrm{vol}(\Gamma\backslash\mathbf{H})}{2}-a+2c'-2\right)$ and $\beta_0:=\frac{1}{2}$.

For the case of $s=-1$, we can express

$$\mathrm{Res}_{s=-1}\left(\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)\right)=\alpha_1x,$$

with some constant α_1.

For the case of $s=-2$, we can express

$$\mathrm{Res}_{s=-2}\left(\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)\right)=\alpha_2,$$

with some constant α_2.

Gathering together these residues,

(3.15)

$$\sum_{z\in R(A,T)} \mathrm{Res}_{s=z}\left(\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)\right)$$
$$=\sum_{n=0}^{M}\frac{x^{s_n}+2}{s_n(s_n+1)(s_n+2)}+\sum_{n=0}^{M}\frac{x^{\tilde{s}_n}+2}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)}$$
$$+\sum_{t_n\geq 0}\frac{x^{s_n}+2}{s_n(s_n+1)(s_n+2)}+\sum_{t_n\geq 0}\frac{x^{\tilde{s}_n}+2}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)}$$
$$+\alpha_0x^2+\beta_0x^2\log x+\alpha_1x+\alpha_2.$$

Now we estimate $\int_1^3 \frac{x^{\sigma+2}}{T^3}|\frac{Z'}{Z}(\sigma+iT)|d\sigma$. From Lemma 3.5, we have

$$\int_1^3\frac{x^{\sigma+2}}{T^3}\left|\frac{Z'}{Z}(\sigma+iT)\right|d\sigma \ll \int_1^{1+\varepsilon}\frac{x^{\sigma+2}}{T^3}T^2d\sigma+\int_{1+\varepsilon}^{2}\frac{x^{\sigma+2}}{T^3}T^{2(2-\sigma)}d\sigma+\int_2^3\frac{x^{\sigma+2}}{T^3}\frac{1}{\varepsilon}d\sigma.$$

Then we get

$$\int_1^3\frac{x^{\sigma+2}}{T^3}\left|\frac{Z'}{Z}(\sigma+iT)\right| \ll \frac{x^{3+\varepsilon}}{T\log x}+\frac{\frac{x^4}{T^3}-\frac{x^{3+\varepsilon}}{T^{1+2\varepsilon}}}{\log x-2\log T}+\frac{x^4(x-1)}{T^3\log x}\ll\frac{x^5}{T(\log x-2\log T)}.$$

Now (3.14) becomes

(3.16)

$$\Psi_2(x)+O\left(\frac{x^5}{T^2}\right)=O(x^{2-A})+O\left(\frac{x^3}{T\log x}\right)+O\left(\frac{x^5}{T(\log x-2\log T)}\right)$$
$$+\sum_{n=0}^{M}\frac{x^{s_n}+2}{s_n(s_n+1)(s_n+2)}+\sum_{n=0}^{M}\frac{x^{\tilde{s}_n}+2}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)}$$
$$+\sum_{t_n\geq 0}\frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)}+\sum_{t_n\geq 0}\frac{x^{\tilde{s}_n+2}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)}$$
$$+\alpha_0x^2+\beta_0x^2\log x+\alpha_1x+\alpha_2.$$

As both A and T go to ∞ in (3.16), we obtain

(3.17)

$$\Psi_2(x)=\alpha_0x^2+\beta_0x^2\log x+\alpha_1x+\alpha_2+\sum_{n=0}^{M}\frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)}+\sum_{n=0}^{M}\frac{x^{\tilde{s}_n+2}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)}$$
$$+\sum_{t_n\geq 0}\frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)}+\sum_{t_n\geq 0}\frac{x^{\tilde{s}_n+2}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)}.$$

Recall $\Psi_2(x)=\int_1^x\Psi_1(x)$, and we obtain Theorem 3.6. □

Our next goal is to show an Ω-result for

$$P(x) := \Psi_\Gamma(x) - \left(\alpha + \beta \log x + \beta + \sum_{n=0}^{M} \frac{x^{s_n}}{s_n} + \sum_{n=0}^{M} \frac{x^{\tilde{s}_n}}{\tilde{s}_n}\right). \tag{3.18}$$

Definition 3.10. *We define*

$$P_1(x) := \int_0^x P(t)dt, \tag{3.19}$$

$$P_2(x) := \int_0^x P_1(t)dt, \tag{3.20}$$

and further

$$\begin{aligned} \mathcal{P}(x) &:= P(x) - N(0)x, \\ \mathcal{P}_1(x) &:= P_1(x) - \frac{1}{2}N(0)x^2, \\ \mathcal{P}_2(x) &:= P_2(x) - \frac{1}{6}N(0)x^3. \end{aligned} \tag{3.21}$$

We have

$$\mathcal{P}_1(x) = b_1 + \int_2^x \mathcal{P}(t)dt, \tag{3.22}$$

and

$$\mathcal{P}_2(x) = b_2 + \int_2^x \mathcal{P}_1(t)dt. \tag{3.23}$$

with constants b_1 and b_2.

Lemma 3.11. *There exists $b_3 \in \mathbf{C}$ such that*

$$b_3 + \int_1^v \frac{\mathcal{P}(e^u)}{e^u}du = \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)}.$$

Proof. Put

$$F(v) := \beta_1 + \int_1^v \frac{\mathcal{P}(e^u)}{e^u}du \quad \text{for } v \geq 1,$$

where $\beta_1 \in \mathbf{C}$ is unspecified temporarily. By changing of variables with $x = e^u$. we have

$$F(v) = \beta_1 + \int_e^{e^v} \frac{\mathcal{P}(x)}{x^2}dx.$$

Now by integration by parts and (3.22), $F(v)$ is written with a constant b_4 as

$$F(v) = \beta_1 + b_4 + \frac{\mathcal{P}(e^v)}{e^{2v}} + 2\int_e^{e^v} \frac{\mathcal{P}_1(x)}{x^3}dx.$$

We use integration by parts again and from (3.23), it follows that

$$F(v) = \beta_1 + b_5 + \frac{\mathcal{P}_1(e^v)}{e^{2v}} + 2\frac{\mathcal{P}_2(e^v)}{e^{3v}} + 6\int_e^{e^v} \frac{\mathcal{P}_2(x)}{x^4}dx \tag{3.24}$$

with some constant b_5.

Applying Theorem 3.6 leads to

$$\mathcal{P}_1(x) = \sum_{t_n>0} \frac{x^{s_n+1}}{s_n(s_n+1)} + \sum_{t_n>0} \frac{x^{\tilde{s}_n+1}}{\tilde{s}_n(\tilde{s}_n+1)}.$$

Thus

$$\frac{\mathcal{P}_1(e^v)}{e^{2v}} = \sum_{t_n>0} \left(\frac{e^{(s_n-1)v}}{s_n(s_n+1)} + \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n+1)} \right).$$

Similarly, from (3.17) and (3.23), we obtain

$$\frac{\mathcal{P}_2(e^v)}{e^{3v}} = \sum_{t_n>0} \left(\frac{x^{s_n-1}}{s_n(s_n+1)(s_n+2)} + \frac{x^{\tilde{s}_n-1}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)} \right).$$

Hence

$$\int_e^{e^v} \frac{\mathcal{P}_2(x)}{x^4}dx = \sum_{t_n>0} \left(\frac{e^{(s_n-1)v}}{s_n(s_n+1)(s_n+2)(s_n-1)} + \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)(\tilde{s}_n-1)} \right).$$

From these calculations we deduce from (3.24) that

$$\begin{aligned} F(v) &= \beta_1 + b_5 + \sum_{t_n>0} \left(\frac{e^{(s_n-1)v}}{s_n(s_n+1)} + \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n+1)} \right) \\ &\quad + 2\sum_{t_n>0} \left(\frac{e^{(s_n-1)v}}{s_n(s_n+1)(s_n+2)} + \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)} \right) \\ &\quad + 6\sum_{t_n>0} \left(\frac{e^{(s_n-1)v}}{s_n(s_n+1)(s_n+2)(s_n-1)} + \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)(\tilde{s}_n-1)} \right) \\ &= \beta_1 + b_5 + \sum_{t_n>0} \left(\frac{e^{(s_n-1)v}}{s_n(s_n-1)} + \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)} \right). \end{aligned}$$

By taking $\beta_1 = -b_5 =: b_3$, we have the lemma. □

In what follows we put

$$F(v) = b_3 + \int_1^v \frac{\mathcal{P}(e^u)}{e^u}du.$$

Lemma 3.12. *There exists $b_6 \in \mathbf{C}$ such that*

$$b_6 + \int_1^v F(u)du = \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)^2} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)^2}.$$

Proof. Put

$$G(v) := \beta_2 + \int_1^v F(u)du \quad \text{for } v \geq 1,$$

where $\beta_2 \in \mathbf{C}$ is unspecified temporarily. By doing the same as in Lemma 3.11, we obtain

$$G(v) = \beta_2 + b_7 + \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)^2} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)^2},$$

where b_7 is some constant. Choosing $\beta_2 = -b_7 := b_6$ yields the lemma. $\square$

In what follows we put

$$G(u) = b_6 + \int_1^v F(u)du.$$

Similarly, we find the following lemma.

Lemma 3.13. *There exists $b_8 \in \mathbf{C}$ such that*

$$b_8 + \int_1^v G(u)du = \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)^3} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)^3}.$$

The proof is similar to Lemma 3.12. Now in what follows we put

$$H(u) = b_8 + \int_1^v G(u)du,$$

which is uniformly convergent for all $v \in \mathbf{R}$. We can therefore extend the definition of $H(v)$ to all $\mathbf{R}$ by using the series representation.

Here we introduce the following lemma.

Lemma 3.14. *Let $k(x) = (\frac{\sin \pi x}{\pi x})^2$. Then*

a) *$k(x)$ is a C^∞-function on $\mathbf{R}$;*
b) *$k(x), k'(x), k''(x), k'''(x)$ are all $O(x^{-2})$ when $|x| \to \infty$;*
c) *$\int_{-\infty}^{\infty} k(x)e^{iux}dx = \max[0, 1 - \frac{|u|}{2\pi}]$.*

Proof. Every statement except for $k'''(x)$ in b) is proved by Hejhal [4, p. 264, Lemma 16.9]. The relevant property of $k'''(x)$ is also deduced by the same method. $\square$

Now we want to estimate

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv$$

for large values of r, A and N.

Lemma 3.15. *Let A be a positive constant. We have*

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv$$
$$= -\frac{2}{N} \sum_{0<t_n \leq 2\pi N} \frac{\sin(t_n r)}{t_n} \left(1 - \frac{t_n}{2\pi N}\right) + O\left(\frac{1}{A^3}\right) + O\left(\frac{1}{r^3}\right) + O(1).$$

Proof. For convenience, we assume A and N are integers. Using Lemma 3.11, it follows that

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv = O\left(\frac{1}{N^2 r^2}\right) - N \int_1^{r+A} F(v)k'(N(v-r))dv.$$

We integrate by parts using Lemma 3.12. Then we have

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv = O\left(\frac{1}{Nr^2}\right) + N^2 \int_1^{r+A} G(v)k''(N(v-r))dv.$$

And using Lemma 3.13 yields

(3.25)
$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv = O\left(\frac{1}{Nr^2}\right) - N^3 \int_1^{r+A} H(v)k'''(N(v-r))dv.$$

The function $H(v)$ has been defined for all $v \in \mathbf{R}$. Since $H(v)$ is uniformly bounded, we can estimate as follows:

$$N^3 \int_{r+A}^{\infty} |H(v)k'''(N(v-r))|dv = O\left(\frac{1}{A^3}\right),$$

and

$$N^3 \int_{-\infty}^{1} |H(v)k'''(N(v-r))|dv = O\left(\frac{1}{r^3}\right).$$

Therefore (3.25) becomes

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv = O\left(\frac{1}{A^3}\right) + O\left(\frac{1}{r^3}\right) - N^3 \int_{-\infty}^{\infty} H(v)k'''(N(v-r))dv.$$

Since the series representation for $H(v)$ converges uniformly on $\mathbf{R}$, we can substitute

$$H(v) = \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)^3} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)^3}$$

and commute integration and summation. So, after integrating term-by-term, we obtain

(3.26)
$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv$$
$$= O\left(\frac{1}{A^3}\right) + O\left(\frac{1}{r^3}\right) - \sum_{t_n>0} \int_{-\infty}^{\infty} \frac{1}{e^v}\left(\frac{e^{s_n v}}{s_n} + \frac{e^{\tilde{s}_n v}}{\tilde{s}_n}\right) k(N(v-r))dv.$$

Here, by considering $s_n = 1 + it_n$, $\tilde{s}_n = 1 - it_n$ and changing of variables with $X := N(v-r)$, we have

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv$$
$$= O\left(\frac{1}{A^3}\right) + O\left(\frac{1}{r^3}\right) - \frac{2}{N}\sum_{t_n>0}\int_{-\infty}^{\infty} \mathrm{Re}\left(\frac{e^{it_n r}e^{i(\frac{t_n}{N})X}}{1+it_n}\right) k(X)dX.$$

Let $\tilde{k}(u) := \int_{-\infty}^{\infty} k(x)e^{iux}dx$. It holds

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv = O\left(\frac{1}{A^3}\right) + O\left(\frac{1}{r^3}\right) - \frac{2}{N}\sum_{t_n>0} \mathrm{Re}\left(\frac{e^{it_n r}}{1+it_n}\right)\tilde{k}\left(\frac{t_n}{N}\right).$$

From Lemma 3.14 (c),

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv$$
$$= O\left(\frac{1}{A^3}\right) + O\left(\frac{1}{r^3}\right) - \frac{2}{N}\sum_{0<t_n\le 2\pi N} \frac{\cos(t_n r) + t_n \sin(t_n r)}{1+{t_n}^2}\left(1 - \frac{t_n}{2\pi N}\right).$$

By Proposition 3.4, we see

$$\sum_{0<t_n\le R} \frac{1}{|s_n|^2} = O(R).$$

Therefore,

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv$$
$$= -\frac{2}{N}\sum_{0<t_n\le 2\pi N} \frac{t_n \sin(t_n r)}{1+{t_n}^2}\left(1 - \frac{t_n}{2\pi N}\right) + O\left(\frac{1}{A^3}\right) + O\left(\frac{1}{r^3}\right) + O(1).$$

Now

$$\frac{t_n}{1+{t_n}^2} = \frac{1}{t_n} - \frac{1}{t_n({t_n}^2+1)}.$$

Hence

$$\int_1^{r+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r))dv$$
$$= -\frac{2}{N}\sum_{0<t_n\le 2\pi N} \frac{\sin(t_n r)}{t_n}\left(1 - \frac{t_n}{2\pi N}\right) + O\left(\frac{1}{A^3}\right) + O\left(\frac{1}{r^3}\right) + O(1).$$

This completes the proof of the lemma. □

Now we show the following lemma.

Lemma 3.16. *For N large, there exist some constants c_1 and c_2 which satisfy*

$$e^{c_1 N^3} \le \prod_{0<t_n\le 2\pi N} \left(1+\frac{100\pi N}{t_n}\right) \le e^{c_2 N^3}.$$

Proof. It is obvious that

$$\prod_{0<t_n\le 2\pi N} \left(\frac{100\pi N}{t_n}\right) \le \prod_{0<t_n\le 2\pi N} \left(1+\frac{100\pi N}{t_n}\right) \le \prod_{0<t_n\le 2\pi N} \left(\frac{200\pi N}{t_n}\right).$$

We can find c_3 and c_4, which satisfies

(3.27)

$$e^{c_3 N^3} \prod_{0<t_n\le 2\pi N} \left(\frac{2\pi N}{t_n}\right) \le \prod_{0<t_n\le 2\pi N} \left(1+\frac{100\pi N}{t_n}\right) \le e^{c_4 N^3} \prod_{0<t_n\le 2\pi N} \left(\frac{2\pi N}{t_n}\right).$$

Here by integration by parts,

$$\sum_{0<t_n\le 2\pi N} \log\left(\frac{2\pi N}{r_n}\right) = O(\log N) + \int_x^{2\pi N} \log\left(\frac{2\pi N}{x}\right) dN(x) = O(N^3) \tag{3.28}$$

From (3.27) and (3.28), the lemma follows. □

Lemma 3.17. [4, p. 266 Lemma 16.10] *Let $a_1, \ldots, a_n$ be real numbers. Suppose that $T_0, \delta_1, \ldots, \delta_n$ are positive numbers. There will then exist integers $x_1, \ldots, x_n$ and a number r such that:*

$$|ta_k - x_k| \le \delta_k \quad \text{for} \quad 1 \le k \le n$$

$$T_0 \le r \le T_0 \prod_{k=1}^{n} \left(1+\frac{1}{\delta_k}\right). \tag{3.29}$$

By applying Lemma 3.17, we have the following lemma.

Lemma 3.18. *There exists r_0 such that:*

$$r_0 t_n = 2\pi I + \varepsilon_n \quad \text{for} \quad 0 < t_n \le 2\pi N, \tag{3.30}$$

where I is an integer and $|\varepsilon_n| \le \frac{t_n}{50}$, and

$$e^{c_5 N^3} \le r_0 \le e^{2c_5 N^3}. \tag{3.31}$$

Proof. We obtain (3.30) by applying Lemma 3.17 with $k = n$, $a_n = \frac{t_n}{2\pi}$, $\delta_n = \frac{t_n}{100\pi N}$, and with x_k an integer.

We now set $T_0 = e^{c_5 N^3}$ in (3.29) and apply Lemma 3.16. This yields (3.31). □

From these lemmas, the following result holds.

Theorem 3.19. *We have*

$$P(x) = \Omega_\pm \left(x(\log\log x)^{\frac{1}{3}}\right).$$

Proof. In view of Lemma 3.15, we restrict ourselves to r_n such that $0 < t_n \le 2\pi N$. Then we define

$$r_1 = r_0 - \frac{1}{4\pi N}.$$

From (3.30) in Lemma 3.18,

$$r_1 t_n = 2\pi I + \varepsilon_n - \frac{t_n}{4\pi N},$$

where $-\frac{t_n}{2\pi N} \le \varepsilon_n - \frac{t_n}{4\pi N} \le \left(\frac{1}{50} - \frac{t_n}{4\pi N}\right) t_n$.

Then for N large there exists $c_6 > 0$ which satisfies $-\sin(r_1 t_n) \ge c_6 \frac{t_n}{N}$. Thus for N large, there exists $c_7 > 0$ such that

$$-\frac{2}{N} \sum_{0<t_n \le 2\pi N} \frac{\sin(t_n r_1)}{t_n} \left(1 - \frac{t_n}{2\pi N}\right) \ge \frac{c_6}{N} \sum_{0<t_n\le 2\pi N} 1 \ge c_7.$$

Referring back to Lemma 3.15, we obtain

$$\int_1^{r_1+A} \frac{\mathcal{P}(e^v)}{e^v} k(N(v-r_1))dv \ge c_8 \tag{3.32}$$

with some $c_8 > 0$, where A and N are kept sufficiently large. The number A is independent of N.

By (3.31) in Lemma 3.18, we have

$$e^{c_5 N^3} - \frac{1}{4\pi N} \le r_1 \le e^{2c_5 N^3}.$$

We can find c_9 and c_{10} such that

$$c_9 N \le (\log r_1)^{\frac{1}{3}} \le c_{10} N. \tag{3.33}$$

Let

$$\mathcal{M} := \sup \left\{ \frac{\mathcal{P}(e^v)}{e^v} \middle| 1 \le v \le A + r_1 \right\}.$$

Using (3.32), we immediately deduce that:

$$\mathcal{M} \int_{-\infty}^{\infty} k(N(v-r_1))dv \ge \mathcal{M} \int_1^{r_1+A} k(N(v-r))dv \ge c_8.$$

Since $\int_{-\infty}^{\infty} k(N(v-r_1))dv = O(\frac{1}{N})$, it follows that

$$\mathcal{M} \ge c_8{}' N$$

with some $c_8{}' > 0$. By (3.33) and definition for $\mathcal{M}$, we have

$$\sup_{1\le v \le r_1+A} \frac{\mathcal{P}(e^v)}{e^v} \ge c_8{}' N \ge c_8{}''(\log r_1)^{\frac{1}{3}},$$

with some $c_8'' > 0$. It follows that

$$\overline{\lim_{v\to\infty}} \frac{\mathcal{P}(e^v)}{e^v(\log v)^{\frac{1}{3}}} \geq c_8''.$$

Hence putting $x = e^v$, we obtain

$$\overline{\lim_{x\to\infty}} \frac{\mathcal{P}(x)}{x(\log\log x)^{\frac{1}{3}}} \geq c_8'',$$

and $\mathcal{P}(x) = \Omega_+\left(x(\log\log x)^{\frac{1}{3}}\right)$.

The Ω_--result is proved similarly by using $r_2 = r_0 + \frac{1}{4\pi N}$. Then we have

$$\mathcal{P}(x) = \Omega_\pm\left(x(\log\log x)^{\frac{1}{3}}\right).$$

From the definition of $\mathcal{P}(x)$ in (3.21), this concludes the proof of this theorem. □

Theorem 3.20. [2, p. 224 Lemma 7.1] *For* $x \to \infty$,

$$\sum_{N(P_0)\leq x} \frac{\log N(P_0)}{N(P_0)} - \sum_{N(P)\leq x} \frac{\log N(P_0)}{m(P)|a(P)-a(P)^{-1}|^2} = O(\log x).$$

From Definition 2.5 and Theorem 3.20, we have

$$\sum_{N(P_0)\leq x} \log N(P_0) - \Psi_\Gamma(x) = O(x\log x).$$

Let

$$P_0(x) := \sum_{N(P_0)\leq x} \log N(P_0) - \sum_{n=0}^{M} \frac{x^{s_n}}{s_n} - \sum_{n=0}^{M} \frac{x^{\tilde{s}_n}}{\tilde{s}_n},$$

then we can express

$$P(x) = P_0(x) + O(x\log x). \tag{3.34}$$

Whereas

$$\begin{aligned}
\int_2^x \frac{dP_0(t)}{\log t} &= \sum_{N(P_0)\leq t} 1 - \int_2^x \left(\sum_{n=0}^{M} \frac{t^{s_n-1}}{\log t} + \sum_{n=0}^{M} \frac{t^{\tilde{s}_n-1}}{\log t}\right) dt \\
&= \sum_{N(P_0)\leq x} 1 - \sum_{n=0}^{M} \int_2^x \frac{t^{s_n-1}}{\log t} dt + \sum_{n=0}^{M} \int_2^x \frac{t^{\tilde{s}_n-1}}{\log t} dt \\
&= \pi_\Gamma(x) - \left(\sum_{n=0}^{M} \mathrm{li}(x^{s_n}) + \sum_{n=0}^{M} \mathrm{li}(x^{\tilde{s}_n})\right) + O(1).
\end{aligned}$$

We define

$$Q(x) := \pi_\Gamma(x) - \left(\sum_{n=0}^{M} \mathrm{li}(x^{s_n}) + \sum_{n=0}^{M} \mathrm{li}(x^{\tilde{s}_n})\right).$$

It follows

$$Q(x) = \int_2^x \frac{dP_0(t)}{\log t} + O(1).$$

P_0 is expressed with P by (3.34), so we have

$$Q(x) = \int_2^x \frac{dP(t)}{\log t} + O\left(\int_2^x \frac{\log t + 1}{\log t} dt\right).$$

From (3.19),

$$Q(x) = \frac{P(x)}{\log x} - \left[\frac{P_1(x)}{t(\log t)^2}\right]_2^x + \int_2^x \frac{P_1(t)}{t^2(\log t)^2} dt + O(x). \tag{3.35}$$

On the other hand, by (3.18), (3.19) and (3.20),

$$P_2(x) = \sum_{t_n \geq 0} \frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)} + \sum_{t_n \geq 0} \frac{x^{\tilde{s}_n+2}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)}. \tag{3.36}$$

Since $\int_0^R \frac{1}{x^3} dN(x) = \log R$ by Proposition 3.4, we have

$$\sum_{0 < t_n \leq R} \frac{1}{|s_n|^3} = O(\log R),$$

and

$$P_2(x) = O(x^3 \log x). \tag{3.37}$$

From (3.20), it leads to

$$P_1(x) = O(x^2 \log x).$$

Now the equation (3.35) is expressed with

$$Q(x) = \frac{P(x)}{\log x} + O\left(\frac{x}{\log x}\right). \tag{3.38}$$

Therefore we reached our main theorem.

Theorem 3.21. *When $x \to \infty$,*

$$\pi_\Gamma(x) = \mathrm{li}(x^2) + \sum_{n=1}^{M} \mathrm{li}(x^{s_n}) + \Omega_\pm\left(\frac{x(\log\log x)^{\frac{1}{3}}}{\log x}\right).$$

4. Ω-RESULT FOR COFINITE GROUPS

In this section, we obtain the Ω-result of $\pi_\Gamma(x)$, where Γ is a cofinite subgroup of G. Then we have to consider the contribution from the elliptic classes, the parabolic classes and the continuous spectra.

The Selberg trace formula is explicitly written by Elstrodt, Grunewald and Mennicke as follows:

Theorem 4.1. [2, p. 297 (5.4)] *Assume that Γ has $h_\Gamma > 0$ classes of cusps represented by $\zeta_1, \ldots, \zeta_{h_\Gamma}$, and $\varphi = \det\Phi$ is the determinant of the scattering matrix. Let h be a function holomorphic in a strip of width strictly greater than 2 around the real axis satisfying the growth condition $h(1+z^2) = O((1+|z|^2)^{\frac{3}{2}-\epsilon})$ for $|z| \to \infty$ uniformly in the strip. Let g be the cosine transform*

$$g(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} h(1+t^2)e^{-itx}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(1+t^2)\cos(xt)dt.$$

Then there is for each cusp ζ_i a number $l_i \in \mathbf{N}$ $(i = 1, \ldots, h_\Gamma)$ and constants

$$c_\Gamma,\ \tilde{c}_\Gamma,\ d_\Gamma,\ d(i,j),\ \alpha(i,j) > 0 \quad (i = 1, \ldots, h_\Gamma,\ j = 1, \ldots, l_i)$$

so that the following identity holds with all sums being absolutely convergent:

$$\begin{aligned}\sum_{n=0}^{\infty} h(\lambda_n) =& \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H})}{4\pi^2}\int_{-\infty}^{\infty} h(1+t^2)t^2dt \\ &+ \sum_{\{R\}} \frac{\pi \log N(P_0)}{m(R)\sin^2\left(\frac{\pi k}{m(R)}\right)}g(0) + \sum_{\{P\}} \frac{4\pi g(\log N(P))\log N(P_0)}{m(P)|a(P)-a(P)^{-1}|^2} \\ &+ c_\Gamma g(0) + \tilde{c}_\Gamma h(1) - \frac{\mathrm{tr}\Phi(0)h(1)}{4} \\ &+ \frac{1}{4\pi}\int_{-\infty}^{\infty} h(1+t^2)\frac{\varphi'}{\varphi}(it)dt - d_\Gamma\int_{-\infty}^{\infty} h(1+t^2)\frac{\Gamma'}{\Gamma}(1+it)dt \\ &+ \sum_{i=1}^{h_\Gamma}\sum_{j=1}^{l_i} d(i,j)\int_0^{\infty} g(x)\frac{\sinh x}{\cosh x - 1 + \alpha(i,j)}dx.\end{aligned}$$

The first sum in the second line extends over all Γ-conjugacy classes of elliptic elements in Γ which do not stabilize a cusp. The second sum extends over all hyperbolic or loxodromic conjugacy classes.

If the stabilizer Γ_{ζ_i} of the cusp ζ_i is torsion free then $d(i,j) = 0$ for $j = 1, \ldots, l_i$.

From the functional equation, the following lemma holds.

Lemma 4.2. *Assume* $\sum_{\gamma_n>0} \frac{e^{(\beta_n-1)v}}{\gamma_n{}^2} = O\left(\frac{1}{1+v^3}\right)$, *where* $\rho_n = \beta_n + i\gamma_n$ *are poles of the scattering determinant. Then the Selberg zeta function satisfies,*

$$\frac{Z'}{Z}(s) + \frac{Z'}{Z}(2-s) = O(|s|^2).$$

Proof. From Gangolli [3, Theorem 4.4], we have the functional equation:

$$(4.1)\quad Z(2-s) = Z(s)\left(\frac{\Gamma(2-s)}{\Gamma(s)}\right)^{4\kappa h_\Gamma}[\varphi(1-s)]^{4\kappa}$$
$$\prod_{k=1}^{l}\left(\frac{s-1-q_k}{1-s-q_k}\right)^{4\kappa b_k}\exp[\int_0^{s-1}4\pi\kappa\mathrm{vol}(\Gamma\backslash\mathbf{H})t^2dt + \kappa_1(s-1)],$$

where κ and κ_1 are constants defined through the process described in [3], and q_k $(1 \le k \le l)$ are the finitely many poles of φ in the interval $(0,1]$ with order b_k.

By (4.1),

$$\frac{Z'}{Z}(s) + \frac{Z'}{Z}(2-s) = 4\kappa h_\Gamma\left(\frac{\Gamma'}{\Gamma}(2-s) + \frac{\Gamma'}{\Gamma}(s)\right) + 4\kappa\frac{\varphi'}{\varphi}(1-s)$$
$$-\sum_{k=1}^{l}(4\kappa b_k)\left(\frac{1}{s-1-q_k} + \frac{1}{1-s-q_k}\right) - 4\pi\kappa\mathrm{vol}(\Gamma\backslash\mathbf{H})(s-1)^2 + \kappa_1.$$

We estimate each term in the right hand side.

From the property of the Γ-function [8, 8.362.2],

$$(4.2)\qquad \frac{\Gamma'}{\Gamma}(2-s) + \frac{\Gamma'}{\Gamma}(s) = O(\log s).$$

From Elstrodt, Mennicke and Grunewald [2, p. 289 (4.25)],

$$\frac{\varphi'}{\varphi}(1-s) = \sum_{\rho}\frac{-2\beta_n}{\beta_n{}^2 + (t-\gamma_n)^2},$$

where $\beta_n \in [0,1]$. When $\sum_{\gamma_n>0} \frac{e^{(\beta_n-1)v}}{\gamma_n{}^2} = O\left(\frac{1}{1+v^3}\right)$, the sum $\sum_{\gamma_n>0}\frac{1}{\gamma_n{}^2}$ converges, and we see that

$$\#\{\gamma_n \mid 1 \le \gamma_n \le T\} = o(T^2).$$

Then

$$(4.3)\qquad \frac{\varphi'}{\varphi}(1-s) \ll \sum_{\gamma_n>0}\frac{1}{(t-\gamma_n)^2}$$
$$\ll \sum_{\gamma_n>0}\frac{1}{\gamma_n{}^2} = O(1).$$

The lemma follows from (4.2) and (4.3). □

From Theorem 4.1, we have the following proposition.

Proposition 4.3. *Assume* $\sum_{\gamma_n>0} \frac{e^{(\beta_n-1)v}}{\gamma_n{}^2} = O\left(\frac{1}{1+v^3}\right)$, *where* $\rho_n = \beta_n + i\gamma_n$ *are poles of the scattering determinant. Then we have for* $\mathrm{Re}(s) \geq 1$,

$$\frac{Z'}{Z}(s) = \frac{1}{s-2} + \sum_{|s-s_n|<1} \frac{1}{s-s_n} + \sum_{|s-\tilde{s}_n|<1} \frac{1}{s-\tilde{s}_n} + O(|s|^2+1), \tag{4.4}$$

where $s_n = 1 + it_n$ *and* $\tilde{s}_n = 1 - it_n$ *run over the zeros of* $Z(s)$ *on* $\mathrm{Re}(s) = 1$.

Proof. In Theorem 4.1, we take the test function

$$h(1+t^2) := \frac{1}{t^2+1+s(s-2)} - \frac{1}{t^2+\beta^2} \quad \left(\beta > \frac{1}{2}, s > 2\right),$$

and

$$g(x) = \frac{1}{2s-2}e^{-(s-1)|x|} - \frac{1}{2\beta}e^{-\beta|x|}.$$

Then

(4.5)

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{1}{t_n{}^2+(s-1)^2} - \frac{1}{t_n{}^2+\beta^2}\right) \\
&= \frac{\mathrm{vol}(\Gamma)}{8\pi}(\beta - s + 1) \\
&+ \sum_{\{R\}} \frac{\pi \log N(P_0)}{m(R)\sin^2\left(\frac{\pi k}{m(R)}\right)}\left(\frac{1}{2s-2} - \frac{1}{2\beta}\right) + \frac{2\pi}{s-1}\frac{Z'}{Z}(s) - \frac{2\pi}{\beta}\frac{Z'}{Z}(\beta+1) \\
&+ c_\Gamma\left(\frac{1}{2s-2} - \frac{1}{2\beta}\right) + \tilde{c}_\Gamma\left(\frac{1}{(s-1)^2} - \frac{1}{\beta^2}\right) - \frac{\mathrm{tr}\Phi(0)}{4}\left(\frac{1}{(s-1)^2} - \frac{1}{\beta^2}\right) \\
&+ \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\frac{1}{t^2+(s-1)^2} - \frac{1}{t^2+\beta^2}\right)\frac{\varphi'}{\varphi}(it)dt \\
&- d_\Gamma \int_{-\infty}^{\infty}\left(\frac{1}{t^2-(s-1)^2} - \frac{1}{t^2+\beta^2}\right)\frac{\Gamma'}{\Gamma}(1+it)dt \\
&+ \sum_{i=1}^{\kappa}\sum_{j=1}^{l_i} d(i,j)\int_0^{\infty} g(x)\frac{\sinh x}{\cosh x - 1 + \alpha(i,j)}dx.
\end{aligned}$$

Since $\frac{\varphi'}{\varphi}(it) = O(1)$ and $\frac{\Gamma'}{\Gamma}(1+it) = O(\log t)$[8, 8.362.2], the proposition follows from (3.5). □

Applying Lemma 4.2 and Proposition 4.3, we have same argument as in Theorem 3.6 under the condition $\sum_{\gamma_n>0} \frac{e^{(\beta_n-1)v}}{\gamma_n{}^2} = O\left(\frac{1}{1+v^3}\right)$. For the residues in (3.14), the following terms are added to the right hand side of (3.15):

$$\mathrm{Res}_{s=\rho_n}\left(\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)\right) = \frac{\mu_1(n)x^{\rho_n+2}}{\rho_n(\rho_n+1)(\rho_n+2)} \quad (n \geq 0),$$

$$\mathrm{Res}_{s=\tilde{\rho}_n}\left(\frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)\right) = \frac{\mu_1(n)x^{\tilde{\rho}_n+2}}{\tilde{\rho}_n(\tilde{\rho}_n+1)(\tilde{\rho}_n+2)} \quad (n \geq 0),$$

where $\mu_1(n)$ is the multiplicity of ρ_n.

Then we have the following theorem.

Theorem 4.4. *Suppose that* $\sum_{\gamma_n>0} \frac{e^{(\beta_n-1)v}}{\gamma_n{}^2} = O\left(\frac{1}{1+v^3}\right)$, *then for constants* α, β, α_1, *we have*

$$\begin{aligned}\Psi_1(x) = \alpha x + \beta x\log x + \alpha_1 + \sum_{n=0}^{M}\frac{x^{s_n+1}}{s_n(s_n+1)} + \sum_{n=0}^{M}\frac{x^{\tilde{s}_n+1}}{\tilde{s}_n(\tilde{s}_n+1)} \\ + \sum_{t_n\geq 0}\frac{x^{s_n+1}}{s_n(s_n+1)} + \sum_{t_n\geq 0}\frac{x^{\tilde{s}_n+1}}{\tilde{s}_n(\tilde{s}_n+1)} + \sum_{\gamma_n\geq 0}\frac{x^{\rho_n+1}}{\rho_n(\rho_n+1)} + \sum_{\gamma_n>0}\frac{x^{\tilde{\rho}_n+1}}{\tilde{\rho}_n(\tilde{\rho}_n+1)},\end{aligned}$$

where $s_n = 1+it_n$ *are the zeros of* $Z(s)$, *and* $\rho_n = \beta_n + i\gamma_n$ *are poles of the scattering determinant.* $\tilde{s}_n$ *and* $\tilde{\rho}_n$ *are the conjugacy elements for* s_n *and* ρ_n, *respectively.*

Our next goal is to show an Ω-result for $P(x)$ in (3.18).

Definition 4.5. *We define*

$$\mathcal{P}(x) := P(x) - N(0)x,$$

The following calculations are analogous to those in the previous section.

Lemma 4.6. *There exists* $d_1 \in \mathbf{C}$ *such that*

$$\begin{aligned}d_1 + \int_1^v \frac{\mathcal{P}(e^u)}{e^u}du \\ = \sum_{t_n>0}\frac{e^{(s_n-1)v}}{s_n(s_n-1)} + \sum_{t_n>0}\frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)} + \sum_{\gamma_n>0}\frac{e^{(\rho_n-1)v}}{\rho_n(\rho_n-1)} + \sum_{\gamma_n>0}\frac{e^{(\tilde{\rho}_n-1)v}}{\tilde{\rho}_n(\tilde{\rho}_n-1)}.\end{aligned}$$

In what follows we put

$$F(v) = d_1 + \int_1^v \frac{\mathcal{P}(e^u)}{e^u}du.$$

Lemma 4.7. *There exists $d_2 \in \mathbf{C}$ such that*

$$d_2 + \int_1^v F(u)du$$
$$= \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)^2} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)^2} + \sum_{\gamma_n>0} \frac{e^{(\rho_n-1)v}}{\rho_n(\rho_n-1)^2} + \sum_{\gamma_n>0} \frac{e^{(\tilde{\rho}_n-1)v}}{\tilde{\rho}_n(\tilde{\rho}_n-1)^2}.$$

In what follows we put

$$G(v) = d_2 + \int_1^v F(u)du.$$

Lemma 4.8. *There exists $d_3 \in \mathbf{C}$ such that*

$$d_3 + \int_1^v G(u)du$$
$$= \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)^3} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)^3} + \sum_{\gamma_n>0} \frac{e^{(\rho_n-1)v}}{\rho_n(\rho_n-1)^3} + \sum_{\gamma_n>0} \frac{e^{(\tilde{\rho}_n-1)v}}{\tilde{\rho}_n(\tilde{\rho}_n-1)^3}.$$

In what follows we put

$$H(v) = d_3 + \int_1^v G(u)du.$$

Suppose that

$$\sum_{\gamma_n>0} \frac{e^{(\beta_n-1)v}}{{\gamma_n}^2} = O\left(\frac{1}{1+v^3}\right),$$

then we can express $F(v)$, $G(v)$ and $H(v)$ as

$$F(v) = d_1 + \int_1^v \frac{\mathcal{P}(e^u)}{e^u} du$$
$$= \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)} + O\left(\frac{1}{1+v^3}\right),$$

$$G(v) = d_2 + \int_1^v F(u)du$$
$$= \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)^2} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)^2} + O\left(\frac{1}{1+v^2}\right),$$

and

$$H(v) = d_3 + \int_1^v G(u)du$$
$$= \sum_{t_n>0} \frac{e^{(s_n-1)v}}{s_n(s_n-1)^3} + \sum_{t_n>0} \frac{e^{(\tilde{s}_n-1)v}}{\tilde{s}_n(\tilde{s}_n-1)^3} + O\left(\frac{1}{1+v}\right).$$

When $v \to \infty$, all O-terms are $O(1)$. So, we can include this case in the case of cocompact groups. Then we have the following theorem by substituting with $x = e^v$.

Theorem 4.9. *When* $\sum\limits_{\gamma_n>0} \frac{x^{\beta_n-1}}{\gamma_n{}^2} = O\left(\frac{1}{1+(\log x)^3}\right)$, *we have*

$$P(x) = \Omega_\pm \left(x(\log\log x)^{\frac{1}{3}}\right).$$

On the other hand, about $P(x)$, we can express that

$$\begin{aligned} P_2(x) &= \sum_{t_n\geq 0} \frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)} + \sum_{t_n\geq 0} \frac{x^{\tilde{s}_n+2}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)} \\ &+ \sum_{\gamma_n\geq 0} \frac{x^{\rho_n+2}}{\rho_n(\rho_n+1)(\rho_n+2)} + \sum_{\gamma_n>0} \frac{x^{\tilde{\rho}_n+2}}{\tilde{\rho}_n(\tilde{\rho}_n+1)(\tilde{\rho}_n+2)} \end{aligned}$$

by (3.18), (3.19) and (3.20).

Since

$$\sum_{\gamma_n>0} \frac{x^{\beta_n-1}}{\gamma_n{}^2} = O\left(\frac{1}{1+(\log x)^3}\right),$$

we rewrite $P_2(x)$ as follows:

$$P_2(x) = \sum_{t_n\geq 0} \frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)} + \sum_{t_n\geq 0} \frac{x^{\tilde{s}_n+2}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)} + O\left(\int_1^x \frac{x^3}{1+(\log u)^3} du\right).$$

Then we can obtain by (3.36) and (3.37)

$$P_2(x) = O(x^3 \log x).$$

It leads to

$$P_1(x) = O(x^2 \log x).$$

Similarly, from (3.38), we have again

$$Q(x) = \frac{P(x)}{\log x} + O\left(\frac{x}{\log x}\right),$$

where $Q(x) = \pi_\Gamma(x) - \sum\limits_{n=0}^{M} \mathrm{li}(x^{s_n}) - \sum\limits_{n=0}^{M} \mathrm{li}(x^{\tilde{s}_n})$. Then we get our main theorem for cofinite Γ.

Theorem 4.10. *Suppose that*

$$\sum_{\gamma_n>0} \frac{x^{\beta_n-1}}{\gamma_n{}^2} = O\left(\frac{1}{1+(\log x)^3}\right). \tag{4.6}$$

When $x \to \infty$, *we have*

$$\pi_\Gamma(x) = \mathrm{li}(x^2) + \sum_{n=1}^{M} \mathrm{li}(x^{s_n}) + \Omega_\pm\left(\frac{x(\log\log x)^{\frac{1}{3}}}{\log x}\right).$$

Example 4.11. *When Γ is the Bianchi group associated to an imaginary quadratic number field $K = \mathbf{Q}(\sqrt{-D})$ $(D \neq 1, 3)$, i.e.*

$$\Gamma = \Gamma_D = PSL(2, O_K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in O_K, ad - bc = 1 \right\} / \{\pm 1\},$$

where O_K is the ring of integers of K, we deduce that the assumption (4.6) *is satisfied as follows:*

By Efrat and Sarnak [1],

Theorem 4.12. [1, p. 817 Theorem 1] *For Γ_D, let*

$$\xi_H(s) = \left({d_H}^{1/2} / (2\pi)^{h_\Gamma} \right)^s \Gamma(s)^{h_\Gamma} \zeta_H(s),$$

where H is the Hilbert class field of K, d_H is the absolute value of the discriminant of H, $\zeta_H(s)$ is the Dedekind zeta function of H and h_Γ is the class number of K. Then,

$$\varphi(s) = (-1)^{(h_\Gamma - 2^{t-1})/2} {w_K}^{2s-2} \frac{\xi_H(s-1)}{\xi_H(s)},$$

where $w_K = \sqrt{2}/{d_K}^{\frac{1}{4}}$ with d_K being the absolute value of the discriminant of K and t is the number of prime divisors of d_K.

By Suetsuna [11], $\zeta_H(s)$ has no zeros in the region

$$\sigma > 1 - \frac{a}{\log(|t| + 2)}. \quad (a > 0)$$

Then we have

$$\beta_n < 1 - \frac{a}{\log(|\gamma_n| + 2)},$$

and we obtain

$$\frac{1}{(1 - \beta_n)^3} < \left(\frac{\log(|\gamma_n| + 2)}{a} \right)^3. \tag{4.7}$$

On the other hand,

$$\sum_{\gamma_n > 0} \frac{e^{(\beta_n - 1)v}}{{\gamma_n}^2} \leq \left(\sum_{\gamma_n > 0} \frac{6}{{\gamma_n}^2 (1 - \beta_n)^3} \right) \frac{1}{v^3}.$$

From (4.7),

$$\sum_{\gamma_n > 0} \frac{1}{{\gamma_n}^2 (1 - \beta_n)^3} = O(1).$$

By substituting $x = e^v$, the assumption (4.6) is satisfied.

5. O-RESULT FOR COCOMPACT GROUPS

In this section, we obtain the 'explicit formula' of $\Psi_\Gamma(x)$, where Γ is a cocompact group.

We have the following lemma.

Lemma 5.1. [5, p. 75, Theorem G] *If $c > 0, y > 0$, then*

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{y^s}{s}ds = \begin{cases} 0, & (y<1) \\ \frac{1}{2}, & (y=1) \\ 1, & (y>1) \end{cases} \tag{5.1}$$

where, in the case $y = 1$, the integer is to be interpreted as a 'Cauchy principal value', that is to say as the limit of

$$\frac{1}{2\pi i}\int_{c-Ti}^{c+Ti}\frac{y^s}{s}ds, \tag{5.2}$$

when $T \to \infty$. Moreover, if $I(y) = I(y,T) + \Delta(y,T)$, where $I(y)$ and $I(y,T)$ denote the integrals (5.1) *and* (5.2) *respectively, then, for $T > 0$,*

$$|\Delta(y,T)| < \begin{cases} \dfrac{y^c}{\pi T|\log y|} & (y \neq 1) \\ \dfrac{c}{\pi T} & (y=1), \end{cases}$$

$$|\Delta(y,T)| < y^c \quad (\textit{always}).$$

Applying Lemma 5.1, it leads us to the following explicit formula.

Theorem 5.2. *Let $1 \leq T < x^{\frac{1}{2}}$. Then we have*

$$\begin{aligned}\Psi_\Gamma(x) = \frac{1}{2}x^2 + \sum_{n=0}^{M}\frac{1}{s_n}x^{s_n} + \sum_{n=0}^{M}\frac{1}{\tilde{s}_n}x^{\tilde{s}_n} \\ + \sum_{0<t_n\leq T}\frac{1}{s_n}x^{s_n} + \sum_{0<t_n\leq T}\frac{1}{\tilde{s}_n}x^{\tilde{s}_n} + O\left(\frac{x^2}{T}\log x\right),\end{aligned} \tag{5.3}$$

where $s_n = 1 + it_n$ and $\tilde{s}_n = 1 - it_n$ are the zeros of $Z(s)$.

Proof. Suppose $T \geq 1$, let η be in $(0,1)$, and we consider

$$J(\eta) = \frac{1}{2\pi i}\int_C \frac{x^s}{s}\frac{Z'}{Z}(s)ds = J_1(\eta) + c_1^+(\eta) + c_1^-(\eta) + c_2^+(\eta) + c_2^- + c_3(\eta), \tag{5.4}$$

where the integral is taken in the positive sense around the rectangle $R(a,T) := \{z \in \mathbf{C} \mid -\eta \le \mathrm{Re}(z) \le 2+\eta, -T \le \mathrm{Im}(z) \le T\}$, and $J_1(\eta), c_1^{\pm}(\eta), c_2^{\pm}(\eta), c_3(\eta)$ are parts of the following integrals :

$$J_1(\eta) := \frac{1}{2\pi i}\int_{2+\eta-iT}^{2+\eta+iT} \frac{x^s}{s}\frac{Z'}{Z}(s)ds,$$

$$c_1^{+}(\eta) := \frac{1}{2\pi i}\int_{2+\eta+iT}^{1+\eta+iT} \frac{x^s}{s}\frac{Z'}{Z}(s)ds,$$

$$c_2^{+}(\eta) := \frac{1}{2\pi i}\int_{1+\eta+iT}^{-\eta+iT} \frac{x^s}{s}\frac{Z'}{Z}(s)ds,$$

$$c_3(\eta) := \frac{1}{2\pi i}\int_{-\eta+iT}^{-\eta-iT} \frac{x^s}{s}\frac{Z'}{Z}(s)ds,$$

$$c_2^{-}(\eta) := \frac{1}{2\pi i}\int_{-\eta-iT}^{1+\eta-iT} \frac{x^s}{s}\frac{Z'}{Z}(s)ds,$$

$$c_1^{-}(\eta) := \frac{1}{2\pi i}\int_{1+\eta-iT}^{2+\eta-iT} \frac{x^s}{s}\frac{Z'}{Z}(s)ds.$$

First, we consider $J_1(\eta)$. From (2.2), we can express that

$$J_1(\eta) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \sum_{\{P\}} \Lambda_\Gamma(P)\frac{(\frac{x}{N(P)})^s}{s}ds,$$

where $c = 2+\eta$. Then applying to Lemma 5.1 with $y = \frac{x}{N(P)}$, we have

$$J_1(\eta) = \sum_{\{P\}} \Lambda_\Gamma(P) I\left(\frac{x}{N(P)}, T\right).$$

Putting

$$\Psi_{\Gamma_0}(x) = \sum_{\{P\}} \Lambda_\Gamma(P) I\left(\frac{x}{N(P)}\right)$$

and

$$X = \sum_{\{P\}} \Lambda_\Gamma(P)\Delta\left(\frac{x}{N(P)}, T\right),$$

we have

$$J_1(\eta) = \Psi_{\Gamma_0}(x) - X. \tag{5.5}$$

By the definition of $I(y)$,

$$\Psi_{\Gamma_0}(x) = \Psi_\Gamma(x) - \frac{1}{2}\log N(P_0), \tag{5.6}$$

and from the property of $\Delta(y)$, we have

$$X < \sum_{\{P\}} \Lambda_\Gamma(P) \left(\frac{x}{N(P)}\right)^{2+\eta} \min\left\{1, \frac{1}{\pi T |\log \frac{x}{N(P)}|}\right\}. \tag{5.7}$$

We put the right hand side of this inequality to be X_1. Since X_1 is bigger than $\frac{1}{2}\log N(P_0)$, we can express

$$J_1(\eta) - \Psi_\Gamma(x) \ll X_1. \tag{5.8}$$

Next, we estimate $c_i^{\pm}(i = 1, 2, 3)$ by using Lemma 3.5.

By (3.9), we have

$$\begin{aligned}
|c_1^{\pm}(\eta)| &= \left|\frac{1}{2\pi i}\int_{2+\eta\pm iT}^{1+\eta\pm iT} \frac{x^s}{s}\frac{Z'}{Z}(s)ds\right| \\
&\ll \left|\int_{2+\eta\pm iT}^{2\pm iT} \frac{x^\sigma}{|s|}\log|t|ds\right| + \left|\int_{2\pm iT}^{1+\eta\pm iT} \frac{x^\sigma}{|s|}|t|^{2(2-\sigma)}\log|t|ds\right|,
\end{aligned}$$

where $s = \sigma + it$. Then

$$\begin{aligned}
|c_1^{\pm}(\eta)| &\ll \frac{x^{2+\eta}}{T}\log T + T^3 \log T\left(\left(\frac{x}{T^2}\right)^{1+\eta} - \left(\frac{x}{T^2}\right)^2\right) \\
&\ll \left(\frac{x^2}{T} + T^{1-2\eta}x\right)x^\eta \log T.
\end{aligned} \tag{5.9}$$

We also see that (3.10) and (3.8) give

$$\begin{aligned}
|c_2^{\pm}(\eta)| &= \left|\frac{1}{2\pi i}\int_{1+\eta+iT}^{-\eta+iT} \frac{x^s}{s}\frac{Z'}{Z}(s)ds\right| \\
&\ll \int_0^{1+\eta} \frac{x^\sigma}{|s|}T^2 \log T d\sigma + \int_{-\eta}^{0} \frac{x^\sigma}{|s|}(|T|^2 + 1)d\sigma \\
&\ll Tx^{1+\eta}\log T.
\end{aligned} \tag{5.10}$$

From (3.10), an estimation for $c_3(\eta)$ holds as:

$$|c_3(\eta)| = \left|\frac{1}{2\pi i}\int_{-\eta+iT}^{-\eta-iT} \frac{x^s}{s}\frac{Z'}{Z}(s)ds\right| \ll \left|\int_{-\eta+iT}^{-\eta-iT} \frac{x^{-\eta}}{|s|}(|t|^2+1)ds\right| \ll x^{-\eta}T^2. \tag{5.11}$$

On the other hand, from the theory of residues,

(5.12)

$$J(\eta) = \frac{1}{2}x^2 + \sum_{n=0}^{M}\frac{1}{s_n}x^{s_n} + \sum_{n=0}^{M}\frac{1}{\tilde{s}_n}x^{\tilde{s}_n} + \sum_{0<t_n\le T}\frac{1}{s_n}x^{s_n} + \sum_{0<t_n\le T}\frac{1}{\tilde{s}_n}x^{\tilde{s}_n} + O(1).$$

Gathering together (5.4), (5.5), (5.6), (5.8) and (5.12), we obtain

$$\Psi_\Gamma + c_1^{\pm}(\eta) + c_2^{\pm}(\eta) + c_3(\eta) = \frac{1}{2}x^2 + \sum_{n=0}^{M} \frac{1}{s_n} x^{s_n} + \sum_{n=0}^{M} \frac{1}{\tilde{s}_n} x^{\tilde{s}_n} + \sum_{0<t_n\le T} \frac{1}{s_n} x^{s_n} + \sum_{0<t_n\le T} \frac{1}{\tilde{s}_n} x^{\tilde{s}_n} + O(X_1) + O(1).$$

Then, from (5.9), (5.10) and (5.11), we have

(5.13)

$$\Psi_\Gamma(x) = \frac{1}{2}x^2 + \sum_{n=0}^{M} \frac{1}{s_n} x^{s_n} + \sum_{n=0}^{M} \frac{1}{\tilde{s}_n} x^{\tilde{s}_n} + \sum_{0<t_n\le T} \frac{1}{s_n} x^{s_n} + \sum_{0<t_n\le T} \frac{1}{\tilde{s}_n} x^{\tilde{s}_n} + O(X_1) + O\left(\left(\frac{x^2}{T} + T^{1-2\eta}x\right) x^{\eta} \log T + T x^{1+\eta} \log T + x^{-\eta} T^2\right).$$

Now we estimate X_1. From the mean value theorem, we have

$$\left|\log \frac{x}{N(P)}\right| > \frac{N(P) - x}{c'} > \frac{|N(P) - x|}{N(P) + x}, \tag{5.14}$$

for $x < c' < N(P)$.

If $|N(P) - x| \ge \frac{1}{4}x$, then

$$\frac{N(P) + x}{|N(P) - x|} \le 1 + \frac{2x}{|N(P) - x|} \le 9, \tag{5.15}$$

and we get $|\log \frac{x}{N(P)}| \ge \frac{1}{9}$.

Next we deal with the case of $|N(P) - x| < \frac{1}{4}x$.

For $\frac{3}{4}x < N(P) < x$, we can write $N(P) = x_1 - r$ $(r > 0)$, where x_1 is the maximum of powers of $N(P)$, not exceeding x. Here r is an integer satisfying $0 < r < \frac{1}{4}x$, and

$$0 < r = |N(P) - x_1| < |N(P) - x| + |x - x_1| \le 2|N(P) - x|.$$

Combining this with (5.14), we have

$$\left|\log \frac{x}{N(P)}\right| > \frac{|N(P) - x|}{N(P) + x} > \frac{\frac{r}{2}}{2x}. \tag{5.16}$$

In the case of $x < N(P) < \frac{5}{4}x$, we have

$$\left|\log \frac{x}{N(P)}\right| > \frac{\frac{r}{2}}{\frac{9}{4}x}. \tag{5.17}$$

Now we put u_P and X' to be as follows:

$$u_P := \left(\frac{x}{N(P)}\right)^c \frac{\Lambda_\Gamma(P)}{\pi T|\log\frac{x}{N(P)}|}$$

and

$$X' := \sum_{\{P\}} u_P,$$

respectively. Then, we have

$$X' = \frac{x^c}{\pi T}\sum_{\{P\}}\frac{\Lambda_\Gamma(P)}{N(P)^c|\log\frac{x}{N(P)}|}. \tag{5.18}$$

Putting $c = 2+\eta$, this can be written as:

$$X' \leq \frac{x^{2+\eta}}{\pi T}\left\{\sum_{|N(P)-x|\geq\frac{1}{4}x} + \sum_{|N(P)-x|<\frac{1}{4}x}\right\}\frac{\Lambda_\Gamma(P)}{N(P)^{2+\eta}|\log\frac{x}{N(P)}|} + u_{x_1},$$

where u_{x_1} is the case of $N(P) = x_1$.

By (5.15), (5.16) and (5.17), we have

$$\begin{aligned} X' - u_{x_1} &\ll \frac{x^{2+\eta}}{T}\left\{\sum_{|N(P)-x|\geq\frac{1}{4}x}\frac{\Lambda_\Gamma(P)}{N(P)^{2+\eta}} + \sum_{0<r<\frac{1}{4}x}\frac{\Lambda_\Gamma(P)}{x^{2+\eta}}\frac{x}{r}\right\} \\ &\ll \frac{x^{2+\eta}}{T} + \frac{x}{T}\sum_r\frac{\Lambda_\Gamma(P)}{r}. \end{aligned} \tag{5.19}$$

The summation over r is separated into two parts, $0 < r < \frac{1}{10}x$ and $\frac{1}{10}x < r < \frac{1}{4}x$. In X', the summation over $0 < r < \frac{1}{10}x$ and u_{x_1} are larger than 1. By (5.7), (5.18) and (5.19)

$$X_1 \ll \frac{x^{2+\eta}}{T} + \frac{x}{T}\sum_{\frac{1}{10}x<r<\frac{1}{4}x}\frac{\Lambda_\Gamma(P)}{r} + x^{2+\eta}\log x.$$

Since $\#\{r|\frac{1}{10}x < r < \frac{1}{4}x\} \ll x^2$ by the prime geodesic theorem, we have

$$\begin{aligned} X_1 &\ll \frac{x^{2+\eta}}{T} + \frac{x}{T}\left(\frac{10}{x}x^2\log x\right) + x^{2+\eta}\log x \\ &\ll \frac{x^{2+\eta}}{T}\log x. \end{aligned} \tag{5.20}$$

Combining (5.13) with (5.20), we have

$$\Psi_\Gamma(x) = \frac{1}{2}x^2 + \sum_{n=0}^{M} \frac{1}{s_n} x^{s_n} + \sum_{n=0}^{M} \frac{1}{\tilde{s}_n} x^{\tilde{s}_n} + \sum_{0<t_n\le T} \frac{1}{s_n} x^{s_n} + \sum_{0<t_n\le T} \frac{1}{\tilde{s}_n} x^{\tilde{s}_n}$$
$$+ O\left(\left(\frac{x^{2+\eta}}{T} \log x \right) + \left(\frac{x^2}{T} + Tx \right) x^\eta \log T + \frac{T^2}{x^\eta} \right).$$

Since $1 \le T \le x^{\frac{1}{2}}$, we get the error term with

$$O\left(\frac{x^{2+\eta}}{T} \log x \right),$$

where $\eta = (\log T)^{-1}$. If $T \ge x^{\frac{1}{3}}$, then the factor x^η is bounded. The theorem follows. In the case of $T < x^{\frac{1}{3}}$,

$$\Psi_\Gamma(x) = \frac{1}{2}x^2 + \sum_{n=0}^{M} \frac{1}{s_n} x^{s_n} + \sum_{n=0}^{M} \frac{1}{\tilde{s}_n} x^{\tilde{s}_n} + \sum_{0<t_n\le T} \frac{1}{s_n} x^{s_n} + \sum_{0<t_n\le T} \frac{1}{\tilde{s}_n} x^{\tilde{s}_n} + O\left(x^{\frac{5}{3}} \log x \right).$$

From Proposition 3.4, we have

$$\left| \sum_{T<|t_n|<x^{\frac{1}{3}}} \frac{x^{s_n}}{s_n} \right| \ll \frac{x}{T} T^3 = xT^2 < x^{\frac{5}{3}}.$$

Since (5.3) holds for $T = x^{\frac{1}{3}}$, the theorem follows for $T < x^{\frac{1}{3}}$. It completes the proof. $\square$

From this explicit formula, we have an estimation of $\Psi_\Gamma(x)$.

Corollary 5.3. *When $x \to \infty$,*

$$\Psi_\Gamma(x) = \frac{1}{2}x^2 + \sum_{n=0}^{M} \frac{1}{s_n} x^{s_n} + \sum_{n=0}^{M} \frac{1}{\tilde{s}_n} x^{\tilde{s}_n} + O(x^{\frac{5}{3}+\varepsilon}),$$

where $s_n = 1 + it_n$ and $\tilde{s}_n = 1 - it_n$ are the zeros of $Z(s)$.

Proof. We consider the summation over t_n of explicit formula in Theorem 5.2. By using Proposition 3.4, we have

$$\sum_{|t_n|\le T} \frac{x^{s_n}}{s_n} \ll \frac{x \sum |x^{it_n}|}{T} \ll \frac{x}{T} T^3 = T^2 x.$$

So, we can express

$$\Psi_\Gamma(x) = \frac{1}{2}x^2 + \sum_{n=0}^{M} \frac{1}{s_n} x^{s_n} + \sum_{n=0}^{M} \frac{1}{\tilde{s}_n} x^{\tilde{s}_n} + O\left(T^2 x + \frac{x^2}{T} \log x \right).$$

By choosing $T = x^{\frac{1}{3}}$, we obtain

$$\Psi_\Gamma(x) = \frac{1}{2}x^2 + \sum_{n=0}^{M} \frac{1}{s_n} x^{s_n} + \sum_{n=0}^{M} \frac{1}{\tilde{s}_n} x^{\tilde{s}_n} + O(x^{\frac{5}{3}+\varepsilon}).$$

□

Next we apply this to the prime geodesic theorem.

Theorem 5.4. *When $x \to \infty$,*

$$\pi_\Gamma(x) = \mathrm{li}(x^2) + \sum_{n=1}^{M} \mathrm{li}(x^{s_n}) + O(x^{\frac{5}{3}+\varepsilon})$$

Proof. Let

$$\Pi(x) := \sum_{2 \le N(p) \le x} \frac{\Lambda(p)}{\log N(p)}.$$

This is expressed in terms of $\pi_\Gamma(x)$:

$$\Pi(x) = \sum_{m=1}^{M} \frac{1}{m} \pi_\Gamma(x^{\frac{1}{m}}),$$

where $M = [\frac{\log x}{\log 2}]$. So we have

$$\Pi(x) - \pi_\Gamma(x) = \sum_{m=2}^{M} \frac{\pi_\Gamma(x^{\frac{1}{m}})}{m} = O(x). \tag{5.21}$$

From Theorem 3.8, we have

$$\Pi(x) = \int_2^x \frac{\Psi_\Gamma(u)}{u \log^2 u} du + \frac{\Psi_\Gamma(x)}{\log x}. \tag{5.22}$$

By the definition of $\mathrm{li}(x^2)$ and integration by parts, we obtain

$$\mathrm{li}(x^2) = \int_2^{x^2} \frac{du}{\log u} = \int_2^{x^2} \frac{du}{\log^2 u} + \frac{x^2}{2 \log x} - \frac{2}{\log 2}. \tag{5.23}$$

Gathering together (5.21), (5.22) and (5.23), we have

$$\pi_\Gamma(x) - \mathrm{li}(x^2) = \int_2^x \frac{\Psi_\Gamma(u) du}{u \log^2 u} + \frac{2\Psi_\Gamma(x) - x^2}{2 \log x} - \int_2^{x^2} \frac{du}{\log^2 u} + O(x). \tag{5.24}$$

Substituting Corollary 5.3 to (5.24) leads to

$$\pi_\Gamma(x) = \mathrm{li}(x^2) + O\left(\frac{x^{\frac{5}{3}+\varepsilon}}{\log x}\right) = \mathrm{li}(x^2) + O(x^{\frac{5}{3}+\varepsilon}).$$

□

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6. O-RESULT FOR BIANCHI GROUPS

In this section, we obtain the O-result of $\pi_\Gamma(x)$, where Γ is a Bianchi group.

In case Γ is essentially cuspidal, the order of the scattering determinant is less than three, and every estimate in the proof of Theorem 5.2 goes through.

In particular we have the following theorem as a consequence of Theorem 4.12.

Theorem 6.1. *Suppose that Γ is a Bianchi group. Let $1 \leq T < x^{\frac{1}{2}}$. Then we have*

$$\Psi_\Gamma(x) = \frac{1}{2}x^2 + \sum_{n=0}^{M} \frac{x^{s_n}}{s_n} + \sum_{n=0}^{M} \frac{x^{\tilde{s}_n}}{\tilde{s}_n} + \sum_{0<t_n\leq T} \frac{x^{s_n}}{s_n} + \sum_{0<t_n\leq T} \frac{x^{\tilde{s}_n}}{\tilde{s}_n} + O\left(\frac{x^2}{T}\log x\right),$$

as $x \to \infty$, where $s_n = 1 + it_n$ and $\tilde{s}_n = 1 - it_n$ are the zeros of $Z(s)$ coming from discrete spectra.

We apply this to the prime geodesic theorem.

Theorem 6.2. *Suppose that Γ is a Bianchi group. When $x \to \infty$,*

$$\pi_\Gamma(x) = \mathrm{li}(x^2) + \sum_{n=0}^{M} \mathrm{li}(x^{s_n}) + O(x^{\frac{5}{3}+\varepsilon})$$

Proof. From Theorem 6.1, the same result is established as that in Corollary 5.3:

$$\Psi_\Gamma(x) = \frac{1}{2}x^2 + \sum_{n=0}^{M} \frac{1}{s_n} + \sum_{n=0}^{M} \frac{1}{\tilde{s}_n} + O(x^{\frac{5}{3}+\varepsilon}).$$

as $x \to \infty$. The same argument as in Theorem 5.4 leads to the result. □

This estimate was proved by Sarnak [10, p. 282, Theorem 5.1] when the imaginary quadratic field ($\neq \mathbf{Q}(i), \mathbf{Q}(\sqrt{-3})$) is of class number one. When $\Gamma = PSL(2, \mathbf{Z}[i])$, the error term is reduced to $O(x^{\frac{11}{7}+\varepsilon})$ under the mean-Lindelöf hypothesis (λ-aspect) for automorphic L-functions by Koyama [7].

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