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Pole Loci of Solutions of a Degenerate Garnier System

by

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Abstract. We treat a certain type of degenerate Garnier system such that all the solutions are meromorphic on $\mathbf{C}^2$. This is regarded as a two-variables version of the first Painlevé equation. It is shown that, for every solution, each pole locus is expressible by an analytic function which satisfies a fourth order non-linear ordinary differential equation. We also give analytic expressions of solutions near their pole loci.
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1. Introduction

Consider the linear differential equation

$$\frac{d^2y}{dx^2} - \frac{1}{x-\lambda}\frac{dy}{dx} - \left(4x^3 + 2tx + 2H - \frac{\mu}{x-\lambda}\right)y = 0, \qquad (1.1)$$
$$H = \frac{1}{2}\mu^2 - 2\lambda^3 - t\lambda,$$

where t, λ, μ are complex parameters. This equation has an irregular singular point at $x = \infty$ and a non-logarithmic regular singular point at $x = \lambda$ with the characteristic exponents $(0, 2)$. The isomonodromic deformation of (1.1) concerning the parameter t is governed by the Hamiltonian system

$$\frac{d\lambda}{dt} = \frac{\partial H}{\partial \mu}, \qquad \frac{d\mu}{dt} = -\frac{\partial H}{\partial \lambda},$$

which is equivalent to the first Painlevé equation

$$\frac{d^2\lambda}{dt^2} = 6\lambda^2 + t, \qquad (1.2)$$

that is to say, there exists a fundamental system of solutions of (1.1) whose Stokes multipliers around $x = \infty$ are independent of t, if and only if $\lambda = \lambda(t), \mu = \mu(t)$

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satisfy the Hamiltonian system above (see [10]). In [6], H. Kimura treated the same problem concerning the linear differential equation

$$\frac{d^2y}{dx^2}-\Big(\sum_{k=1,2}\frac{1}{x-\lambda_k}\Big)\frac{dy}{dx} -\Big(9x^5+9t_1x^3+3t_2x^2+3K_2x+3K_1-\sum_{k=1,2}\frac{\mu_k}{x-\lambda_k}\Big)y=0 \qquad (1.3)$$

having two non-logarithmic singular points $x=\lambda_k$ $(k=1,2)$ with the characteristic exponents $(0,2)$. By this condition, K_j $(j=1,2)$ are determined to be certain rational functions of t_k,λ_k,μ_k $(k=1,2)$. He proved that the isomonodromic deformation of (1.3) yields the completely integrable Hamiltonian system

$$\frac{\partial\lambda_k}{\partial t_j}=\frac{\partial K_j}{\partial\mu_k},\quad \frac{\partial\mu_k}{\partial t_j}=-\frac{\partial K_j}{\partial\lambda_k}\quad (j=1,2;\ k=1,2), \qquad (1.4)$$

and that, by a symplectic transformation $q_i=q_i(t,\lambda)$, $p_i=p_i(t,\lambda,\mu)$, $s_i=s_i(t)$ $(i=1,2)$, $t=(t_1,t_2)$, $\lambda=(\lambda_1,\lambda_2)$, $\mu=(\mu_1,\mu_2)$, system (1.4) is changed into a degenerate Garnier system of the form

$$\frac{\partial q_k}{\partial s_j}=\frac{\partial H_j}{\partial p_k},\quad \frac{\partial p_k}{\partial s_j}=-\frac{\partial H_j}{\partial q_k}\quad (j=1,2;\ k=1,2) \qquad (1.5)$$

with

$$\begin{aligned}3H_1=&\Big(q_2^2-q_1-\frac{s_1}{3}\Big)p_1^2+2q_2p_1p_2+p_2^2+9\Big(q_1+\frac{s_1}{3}\Big)q_2\Big(q_2^2-2q_1+\frac{s_1}{3}\Big)-3s_2q_1,\\ 3H_2=&q_2p_1^2+2p_1p_2+9\Big(q_2^4-3q_1q_2^2+q_1^2-\frac{s_1}{3}q_1-\frac{s_2}{3}q_2\Big).\end{aligned}$$

(For Garnier systems see also [1], [7].) This system has the Painlevé property, which was proved by using the results of [9], [4], [5].

Theorem 1.0 ([11; Theorem A]). *For every solution* $\Xi=(q_1,q_2,p_1,p_2)$ *of* (1.5), *each entry is meromorphic on* $\mathbf{C}^2$.

Furthermore, (1.5) is located at the extremity of the degeneration scheme given in [6]. By these facts, system (1.5) may be regarded as a two-variables version of the first Painlevé equation (1.2). Each entry of a solution of (1.5) admits a pole along an analytic set in $\mathbf{C}^2$. We call each irreducible component of the analytic set *a pole locus*. For example, each irreducible component of the analytic set $\{(s_1,s_2)\in\mathbf{C}^2\,|\,1/q_i(s_1,s_2)=0\}$ is a pole locus of $q_i(s_1,s_2)$. For an arbitrary solution $\Xi=(q_1,q_2,p_1,p_2)$ of (1.5), a pole locus of $q_2(s_1,s_2)$ is dominant (cf. Lemma 2.2,(2)), which we call a *pole locus of* Ξ. (For basic properties of an analytic set, see [2].)

The purpose of this paper is to examine pole loci of solutions of (1.5). Our main results are stated as follows:

Theorem 1.1. *Each entry of an arbitrary solution of* (1.5) *is transcendental and admits at least one pole locus.*

Theorem 1.2. (1) *Let* Ξ *be an arbitrary solution of* (1.5). *Then each pole locus of* Ξ *coincides with an analytic set in* $\mathbf{C}^2$ *expressible in the form*

$$(\pi, f)(\mathcal{R}) = \big\{(s_1, s_2) = (\pi(s), f(s)) \bigm| s \in \mathcal{R}\big\} \,(\subset \mathbf{C}^2),$$

where

(i) $\pi : \mathcal{R} \to \mathbf{C}$, $\pi(s) = s_1$ *is the branching Riemann surface of* $f(s_1)$,

(ii) $f(s_1)$ *is a solution of the differential equation*

$$y^{(4)} = -40(y')^3 y'' - 12 s_1 y' y'' - \frac{4}{3} y y'' - \frac{20}{3}(y')^2 - \frac{2}{3} s_1 \tag{1.6}$$

$('= d/ds_1)$.

(2) *Conversely, for every solution* $y = \phi(s_1)$ *of* (1.6), *there exists a solution* Ξ_* *of* (1.5) *such that the analytic set*

$$(\pi_*, \phi)(\mathcal{R}_*) = \big\{(s_1, s_2) = (\pi_*(s), \phi(s)) \bigm| s \in \mathcal{R}_*\big\} \,(\subset \mathbf{C}^2)$$

is a pole locus of Ξ_*, *where* $\pi_* : \mathcal{R}_* \to \mathbf{C}$, $\pi_*(s) = s_1$ *is the branching Riemann surface of* $\phi(s_1)$.

Theorem 1.3. (1) *For every* $(a, B_0, B_1, B_2) \in \mathbf{C}^4$, *equation* (1.6) *admits a solution* $\psi(s_1)$ *expanded into the convergent Puiseux series*

$$\psi(s_1) = \Phi(B_0, B_1, B_2, 3^{1/3}(s_1 - a)^{1/3})$$

with

$$\Phi(B_0, B_1, B_2, \sigma) = C_0 - \sigma + \sum_{j \geq 5} C_j \sigma^j, \tag{1.7}$$
$$C_0 = B_0, \quad C_9 = B_1, \quad C_{11} = B_2$$

around $s_1 = a$, *where* C_j $(j \geq 5,\ j \neq 9, 11)$ *are uniquely determined polynomials in* a, B_0, B_1, B_2.

(2) *Let* $y = g(s_1)$ *be an arbitrary solution of* (1.6). *Suppose that, for* $s_1 = a^* \in \mathbf{C}$, *there exists a sequence* $\big\{\alpha_\nu \bigm| \nu \in \mathbf{N}\big\}$ *with the properties* :

(i) $\alpha_\nu \to a^*$ *as* $\nu \to \infty$;

(ii) $\big\{g(\alpha_\nu) \bigm| \nu \in \mathbf{N}\big\}$ *is bounded.*

Then, around $s_1 = a^*$, $g(s_1)$ *is either analytic or expressible in the form*

$$g(s_1) = \Phi(B_0^*, B_1^*, B_2^*, 3^{1/3}(s_1 - a^*)^{1/3}), \tag{1.8}$$

where B_0^*, B_1^*, B_2^* *are some complex constants.*

Theorem 1.4. *Let Ξ be an arbitrary solution of* (1.5), *and* (a_1, a_2) *be an arbitrary point on a pole locus of* Ξ. *Then around* $(s_1, s_2) = (a_1, a_2)$, *the second entry* $q_2(s_1, s_2)$ *of* Ξ *is expressible by either of the following*:

$$q_2(s_1, s_2) = \frac{Y(s_1, s_2)}{(s_2 - f(s_1))^2},$$
$$q_2(s_1, s_2) = Y_*(s_1, s_2) \sum_{l=0}^{2} \frac{1}{(s_2 - \Phi_l(s_1))^2},$$
$$\Phi_l(s_1) = \Phi(a_2, B, B', 3^{1/3} e^{2l\pi i/3} (s_1 - a_1)^{1/3}).$$

Here

(i) $f(s_1)$ *is a solution of* (1.6) *analytic at* $s_1 = a_1$ *and satisfying* $f(a_1) = a_2$;

(ii) $\Phi(\cdot, \cdot, \cdot, \sigma)$ *is a convergent series of the form* (1.7), *and* B, B' *are some complex constants*;

(iii) $Y(s_1, s_2)$ *and* $Y_*(s_1, s_2)$ *are analytic at* $(s_1, s_2) = (a_1, a_2)$ *and satisfy* $Y(a_1, a_2) = Y_*(a_1, a_2) = 1$.

Remark 1.1. By Theorem 1.3, for an arbitrary solution $g(s_1)$ of (1.6), if there exists a singular point ω_0 around which $g(s_1)$ does not admit an expression of the form (1.8), then $|g(s_1)| \to \infty$ as $s_1 \to \omega_0$ along an arbitrary curve terminating in ω_0.

Remark 1.2. Theorem 1.4 implies that, in Theorem 1.2, the mapping (π, f) : $\mathcal{R} \to \mathbf{C}^2$, $(\pi, f)(s) = (\pi(s), f(s))$ (or $(\pi_*, \phi) : \mathcal{R}_* \to \mathbf{C}^2$) is an imbedding.

In Section 2, we sum up several lemmas. Using them, we prove Theorems 1.1, 1.2, 1.3 and 1.4 in Sections 6, 3, 4 and 5, respectively. The proofs depend much on the Painlevé property of (1.5).

2. Preliminaries

We begin with the following lemma which is an immediate consequence of the Weierstrass preparatory theorem ([2; vol II]).

Lemma 2.1. *For an analytic function* $\varphi(z_1, z_2)$, *we put* $D_0 = \{(z_1, z_2) \,|\, \varphi(z_1, z_2) = 0\}$. *Suppose that* $(z_1^0, z_2^0) \in D_0$ *and that* $\varphi(z_1^0, z_2) \not\equiv 0$ *around* $z_2 = z_2^0$. *Then, there exists a sufficiently small positive constant* ε *such that, in the polydisk* $|z_j - z_j^0| < \varepsilon$ $(j = 1, 2)$, D_0 *coincides with the analytic set defined by*

$$(z_2 - z_2^0)^{m_0} + h_{m_0-1}(z_1)(z_2 - z_2^0)^{m_0-1} + \cdots + h_1(z_1)(z_2 - z_2^0) + h_0(z_1) = 0$$

$(m_0 \in \mathbf{N})$, *where* $h_l(z_1)$ $(0 \le l \le m_0 - 1)$ *are analytic for* $|z_1 - z_1^0| < \varepsilon$ *and satisfy* $h_l(z_1^0) = 0$.

From the Hamiltonian system restricted to the s_2-plane

$$\frac{\partial q_k}{\partial s_2} = \frac{\partial H_2}{\partial p_k}, \qquad \frac{\partial p_k}{\partial s_2} = -\frac{\partial H_2}{\partial q_k} \qquad (k = 1, 2), \tag{2.1}$$

we derive the following:

Lemma 2.2. *Let $\Xi = (q_1, q_2, p_1, p_2)$ be an arbitrary solution of* (1.5). *Then,*
(1) $\eta = q_2(s_1, s_2)$ *satisfies*

$$\frac{\partial^4\eta}{\partial s_2^4} = 20\eta\frac{\partial^2\eta}{\partial s_2^2} + 10\left(\frac{\partial\eta}{\partial s_2}\right)^2 - 40\eta^3 - 8s_1\eta - \frac{8}{3}s_2; \tag{2.2}$$

(2) q_1, p_1, p_2 *are expressed as*

$$q_1 = -\frac{1}{4}\frac{\partial^2 q_2}{\partial s_2^2} + \frac{3}{2}q_2^2 + \frac{s_1}{6}, \qquad p_1 = \frac{3}{2}\frac{\partial q_2}{\partial s_2}, \qquad p_2 = -\frac{3}{8}\frac{\partial^3 q_2}{\partial s_2^3} + 3q_2\frac{\partial q_2}{\partial s_2}.$$

Remark 2.1. We can regard (2.2) as a fourth order non-linear ordinary differential equation with respect to s_2 containing a complex parameter s_1.

Lemma 2.3. *Let $\eta = q(s_2)$ be an arbitrary solution of* (2.2) *with $s_1 = a_1$ ($\in \mathbf{C}$). Then, $q(s_2)$ is meromorphic on $\mathbf{C}$, and system* (1.5) *admits a solution $\Xi^*(s_1, s_2) = (q_1^*, q_2^*, p_1^*, p_2^*)$ with the properties*:
(i) *every entry of $\Xi^*(s_1, s_2)$ is meromorphic on $\mathbf{C}^2$*;
(ii) $q_2^*(a_1, s_2) = q(s_2)$.

Proof. The solution $q(s_2)$ is analytic at some point $s_2 = a_2^0 \in \mathbf{C}$. Note that

$$(q_1, q_2, p_1, p_2) = \Big(-\frac{1}{4}\frac{d^2q}{ds_2^2} + \frac{3}{2}q^2 + \frac{a_1}{6},\ q,\ \frac{3}{2}\frac{dq}{ds_2},\ -\frac{3}{8}\frac{d^3q}{ds_2^3} + 3q\frac{dq}{ds_2}\Big)$$

satisfies (2.1) with $s_1 = a_1$ (cf. Lemma 2.2,(2)). By the complete integrability of (1.5), there exists a solution $\Xi^*(s_1, s_2)$ of (1.5) whose second entry $q_2^*(s_1, s_2)$ satisfies $q_2^*(a_1, s_2) = q(s_2)$. Combining this fact with Theorem 1.0, we have the lemma. □

For each solution of (2.2), Laurent series expansions around movable poles are known ([11; Theorem C]):

Lemma 2.4. *For every $(a, b_0, b_1, b_2) \in \mathbf{C}^4$, equation* (2.2) *admits two kinds of solutions expressible by the convergent Laurent series*:

$$q(a, b_0, b_1, b_2; s_2) = (s_2 - a)^{-2} + b_0 + \sum_{j\ge 2} c_j(s_2 - a)^j, \tag{2.3}$$

$$c_3 = b_1,\ c_6 = b_2,$$

$$\tilde{q}(a, b_0, b_1; s_2) = 3(s_2 - a)^{-2} + \sum_{j\ge 2} \tilde{c}_j(s_2 - a)^j, \tag{2.4}$$

$$\tilde{c}_6 = b_0,\ \tilde{c}_8 = b_1$$

around $s_2 = a$. Here c_j ($j \ne 3, 6$) (or $\tilde{c}_j$ ($j \ne 6, 8$)) are uniquely determined polynomials in a, b_0, b_1, b_2, s_1 (or a, b_0, b_1, s_1). Conversely, every solution of (2.2) *with a movable pole at $s_2 = a$ is expressible in the form* (2.3) *or* (2.4) *with some $(b_0, b_1, b_2) \in \mathbf{C}^3$ or $(b_0, b_1) \in \mathbf{C}^2$, respectively.*

The first several coefficients of (2.3) are computed recursively:

Lemma 2.5. *In series* (2.3),

$$c_2 = -3b_0^2 - \frac{s_1}{5}, \qquad c_4 = -10b_0^3 - \frac{4}{7}s_1 b_0 + \frac{a}{21}, \qquad c_5 = \frac{3}{2}b_0 b_1 + \frac{1}{30},$$
$$c_7 = \frac{2}{105}b_0 - \frac{2}{35}s_1 b_1, \qquad c_8 = \frac{3}{11}c_2 c_4 - \frac{3}{22}b_0 c_2^2 + \frac{9}{22}b_0 b_2 + \frac{9}{88}b_1^2,$$
$$c_9 = \frac{55}{243}c_2 c_5 + \frac{25}{81}b_0 c_7 + \frac{5}{27}b_1 c_4 - \frac{10}{81}b_0 b_1 c_2.$$

Combining Lemma 2.2,(2) with $\partial q_2/\partial s_1 = \partial H_1/\partial p_2 = (2/3)(q_2 p_1 + p_2)$, we have the relation below:

Lemma 2.6. *For every solution* $\Xi = (q_1, q_2, p_1, p_2)$ *of* (1.5),

$$\frac{\partial q_2}{\partial s_1} = 3q_2 \frac{\partial q_2}{\partial s_2} - \frac{1}{4}\frac{\partial^3 q_2}{\partial s_2^3}. \tag{2.5}$$

Suppose that a solution $\Xi = (q_1, q_2, p_1, p_2)$ of (1.5) admits a pole locus D_∞.

Lemma 2.7. *The pole locus* D_∞ *does not admit an expression* $s_1 - s_1^0 \equiv 0$, $s_1^0 \in \mathbf{C}$ *around any point belonging to* D_∞.

Proof. Suppose the contrary. Then there exists a point $(s_1, s_2) = (s_1^0, s_2^0)$, around which $q_2(s_1, s_2)$ is expressible in the form $q_2(s_1, s_2) = h(s_1, s_2)/(s_1 - s_1^0)^{n_0}$, $n_0 \in \mathbf{N}$, where $h(s_1, s_2)$ is analytic and satisfies $h(s_1^0, s_2^0) \neq 0$. Substituting this into (2.2) and noting the multiplicity of the pole $s_1 = s_1^0$, we arrive at a contradiction. □

Lemma 2.8. *Suppose that the pole locus* D_∞ *intersects no other pole loci at* $(a_1, a_2) \in D_\infty$, *and that, around* (a_1, a_2), *it is expressible in the form* $s_2 = \varphi(s_1)$, *where* $\varphi(s_1)$ *is analytic at* $s_1 = a_1$, *and satisfies* $\varphi(a_1) = a_2$. *Then, around* (a_1, a_2),

$$q_2(s_1, s_2) = (s_2 - \varphi(s_1))^{-2}\big(1 + O(s_2 - \varphi(s_1))\big).$$

Proof. Observing that $q_2(s_1, s_2)$ satisfies (2.2) for each s_1, from Lemma 2.4, we obtain

$$q_2(s_1, s_2) = (s_2 - \varphi(s_1))^{-2} Q(s_1, s_2) \tag{2.6}$$

or

$$q_2(s_1, s_2) = 3(s_2 - \varphi(s_1))^{-2} Q(s_1, s_2), \tag{2.7}$$

where $Q(s_1, s_2)$ is analytic around $(s_1, s_2) = (a_1, a_2)$ and satisfies $Q(a_1, a_2) = 1$. Substituting (2.7) into (2.5) and comparing both sides, we arrive at a contradiction. This implies that $q_2(s_1, s_2)$ is expressible in the form (2.6). □

Lemma 2.9. *For an arbitrary point* $(a_1, a_2) \in D_\infty$, *we have either of the following*:

$$q_2(s_1, s_2) = \frac{u(s_1, s_2)}{(s_2 - \varphi(s_1))^2}, \tag{2.8}$$

$$q_2(s_1, s_2) = u_*(s_1, s_2) \sum_{l=0}^{2} \frac{1}{(s_2 - \chi_l(s_1))^2}, \tag{2.9}$$

$$\chi_l(s_1) = \chi(e^{2l\pi i/3}(s_1 - a_1)^{1/3}).$$

Here

(i) $\varphi(s_1)$ *is analytic at* $s_1 = a_1$ *and satisfies* $\varphi(a_1) = a_2$;

(ii) $\chi(\tau)$ *is analytic at* $\tau = 0$ *and satisfies* $\chi(0) = a_2$;

(iii) $u(s_1, s_2)$ *and* $u_*(s_1, s_2)$ *are analytic at* $(s_1, s_2) = (a_1, a_2)$ *and satisfy* $u(a_1, a_2) = u_*(a_1, a_2) = 1$.

Proof. By Lemmas 2.1 and 2.7, for sufficiently small $\varepsilon_0 > 0$, the set D_∞ restricted to the polydisk $\Delta : |s_j - a_j| < \varepsilon_0$ $(j = 1, 2)$ is decomposed as below:

$$D_\infty \cap \Delta = D_\infty^1 \cup \cdots \cup D_\infty^m,$$

where

(1) D_∞^μ $(1 \leq \mu \leq m)$ are local irreducible components in Δ passing through (a_1, a_2),

(2) each D_∞^μ are expressed by a $\nu(\mu)$-valued local algebroidal function $s_2 = \psi_\mu(s_1)$ $(\nu(\mu) \in \mathbf{N})$, whose branches are given by

$$\psi_\mu^k(s_1) = \Psi_\mu(e^{2k\pi i/\nu(\mu)}(s_1 - a_1)^{1/\nu(\mu)}), \quad k = 0, 1, ..., \nu(\mu) - 1,$$
$$\Psi_\mu(\tau) = a_2 + \sum_{j \geq 1} \gamma_{\mu,j}\tau^j.$$

Let us consider the function

$$U(s_1, s_2) = q_2(s_1, s_2)\Bigg[\sum_{\mu=1}^{m}\Bigg(\sum_{k=0}^{\nu(\mu)-1} \frac{1}{(s_2 - \psi_\mu^k(s_1))^2}\Bigg)\Bigg]^{-1}. \tag{2.10}$$

Then $U(s_1, s_2)$ is single-valued in Δ. We put $\Delta^* = \Delta - \Pi_0$, $\Pi_0 = \big\{(a_1, s_2) \bigm| |s_2 - a_2| < \varepsilon_0\big\}$. Note that $U(s_1, s_2)$ is analytic in $\Delta^* - (D_\infty^1 \cup \cdots \cup D_\infty^m)$, and that, by Lemma 2.8, $U(s_1, s_2) = 1$ along each D_∞^μ $(1 \leq \mu \leq m)$. Hence $U(s_1, s_2)$ is analytic in Δ^*. Since $U(s_1, s_2)$ is also analytic around each point belonging to $\Pi_0 - \big\{(a_1, a_2)\big\}$, it is analytic in $\Delta - \{(a_1, a_2)\}$. Hence $U(s_1, s_2)$ is analytic in Δ (cf. [2; vol I]). By the continuity at (a_1, a_2), we have

$$U(s_1, s_2) = 1 + O(|s_1 - a_1| + |s_2 - a_2|). \tag{2.11}$$

Put $s_1 = a_1$ in (2.10). Then we have

$$q_2(a_1, s_2) = \Big(\sum_{\mu=1}^{m} \nu(\mu)\Big)(s_2 - a_2)^{-2}\big(1 + O(s_2 - a_2)\big).$$

By Lemma 2.4, either of the following four cases may occur:

(a) $m = 1, \nu(1) = 1$;

(b) $m = 1, \nu(1) = 3$;

(c) $m = 2, \nu(1) = 2, \nu(2) = 1$;

(d) $m = 3, \nu(1) = \nu(2) = \nu(3) = 1$.

In the case (c), we have

$$q_2(s_1,s_2)=U(s_1,s_2)\Big(\frac{1}{(s_2-\psi_-(s_1))^2}+\frac{1}{(s_2-\psi_+(s_1))^2}+\frac{1}{(s_2-\psi_0(s_1))^2}\Big),$$

where

$$\psi_0(s_1)=a_2+\sum_{j\ge1}\gamma_{0,j}(s_1-a_1)^j,$$

$$\psi_-(s_1)=\Psi(-(s_1-a_1)^{1/2}),\quad \psi_+(s_1)=\Psi((s_1-a_1)^{1/2}),\quad \Psi(\tau)=a_2+\sum_{j\ge1}\gamma_j\tau^j.$$

Substitute this into (2.5), and put $s_1=a_1$. Then we see that the right-hand side admits a pole $s_2=a_2$ of multiplicity 5, and that the multiplicity of the pole on the left-hand side does not exceed 4. Hence the case (c) does not occur. By the same way, we can verify that (d) is also impossible. In the cases (a) and (b), from (2.10) with (2.11) we can derive expressions (2.8) and (2.9), respectively. Thus the proof is completed. □

The following is an immediate consequence of Clunie's lemma [8; Lemma 2.4.2].

Lemma 2.10. *Let $g(z)$ be a transcendental meromorphic function satisfying the relation $g^{n+1}=Q(z,g)$ $(n\in\mathbf{N})$, where $Q(z,u)$ is a polynomial in z,u and $u^{(k)}$ $(k=1,2,...)$. If the total degree of $Q(z,u)$ as a polynomial in u and its derivatives is at most n, then $g(z)$ admits infinitely many poles in $\mathbf{C}$.*

3. Proof of Theorem 1.2

3.1. Proof of the assertion (1)

Let D_∞ be a pole locus of Ξ. By Lemma 2.9, we can take a point $(a_1,a_2)\in D_\infty$ such that, around $(s_1,s_2)=(a_1,a_2)$, D_∞ is expressible in the form $s_2=f(s_1)$, where $f(s_1)$ is analytic at $s_1=a_1$ and satisfies $f(a_1)=a_2$. Then, using Lemma 2.4, by the same argument as in the proof of Lemma 2.8, we have

$$q_2(s_1,s_2)=(s_2-f(s_1))^{-2}+b_0(s_1)+\sum_{j\ge2}c_j(s_1)(s_2-f(s_1))^j, \tag{3.1}$$

$$c_3(s_1)=b_1(s_1),\quad c_6(s_1)=b_2(s_1),$$

around $(s_1,s_2)=(a_1,a_2)$, where $b_0(s_1)$, $c_j(s_1)$ are analytic at $s_1=a_1$. Substituting (3.1) into (2.5) and comparing the coefficients of $(s_2-f(s_1))^j$ $(-3\le j\le6)$, we obtain

$$f'=-3b_0,\quad b_0'=\frac{3}{2}b_1,\quad b_1'-4c_4f'=-18b_2+6c_2^2+12b_0c_4,$$

$$b_2'-7c_7f'=21c_2c_5+21b_0c_7+21b_1c_4-105c_9\qquad ('=d/ds_1).$$

From these relations and Lemma 2.5 with $a=f(s_1)$, it follows that

$$f'=-3b_0, \tag{3.2,1}$$

$$b_0'=\frac{3}{2}b_1, \tag{3.2,2}$$

$$b_1'=-18b_2+6\Big(-3b_0^2-\frac{s_1}{5}\Big)^2, \tag{3.2,3}$$

$$b_2'=-42b_0^3b_1-\frac{46}{135}b_0^2-\frac{4}{5}s_1b_0b_1+\frac{2}{27}b_1f+\frac{112}{6075}s_1. \tag{3.2,4}$$

Eliminating b_0, b_1, b_2, we can verify that $f(s_1)$ satisfies (1.6). By $f(s), s \in \mathcal{R}$, we denote the analytic continuation of it, where $\pi : \mathcal{R} \to \mathbf{C}$ is the branching Riemann surface of f. By Lemma 2.1 and the connectedness of D_∞, for every point $(\hat{a}_1, \hat{a}_2) \in D_\infty$, each local irreducible component of D_∞ in a small neighbourhood of $(\hat{a}_1, \hat{a}_2)$ is expressible in the form $\{(\pi(s), f(s)) \mid s \in \mathcal{R}, |s - \hat{a}| < \hat{\delta}\}$, where $\hat{a} \in \mathcal{R}$ is some point satisfying $\pi(\hat{a}) = \hat{a}_1$, and $\hat{\delta}$ is a sufficiently small positive constant. Thus we arrive at the desired expression of D_∞.

3.2. Proof of the assertion (2)

Let $\phi(s_1)$ be a solution of (1.6) which is analytic around $s_1 = a_1$. We set

$$\beta_0 = -\frac{\phi'(a_1)}{3}, \qquad \beta_1 = -\frac{2}{9}\phi''(a_1),$$
$$\beta_2 = \frac{\phi^{(3)}(a_1)}{81} + \frac{1}{3}\Big(\frac{\phi'(a_1)^2}{3} + \frac{a_1}{5}\Big)^2 \tag{3.3}$$

$({}' = d/ds_1)$. Consider equation (2.2) with $s_1 = a_1$ and its solution

$$\begin{aligned} q(s_2) =& q(a_2, \beta_0, \beta_1, \beta_2; s_2) \\ =& (s_2 - a_2)^{-2} + \beta_0 + \sum_{j \geq 2} c_j (s_2 - a_2)^j, \quad c_3 = \beta_1,\ c_6 = \beta_2 \end{aligned} \tag{3.4}$$

around $s_2 = a_2$ (cf. (2.3)). Then by Lemma 2.3, there exists a solution $\Xi^* = (q_1^*, q_2^*, p_1^*, p_2^*)$ of (1.5) such that

$$q_2^*(a_1, s_2) = q(s_2). \tag{3.5}$$

Around $(s_1, s_2) = (a_1, a_2)$, there exists a local irreducible component of a pole locus of Ξ^*, which passes through (a_1, a_2). By (3.4) and Lemma 2.9, it is expressible by $s_2 = f_*(s_1)$, where $f_*(s_1)$ is analytic around $s_1 = a_1$ and satisfies $f_*(a_1) = a_2$. By the same argument as in Section 3.1, we can show that $f_*(s_1)$ satisfies (1.6), and obtain the expression of the second entry of Ξ^*

$$q_2^*(s_1, s_2) = (s_2 - f_*(s_1))^{-2} + b_0^*(s_1) + \sum_{j \geq 2} c_j^*(s_1)(s_2 - f_*(s_1))^j,$$
$$c_3^*(s_1) = b_1^*(s_1),\ c_6^*(s_1) = b_2^*(s_1)$$

(cf. (3.1)), where

$$b_0^*(s_1) = -\frac{f_*'(s_1)}{3}, \qquad b_1^*(s_1) = -\frac{2}{9}f_*''(s_1),$$
$$b_2^*(s_1) = \frac{f_*^{(3)}(s_1)}{81} + \frac{1}{3}\Big(\frac{f_*'(s_1)^2}{3} + \frac{s_1}{5}\Big)^2. \tag{3.6}$$

Then, by (3.5),

$$b_0^*(a_1) = \beta_0, \quad b_1^*(a_1) = \beta_1, \quad b_2^*(a_1) = \beta_2. \tag{3.7}$$

These formulas are derived from the relations corresponding to (3.2,k) ($1 \le k \le 3$). By (3.3), (3.6) and (3.7), we have $f'_*(a_1) = \phi'(a_1)$, $f''_*(a_1) = \phi''(a_1)$, $f^{(3)}_*(a_1) = \phi^{(3)}(a_1)$. By the uniqueness of a solution of the initial value problem associated with (1.6), we have $f_*(s_1) = \phi(s_1)$ around $s_1 = a_1$. This implies that Ξ^* admits a pole locus expressible by the analytic continuation of $\phi(s_1)$. By the same argument as in Section 3.1, this pole locus is globally expressible by ϕ. Thus the proof is completed.

4. Proof of Theorem 1.3

4.1. Proof of the assertion (1)

By the change of the variables $\sigma = 3^{1/3}(s_1 - a)^{1/3}$, $y = B_0 - \sigma + v$, equation (1.6) is taken into

$$\sigma^4 v^{(4\cdot)} - 12\sigma^3 v^{(3\cdot)} + 12\sigma^2 \ddot{v} + 240\sigma\dot{v} = G(\sigma, v, \dot{v}, \ddot{v}), \tag{4.1}$$

$$G(\sigma, v, \dot{v}, \ddot{v}) = 80\sigma(6 - 4\dot{v} + (\dot{v})^2)(\dot{v})^2 - 40\sigma^2(3 - 3\dot{v} + (\dot{v})^2)\dot{v}\ddot{v}$$
$$- 12\sigma^5(a + \sigma^3/3)(\sigma\ddot{v} - 2(\dot{v} - 1))(\dot{v} - 1) - \frac{4}{3}\sigma^7(B_0 - \sigma + v)(\sigma\ddot{v} - 2(\dot{v} - 1))$$
$$- \frac{20}{3}\sigma^8(\dot{v} - 1)^2 - \frac{2}{3}\sigma^{12}(a + \sigma^3/3)$$

($\dot{} = d/d\sigma$). Substitute the formal series $v = \sum_{j\ge5} C_j\sigma^j$ into (4.1) and compare the coefficients of σ^j ($j \ge 5$). This series satisfies (4.1) if and only if

$$j(j-9)(j-11)(j+2)C_j = P_j(a, B_0, C_k; 5 \le k \le j-1), \quad j \ge 5, \tag{4.2}$$

where P_j ($j \ge 5$) are polynomials in a, B_0, C_k. For $5 \le j \le 11$, they are given by

$$P_5 = 24a, \quad P_6 = 0, \quad P_7 = -\frac{8}{3}B_0, \quad P_8 = 4, \quad P_9 = 0,$$
$$P_{10} = 72(a - 50C_5)C_6, \quad P_{11} = -\frac{40}{3}B_0C_5 + 168(a - 50C_5)C_7 - 4320C_6^2.$$

From (4.2) with $j = 5, ..., 10$, we have

$$C_5 = \frac{a}{35}, \quad C_6 = 0, \quad C_7 = -\frac{B_0}{189}, \quad C_8 = \frac{1}{60}, \quad C_9 = B_1, \quad C_{10} = 0,$$

where B_1 is an arbitrary constant. Since, for C_5, C_6, C_7 above, P_{11} vanishes, the coefficient C_{11} can also be taken to be an arbitrary constant B_2. In addition to these coefficients, determining C_j ($j \ge 12$) recursively by (4.2), we obtain a formal solution $v = \sum_{j\ge5} C_j\sigma^j$ ($C_9 = B_1, C_{11} = B_2$) of (4.1). The formal series $u = h(\sigma) = \sum_{k\ge0} C_{k+5}\sigma^k$ satisfies

$$\sigma^4 u^{(4\cdot)} + 8\sigma^3 u^{(3\cdot)} - 48\sigma^2\ddot{u} - 120\sigma\dot{u} + 840u = H(\sigma, u, \sigma\dot{u}, \sigma^2\ddot{u})$$
$$= \sigma^{-5}G(\sigma, \sigma^5 u, \sigma^5\dot{u} + 5\sigma^4 u, \sigma^5\ddot{u} + 10\sigma^4\dot{u} + 20\sigma^3 u), \tag{4.3}$$

where $H(\sigma, \xi_0, \xi_1, \xi_2)$ is a polynomial in $(\sigma, \xi_0, \xi_1, \xi_2)$ such that $H(0, \xi_0, \xi_1, \xi_2) \equiv 24a$. Equation (4.3) is written in the form

$$\sigma\mathbf{u}' = \Lambda\mathbf{u} + \mathbf{p}(\sigma, \mathbf{u}),$$

where

$$\mathbf{u}=\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix},\quad \Lambda=\begin{pmatrix}0&1&0&0\\0&1&1&0\\0&0&2&1\\-840&120&48&-5\end{pmatrix},\quad \mathbf{p}(\sigma,\mathbf{u})=\begin{pmatrix}0\\0\\0\\H(\sigma,u_0,u_1,u_2)\end{pmatrix},$$

$$u_0=u,\quad u_1=\sigma\dot{u},\quad u_2=\sigma^2\ddot{u},\quad u_3=\sigma^3u^{(3\cdot)}.$$

This system admits the formal solution

$$\mathbf{u}(\sigma)={}^t(h(\sigma),\sigma\dot{h}(\sigma),\sigma^2\ddot{h}(\sigma),\sigma^3h^{(3\cdot)}(\sigma))=\sum_{k\geq 0}{}^t\mathbf{c}_k\sigma^k,$$

$\mathbf{c}_k=(c_{k1},c_{k2},c_{k3},c_{k4})\in\mathbf{C}^4$. By the same argument as in [11; §6.2], we can construct a convergent power series $\sum_{k\geq 0}\Gamma_k\sigma^k$ such that $\Gamma_k\geq\max_{1\leq i\leq 4}|c_{ki}|$ $(k\geq 0)$, and show the convergence of $\Phi(B_0,B_1,B_2,\sigma)=B_0-\sigma+\sigma^5h(\sigma)$ around $\sigma=0$.

4.2. Proof of the assertion (2)

For an arbitrary solution $y=g(s_1)$ of (1.6), by Theorem 1.2,(2), there exists a solution Ξ_* of (1.5) such that $s_2=g(s_1)$ represents a pole locus D^*_∞ of Ξ_* which contains $\{(\alpha_\nu,g(\alpha_\nu))\mid\nu\in\mathbf{N}\}$. By supposition, we can choose a sequence $\{\alpha'_\nu\mid\nu\in\mathbf{N}\}\subset\{\alpha_\nu\mid\nu\in\mathbf{N}\}$ in such a way that $g(\alpha'_\nu)$ converges to some constant $B^*_0\in\mathbf{C}$ as $\alpha'_\nu\to a^*$. Then (a^*,B^*_0) belongs to D^*_∞. Hence, by Lemma 2.9, we have $g(a^*)=B^*_0$. Moreover, by the same lemma, if $g(s_1)$ is not analytic at $s_1=a^*$, then it is written in the form

$$g(s_1)=T(3^{1/3}(s_1-a^*)^{1/3}),\quad T(\sigma)=\sum_{j\geq 0}C'_j\sigma^j,\quad C'_0=B^*_0$$

around $s_1=a^*$. Since $g(s_1)$ is a solution of (1.6), $V(\sigma)=T(\sigma)-B^*_0+\sigma=\sum_{j\geq 1}C''_j\sigma^j$ satisfies (4.1) with $(a,B_0)=(a^*,B^*_0)$. Substituting $V(\sigma)$ into (4.1) and comparing the coefficients of σ^j, we derive that $C''_1=\cdots=C''_4=0$, and that

$$j(j-9)(j-11)(j+2)C''_j=P_j(a^*,B^*_0,C''_k;5\leq k\leq j-1),\quad j\geq 5$$

(cf. (4.2)), from which C''_j $(j\geq 5,\ j\neq 9,11)$ are uniquely determined to be polynomials in a^*, B^*_0, $B^*_1=C''_9$, $B^*_2=C''_{11}$. Thus we obtain expression (1.8).

5. Proof of Theorem 1.4

This theorem immediately follows from Lemma 2.9, Theorem 1.2,(1), and Theorem 1.3,(2).

6. Proof of Theorem 1.1

Let $\Xi=(q_1,q_2,p_1,p_2)$ be an arbitrary solution of (1.5). Note that $\eta=q_2(0,s_2)$ satisfies (2.2) with $s_1=0$. Supposing that $q_2(0,s_2)$ is rational in s_2, and substituting the Laurent series expansion around $s_2=\infty$ into (2.2) with $s_1=0$, we can derive a contradiction. Hence, $q_2(0,s_2)$ is a transcendental meromorphic function of s_2. By Lemma 2.10, $q_2(0,s_2)$ admits infinitely many poles $s_2=\rho_\iota$ $(\iota\in\mathbf{N})$. By Lemma

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2.9, for each ρ_ι, $q_2(s_1,s_2)$ admits the pole locus passing through $(0,\rho_\iota)$ expressible by $s_2=f_\iota(s_1)$, where $f_\iota(s_1)$ is analytic or locally algebroidal at $s_1=0$. Since each branch point of $f_\iota(s_1)$ is isolated, for every $N\in\mathbf{N}$, we can choose $s_1=a^{(N)}$ near $s_1=0$ in such a way that, for every $\iota=1,...,N$, the function $f_\iota(s_1)$ is analytic at $s_1=a^{(N)}$. Then, by Lemma 2.2 and (3.1), we have

$$\begin{aligned} q_1(s_1,s_2)&=3b_0(s_1)(s_2-f_\iota(s_1))^{-2}\big(1+O(s_2-f_\iota(s_1))\big),\\ p_1(s_1,s_2)&=-3(s_2-f_\iota(s_1))^{-3}\big(1+O(s_2-f_\iota(s_1))\big),\\ p_2(s_1,s_2)&=3(s_2-f_\iota(s_1))^{-5}\big(1+O(s_2-f_\iota(s_1))\big),\end{aligned}$$

around $(a^{(N)},f_\iota(a^{(N)}))$. Since $f_\iota(s_1)$ satisfies (1.6), we have $b_0(s_1)=-f'_\iota(s_1)/3\not\equiv 0$ (cf. (3.2,1)) and may suppose $b_0(a^{(N)})\neq 0$. Then the functions $q_1(a^{(N)},s_2)$, $p_1(a^{(N)},s_2)$, $p_2(a^{(N)},s_2)$ also have at least N distinct poles $s_2=f_\iota(a^{(N)})$ $(1\le\iota\le N)$. Since N is arbitrary, every entry of Ξ is transcendental. Thus the proof is completed.

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