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Prime Geodesic Theorem for the Picard Manifold under the mean-Lindelöf Hypothesis

by

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Abstract. The error term of the prime geodesic theorem for the Picard manifold is improved under assuming the mean version of the Lindelöf hypothesis for automorphic L-functions. We obtain the bound $O(X^{\frac{11}{7}+\varepsilon})$. The unconditional current best bound is $O(X^{\frac{5}{3}+\varepsilon})$ by Sarnak.

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1. Introduction. In the mid-fifties Selberg [Se] introduced a zeta function for a hyperbolic surface M with fundamental group Γ. It is called the Selberg zeta function of Γ. It is defined by an Euler product, which is analogous to the Riemann zeta function. The product is taken over primitive hyperbolic conjugacy classes p of Γ. Each Euler factor contains the norm $N(p)$ of conjugacy classes, which can be viewed as 'pseudoprimes' in the sense that they have the same asymptotic distribution as the rational primes, namely,

$$\pi_\Gamma(X) = \#\{p\ :\ \text{primitive conjugacy class, } N(p) \leq X\} \sim \mathrm{li}X. \tag{1.1}$$

It is well-known that there is one-to-one correspondence between the set of hyperbolic conjugacy classes of Γ and the set of homotopy classes of M, and that each homotopy class contains just one closed geodesic. Therefore (1.1) is called the prime geodesic theorem, since the norm is equal to $e^{l(p)}$, where $l(p)$ is the length of the geodesic corresponding to p. The formula (1.1) can be proved in an analogous way to that of the classical prime number theorem by using the Selberg zeta function in place of the Riemann zeta function.

The prime geodesic theorem is different, however, from the classical prime number theorem, when we try to have a good error term. In the classical case, the Riemann hypothesis would give us the best possible error term. But in (1.1), we cannot have the sharp estimate, even if we prove the analog of the Riemann hypothesis for the Selberg zeta function. This is because of the abundance of non-trivial zeros, which is clearly explained in [I] Section 1. Actually, even when the analog of the Riemann hypothesis is proved, we have the error term $O(X^{\frac{3}{4}+\varepsilon})$, which is far from the conjectural term $O(X^{\frac{1}{2}+\varepsilon})$.

For the purpose of improving the error term, we need to estimate a certain sum which is taken over a set of non-trivial zeros of the Selberg zeta function. Iwaniec[I] found for $\Gamma = PSL(2, \mathbf{Z})$ that the estimate is closely related to the mean Lindelöf hypothesis of the Rankin-Selberg L-function, and he was able to prove the bound of $O(X^{\frac{35}{48}+\varepsilon})$ for the remainder.

Later, in the development of the theory of arithmetic quantum chaos introduced by Sarnak [S2] we have become aware that the Lindelöf hypothesis of the L-function is closely related to quantum ergodicity [LS]. In their paper [LS], Luo and Sarnak have proved the mean Lindelöf hypothesis for the Rankin-Selberg L-function. As application, they obtained $O(X^{\frac{7}{10}+\varepsilon})$ for $\Gamma = PSL(2, \mathbf{Z})$ as the error term of (1.1). More recently Luo-Rudnick-Sarnak have generalized the result to any congruence subgroup Γ in $PSL(2, \mathbf{Z})$ by proving the bound $\lambda_1 \geq 0.21...$ for the first eigenvalue λ_1 of the Laplacian [LRS].

Turning our eyes to the three dimensional case, we consider arithmetic hyperbolic manifolds with fundamental group $\Gamma \subset PSL(2, \mathbf{C})$. The Selberg zeta function for Γ is analogously defined (See (4.1) below). In this case, the prime geodesic theorem takes the form

$$\pi_\Gamma(X) \sim \mathrm{li} X^2. \tag{1.2}$$

The best error term obtained so far is $O(X^{\frac{5}{3}+\varepsilon})$ by Sarnak[S1] for the case of $\Gamma = PSL(2, O)$ with O the integer ring of an imaginary quadratic field K of class number one. This result can be regarded as the one which corresponds with $O(X^{\frac{3}{4}+\varepsilon})$ in the two dimensional case in the sense that no cancellation in the sum over non-trivial zeros is considered.

Throughout this paper we treat the case of $\Gamma = PSL(2, \mathbf{Z}[\sqrt{-1}])$. The corresponding hyperbolic manifold is 3-dimensional and is called the Picard manifold. The goal of this paper is to improve the error term by using the method of [LS], under some assumption on an estimate of certain L-functions.

The main thoerem is as follows:

Theorem 1.1. *Under assuming the mean version of the λ-aspect of the Lindelöf hypothesis for the second symmetric power L-function (Assumption 3.2 below), we have for the Picard manifold*

$$\pi_\Gamma(X) = \mathrm{li} X^2 + O(X^{\frac{11}{7}+\varepsilon}). \tag{1.3}$$

It seems natural not to limit ourselves to the Picard manifold. The only reason for the restriction is our not having the explicit form of the Kuznetsov formula for more general cases. We have found that the extension made by Miatello-Wallach [MW] is not explicit enough, since the Kloosterman term in their formula is expressed in terms of the I-Bessel function which is hard to deal with. For proving Theorem 1.1, we will employ the explicit Kuznetsov formula obtained by Motohashi [M]. As is pointed out in [M2, p.95], it is a matter of technicality to extend the result to Bianchi groups defined over arbitrary imaginary quadratic fields. We will mention in Remark 4.2 below how to generalize the result to Bianchi groups once we obtain the explicit Kuznetsov formula.

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2. Kuznetsov Formula. A point in the hyperbolic three-dimensional space $\mathbf{H}^3$ is denoted by $v = z + yj$, $z = x_1 + x_2 i \in \mathbf{C}$, $y > 0$. We put $K = \mathbf{Q}(\sqrt{-1})$ and its integer ring $O = \mathbf{Z}[\sqrt{-1}]$. The group $\Gamma = PSL(2, O)$ acts on $\mathbf{H}^3$ and the quotient space $M = \Gamma\backslash\mathbf{H}^3$ is a three dimensional arithmetic hyperbolic manifold which is called the Picard manifold. The Laplacian on M is defined by

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial}{\partial y}.$$

It has the self-adjoint extension on $L^2(M)$. It is known that the spectra of Δ is composed of both discrete and continuous ones. An eigenfunction for a discrete spectrum is called a cusp form. We denote it by $u_j(v)$ $(j = 1, 2, 3, \ldots)$ with eigenvalue λ_j. We shall assume the $u_j(v)$'s to be chosen so that they are simultaneous eigenfunctions of the ring of Hecke operators and L^2-normalized. The Fourier development of $u_j(v)$ is given in [S1] (2.20):

$$u_j(v) = \sum_{n \in O^*} \rho_j(n) y K_{ir_j}(2\pi|n|y) e(\langle n, z\rangle), \tag{2.1}$$

where $\lambda_j = 1 + r_j^2$, $\langle n, z\rangle$ is the standard inner product in $\mathbf{R}^2$ and K is the K-Bessel function.

The Kuznetsov formula was discovered by Kuznetsov[Ku] for $SL(2,\mathbf{Z})$ by calculating the inner product of Poincare series in two different ways. It was generalized by Motohashi [M] for our case:

Theorem 2.1. *Let*

$$S(m,n;c)=\sum_{a\in(O/(c))^*} e(\langle m,a/c\rangle)e(\langle n,a^*/c\rangle)$$

with $aa^*\equiv 1 \pmod c$ *be the Kloosterman sum for* $m,n,c\in O$. *Assume that the function* $h(r)$, $r\in\mathbf{C}$, *is holomorphic in the horizontal strip* $\{r\in\mathbf{C} : |\mathrm{Im}(r)|<\frac{1}{2}+\varepsilon\}$ *and satisfies* $h(r)=h(-r)$, $|h(r)|\ll(1+|r|)^{-3-\varepsilon}$ *for an arbitrary fixed* $\varepsilon>0$. *Then for any non-zero* $m,n\in O$,

$$D+C=U+S \tag{2.2}$$

holds for

$$\begin{aligned}
D&=\sum_{j=1}^{\infty}\frac{r_j\rho_j(n)\overline{\rho_j(m)}}{\sinh \pi r_j}h(r_j)\\
C&=2\pi\int_{-\infty}^{\infty}\frac{\sigma_{ir}(n)\sigma_{ir}(m)}{|mn|^{ir}|\zeta_K(1+ir)|^2}h(r)dr\\
U&=\frac{\delta_{m,n}+\delta_{m,-n}}{\pi^2}\int_{-\infty}^{\infty}h(r)r^2dr\\
S&=\sum_{c\in O^*}\frac{S(m,n;c)}{|c|^2}\int_{-\infty}^{\infty}\frac{r^2}{\sinh\pi r}h(r)\mathcal{J}_{ir}(z)dr,
\end{aligned}$$

where $\sigma_s(n)=\sum_{d|n}d^s$, $z=\frac{2\pi(\overline{mn})^{\frac{1}{2}}}{c}$, *and*

$$\mathcal{J}_\nu(z)=2^{-2\nu}|z|^{2\nu}J_\nu^*(z)J_\nu^*(\overline{z}),$$

where $J_\nu^*(z)$ *is the entire function equal to* $J_\nu(z)(\frac{z}{2})^{-\nu}$ *with* J_ν *being the J-Bessel function of order* ν.

3. Rankin-Selberg L-function. For a Maass-Hecke cusp form $u_j(v)$ with its Fourier development given by (2.1), we have the Rankin-Selberg convolution L-function $L(s, u_j \times u_j)$ and the second symmetric power L-function $L^{(2)}(s, u_j)$ which satisfy the following:

$$L(s, u_j \times u_j) = \zeta_K(2s) \sum_{n \in O^*} \frac{|\lambda_j(n)|^2}{N(n)^s}$$
$$L^{(2)}(s, u_j) = \sum_{n \in O^*} \frac{c_j(n)}{N(n)^s} = \zeta_K(s)^{-1} L(s, u_j \times u_j),$$

with $\rho_j(n) = \sqrt{\frac{\sinh \pi r_j}{r_j}} v_j(n)$, $v_j(n) = v_j(1)\lambda_j(n)$ and $c_j(n) = \sum_{l^2 k = n} \lambda_j(k^2)$. It is known that the both functions converge in $\mathrm{Re}(s) > 1$. The functional equation of $L(s, u_j \times u_j)$ is inherited from Eisenstein series by our unfolding the integral. We compute that

$$\hat{L}(s) := \int_{\Gamma \backslash \mathbf{H}^3} |u_j(v)|^2 E(v, 2s) dv = |\rho_j(1)|^2 \frac{L(s, u_j \times u_j)}{\zeta_K(2s)} \frac{\Gamma(s + ir_j)\Gamma(s - ir_j)\Gamma(s)^2}{8\pi^{2s}\Gamma(2s)}$$

satisfies $\hat{L}(s) = \phi(s)\hat{L}(1-s)$ where $\phi(s)$ is the scattering determinant. We normalize such that $\|u_j\| = 1$ with respect to the Petersson inner product

$$\langle f, g \rangle = \frac{1}{\mathrm{vol}(\Gamma \backslash \mathbf{H}^3)} \int_{\Gamma \backslash \mathbf{H}^3} f(v) \overline{g(v)} dv.$$

The residue R_j of $L(s, u_j \times u_j)$ at the simple pole at $s = 1$ satisfies

$$|v_j(1)|^{-2} \ll R_j \ll |v_j(1)|^{-2}.$$

From multiplicativity of Hecke eigenvalues, we have $|v_j(1)| \neq 0$. The following proposition describes how large it can be.

Proposition 3.1. *The residue R_j of $L(s, u_j \times u_j)$ at $s = 1$ satisfies the following bound:*

$$r_j^{-\varepsilon} \ll R_j \ll r_j^{\varepsilon}.$$

In other words,

$$r_j^{\varepsilon} \gg |v_j(1)| \gg r_j^{-\varepsilon}. \tag{3.1}$$

Proof. The right inequality is already proved in the proofs of [K2] Theorem 2.1 and Lemma 2.2. The left one is a version of [HL] Corollary 0.3. Although they prove explicitly for the case of $PSL(2,\mathbf{R})$, everything goes through in our case. □

In what follows, $X \ll_w Y$ means that $|X| \le CY$ for an implied positive constant C depending on w. We will also denote $T < r_j \le 2T$ by $r_j \sim T$.

Assumption 3.2 (mean-Lindelöf). *The second symmetric power L-function $L^{(2)}(s, u_j)$ attached to a cusp form u_j of $\Gamma = PSL(2, O)$ satisfies*

$$\sum_{r_j \sim T} |L^{(2)}(w, u_j)|^2 \ll |w|^{\delta} T^{3+\varepsilon}$$

for some $\delta > 0$, where $\Delta u_j = (1 + r_j^2) u_j$ and $\mathrm{Re}(w) = \frac{1}{2}$.

Remark 3.3. By standard methods [T], the Riemann Hypothesis for $L^{(2)}(s, u_j)$ implies $L^{(2)}(\frac{1}{2} + it, u_j) \ll_{\varepsilon} (|tr_j| + 1)^{\varepsilon}$.

For our future use, it would be convenient that we describe the mean Lindelöf hypothesis in terms of the Rankin-Selberg L-function:

$$L(s, u_j \otimes u_j) = \sum_{n \in O^*} \frac{|\rho_j(n)|^2}{N(n)^s}.$$

Assumption 3.2 implies that if $\mathrm{Re}(w) = \frac{1}{2}$, then

$$\sum_{r_j \sim T} \frac{r_j}{\sinh \pi r_j} |L(w, u_j \otimes u_j)| \ll_w T^{3+\varepsilon}. \tag{3.2}$$

4. Proofs. The Selberg zeta function for the group Γ is defined by

$$Z_{\Gamma}(s) = \prod_{p \in \mathbf{P}} \prod_{(k,l)} (1 - a(p)^{-2k} \overline{a(p)}^{-2l} N(p)^{-s}), \tag{4.1}$$

where $\mathbf{P}$ is a certain set of primitive hyperbolic conjugacy classes and (k, l) runs through all the pairs of positive integers satisfying that $k \equiv l \pmod{m(p)}$ with $m(p)$ being the order of the torsion part of the centralizer of p. The complex number $a(p)$ is the eigenvalue of p with $|a(p)| > 1$ and $N(p) = |a(p)|^2$. The Selberg zeta function has the determinant expression [K1] Theorem 4.4:

$$Z_\Gamma(s) \times (\text{gamma factors}) = \det{}_D(\Delta - s(2-s)) \times (\text{Eisenstein factors}),$$

where $\det_D$ is the regularized determinant composed of discrete spectra of Δ. Since $\det_D(\Delta - s(2-s)) = 0$ is equivalent to $s = 1 \pm \sqrt{\lambda - 1} i$ for some eigenvalue λ, we see that such zeros of $Z_\Gamma(s)$ are of real part one, except for at most finite number of zeros coming from small eigenvalues less than 1. It is shown in [EGM] that such exceptional eigenvalues do not exist for the case at hand, viz $PSL(2, \mathbf{Z}[i])$.

We have the logarithmic derivative as

$$\frac{Z'}{Z}(s) = \sum_n \frac{\Lambda(n)}{N(n)^s},$$

where

$$\Lambda(n) = \frac{N(n) \log N(p)}{m(p)|a(n) - a(n)^{-1}|^2}$$

with n being a geodesic not necessarily prime and a power of prime p. We define the Ψ-function as

$$\Psi_\Gamma(X) = \sum_{N(n) \leq X} \Lambda(n).$$

Its explicit formula is deduced by Nakasuji. Her theorem is described as follows [N, Theorem 6.2]:

Lemma 4.1. *Let $\Gamma = PSL(2, O)$ where O is the integer ring of an imaginary quadratic field of class number 1. We have for $1 \leq T \leq X^{\frac{1}{2}}$*

$$\Psi_\Gamma(X) = \frac{X^2}{2} + \sum_{|r_j| \leq T} \frac{X^{s_j}}{s_j} + O\left(\frac{X^2}{T} \log X\right),$$

where $s_j = 1 + ir_j$ runs over the zeros of $Z(s)$ on $\mathrm{Re}(s) = 1$ *and the interval* $(0, 2)$, *counted with multiplicities.*

As mentioned above, in our case it holds that all s_j satisfy $\mathrm{Re}(s_j) = 1$.

Remark 4.2. For more general Bianchi groups and their congruence subgroups, the current best estimate of the smallest eigenvalue λ_1 is obtained by Luo, Rudnick and Sarnak [S3] by using the method of [LRS]:

$$\lambda_1 \geq \frac{171}{196} = 0.872... \tag{4.2}$$

The estimate (4.2) gives the bound $\mathrm{Re}(s_1) \leq \frac{18}{13}$. Thus we have

$$\Psi_\Gamma(X) = \frac{X^2}{2} + \sum_{\substack{|r_j| \leq T \\ r_j \in \mathbf{R}}} \frac{X^{s_j}}{s_j} + O\left(X^{\frac{18}{13}} + \frac{X^2}{T} \log x\right), \tag{4.3}$$

where $s_j = 1 + ir_j$ runs over the zeros of $Z(s)$ on $\mathrm{Re}(s) = 1$, counted with multiplicities. Therefore our main theorem holds for general Bianchi manifolds, once we generalize the explicit Kuznetsov formula.

It is known that the asymptotic distribution of such zeros is given by

$$\#\{s_j = 1 + ir_j \ : \ |r_j| \leq T\} \sim \frac{\mathrm{vol}(\Gamma \backslash \mathbf{H}^3)}{6\pi^2} T^3.$$

Therefore by estimating the second term in the left hand side of (4.3) roughly like

$$\left| \sum_{|r_j| \leq T} \frac{X^{s_j}}{s_j} \right| \leq \sum_{|r_j| \leq T} \left| \frac{X^{s_j}}{s_j} \right| \ll XT^2,$$

without considering any cancellation in the sum, we have the error terms $O(XT^2 + X^{2+\varepsilon}T^{-1})$. By putting $T = X^{1/3}$, we have $O(X^{\frac{5}{3}+\varepsilon})$, which coincides with Sarnak's bound [S1].

For improving the error term, we need a sharper estimate of the sum $\sum_{|r_j|\le T} X^{ir_j}$. By the standard argument as in [I] or [LS] Section 6, the bound is the same as that of the smooth version $\sum_{r_j>0} X^{ir_j}\exp(-r_j/T)$.

In the following lemma, we treat a sum involving the Fourier coefficients in order to apply the Kuznetsov formula later.

Lemma 4.3. *Let $\omega \in C^\infty[\sqrt{N}, \sqrt{2N}]$ such that its derivatives satisfy $|\omega^{(p)}(\xi)| \ll N^{-p/2}$ $(p \ge 0)$ and that $\Omega(2) := \int_0^\infty \omega(\xi)\xi d\xi = N$. Then under the assumption (3.2), we have*

$$\sum_{n\in O} \frac{r_j\omega(|n|)|\rho_j(n)|^2}{\sinh \pi r_j} = cN + \mathfrak{r}(r_j, N),$$

with

$$\sum_{|r_j|\ge T} |\mathfrak{r}(r_j, N)| \ll T^{3+\varepsilon} N^{\frac{1}{2}}$$

for some constant $c > 0$.

Proof. We are following the proof of [I] Lemma 8. Consider the Mellin transform

$$\Omega(s) = \int_0^\infty \omega(\xi)\xi^{s-1}d\xi \ll (1+|\tau|)^{-p}N^{\sigma/2}$$

for $s = \sigma + i\tau$, by partial integration p times. We also have the inverse Mellin transform

$$\omega(|n|) = \frac{1}{2\pi i}\int_{(3)} \Omega(s)\frac{ds}{|n|^s}.$$

Hence by the Cauchy theorem, we compute and put $\mathfrak{r}(r_j, N)$ as follows:

$$\begin{aligned}\sum_{n\in O} \omega(|n|)|\rho_j(n)|^2 &= \frac{1}{2\pi i}\int_{(3)} \Omega(s)L(\frac{s}{2}, u_j\otimes u_j)ds \\ &= c\Omega(2)\frac{\sinh \pi r_j}{r_j} + O\left(N^{\frac{1}{2}}\int_0^\infty \frac{|L(\frac{1}{2}+it, u_j\otimes u_j)|}{(1+|t|)^p}dt\right) \\ &= (cN + \mathfrak{r}(r_j, N))\frac{\sinh \pi r_j}{r_j}.\end{aligned}$$

By (3.2), it follows that

$$\sum_{|r_j|\le T} |\mathfrak{r}(r_j,N)| \ll \sum_{|r_j|\le T} \frac{N^{\frac{1}{2}} r_j}{\sinh \pi r_j} \int_0^\infty \frac{|L(\frac{1}{2}+it, u_j\otimes u_j)|}{(1+|t|)^p} dt \ll N^{\frac{1}{2}} T^{3+\varepsilon}.$$

The proof is complete. □

Proof of Theorem 1.1. By Lemma 4.3 we have

$$\frac{1}{N}\sum_{n\in O} \omega(|n|) \sum_{|r_j|\le T} \frac{r_j|\rho_j(n)|^2}{\sinh \pi r_j} X^{ir_j}\exp(-r_j/T)$$
$$= c\sum_{|r_j|\le T} X^{ir_j}\exp(-r_j/T) + O(T^{3+\varepsilon}N^{-\frac{1}{2}}). \quad (4.4)$$

Hereafter we estimate the sum over r_j in the left hand side of (4.4) by using the Kuznetsov formula.

We apply Theorem 2.1 to the test function

$$h(r) = \frac{\sinh(\pi + 2i\beta)r}{\sinh \pi r}$$

with $2\beta = \log X + \frac{i}{T}$. It satisfies the conditions of the theorem and for $r > 0$,

$$h(r) = X^{ir}e^{-r/T} + O(e^{-\pi r}).$$

Then the term D is equal to the relevant sum in (4.4) up to a tiny error term which we can ignore. We need to estimate the other three terms C, U and S. The terms C and U are easily estimated as $Z \ll T^{1+\varepsilon}$ and $U \ll \int_0^T X^{ir} r^2 dr \ll T^2$. For computing the term S, we appeal to the series representation of the J-Bessel function [Er, §7.2.1 (2)]:

$$J_{ir}(z) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\Gamma(ir+l+1)} \left(\frac{z}{2}\right)^{2l+ir}.$$

We see that the series for $z^{-ir}J_{ir}(z)$ converges absolutely, and uniformly for any r and in any bounded domain of z.

Thus we have

$$\mathcal{J}_{ir}(z) = \left|\frac{z}{2}\right|^{2ir} \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{l+m}}{l!m!\Gamma(ir+l+1)\Gamma(ir+m+1)} \left(\frac{z}{2}\right)^{2l} \left(\frac{\bar{z}}{2}\right)^{2m}. \qquad (4.5)$$

From the Stirling formula we have for fixed x:

$$\Gamma(x+iy) = \sqrt{2\pi}|y|^{x-\frac{1}{2}} e^{-\frac{\pi|y|}{2}+iy(\log|y|-1)}(1+O(y^{-1})),$$

as $|y| \to \infty$. So we have

$$\Gamma(ir+l+1)\Gamma(ir+m+1) = 2\pi|r|^{l+m+1} e^{-\pi|r|+2ir(\log|r|-1)}(1+O(r^{-1})),$$

as $|r| \to \infty$.

Therefore

$$\begin{aligned} S &= \sum_{c\in O^*} \frac{S(n,n;c)}{|c|^2} \int_{-\infty}^{\infty} \frac{r^2}{\sinh \pi r} \mathcal{J}_{ir}(z) X^{i|r|} e^{-\frac{|r|}{T}} dr \\ &= \frac{1}{2\pi} \sum_{c\in O^*} \frac{S(n,n;c)}{|c|^2} \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{l+m}}{l!m!} \left(\frac{z}{2}\right)^{2l} \left(\frac{\bar{z}}{2}\right)^{2m} \\ &\qquad \int_{-\infty}^{\infty} \frac{r^2}{\sinh \pi r} \left(\frac{e^{\frac{\pi|r|}{2}}}{|r|^{\frac{l+m+1}{2}}} \left|\frac{z}{2r}\right|^{ir} \right)^2 X^{i|r|} e^{-\frac{|r|}{T}} (1+O(r^{-1}))dr \end{aligned}$$

The absolute value of the last integral is less than

$$A_1 \int_0^{\infty} r^{1-l-m} \left(\frac{|n|X^{1/2}}{r|c|}\right)^{2ir} e^{-\frac{r}{T}}(1+O(r^{-1}))dr \qquad (4.6)$$

for some constant $A_1 > 0$. Putting $Y = \frac{|n|X^{\frac{1}{2}}}{|c|}$ and changing the variable $r \mapsto \frac{r}{Y}$, we see this is smaller than

$$A_2 Y^{2-l-m} \int_0^{\infty} r^{1-l-m} e^{-2iYr\log r - \frac{Yr}{T}}(1+O((Yr)^{-1}))dr$$

for some constant $A_2 > 0$. The integrand has an oscillating factor $e^{ip(r)}$ with $p(r) = 2Yr\log r$. There exists the solution $r = e^{-1}$ of $p'(r) = 0$, and we see by the saddle point method [O, Theorem 13.1] that the integral in (4.6) is bounded by

$$Y^{\frac{3}{2}-l-m} e^{l+m-1-\frac{Y}{eT}}(1+O(Y^{-1})).$$

It is exponentially small unless $\frac{Y}{T} \ll \log T$, namely, $N(c) \gg \frac{N(n)X}{T^2 \log T}$. Thus

$$S \ll \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{l+m} e^{l+m-1}}{l!m!}$$
$$\sum_{N(c) \gg \frac{N(n)X}{T^2 \log T}} \frac{S(n,n;c)}{N(c)} \left(\frac{\bar{n}}{c}\right)^{2l} \overline{\left(\frac{\bar{n}}{c}\right)^{2m}} \left(\frac{N(n)^{\frac{1}{2}} X^{\frac{1}{2}}}{N(c)^{\frac{1}{2}}}\right)^{\frac{3}{2}-l-m} (1 + O(Y^{-1})). \tag{4.7}$$

We will compute the contribution to the sum over c in (4.7) from a general term Y^{-k} $(k \geq 0)$ in the $(1 + O(Y^{-1}))$ part. It has a contribution which is smaller in absolute value than

$$N^{\frac{3}{4}+\frac{l+m-k}{2}} X^{\frac{3}{4}-\frac{l+m+k}{2}} \sum_{N(c) \gg \frac{N(n)X}{T^2 \log T}} \frac{(n,c)d(c)}{N(c)^{\frac{5}{4}+\frac{l+m-k}{2}}} \tag{4.8}$$

by the Weil bound $S(n,n;c) \ll N(c)^{\frac{1}{2}}(n,c)d(c)$. Then we have if $\frac{N(n)X}{T^2 \log T} > 1$,

$$\begin{aligned} S &\ll N^{\frac{3}{4}+\frac{l+m}{2}+\varepsilon} X^{\frac{3}{4}-\frac{l+m}{2}} \left(\frac{N(n)X}{T^2 \log T}\right)^{-\frac{1}{4}-\frac{l+m-k}{2}} \\ &\ll N^{\frac{1}{2}+\varepsilon} T^{\frac{1}{2}+l+m-k+\varepsilon} X^{\frac{1}{2}-(l+m)}. \end{aligned} \tag{4.9}$$

If $\frac{N(n)X}{T^2 \log T} < 1$, the sum over c in (4.8) is $O(1)$, and we have

$$\begin{aligned} N^{\frac{3}{4}+\frac{l+m-k}{2}} X^{\frac{3}{4}-\frac{l+m+k}{2}} &= (NX)^{\frac{1}{2}} N^{\frac{1}{4}+\frac{l+m-k}{2}} X^{\frac{1}{4}-\frac{l+m+k}{2}} \\ &\ll (NX)^{\frac{1}{2}} \left(\frac{T^2 \log T}{X}\right)^{\frac{1}{4}+\frac{l+m-k}{2}} X^{\frac{1}{4}-\frac{l+m+k}{2}} \\ &= N^{\frac{1}{2}} T^{\frac{1}{2}+l+m-k+\varepsilon} X^{\frac{1}{2}-(l+m+k)} \end{aligned} \tag{4.10}$$

Both (4.9) and (4.10) are largest when $k = 0$. In what follows we only treat the case of $k = 0$. Thus

$$S \ll N^{\frac{1}{2}+\varepsilon} T^{\frac{1}{2}+l+m+\varepsilon} X^{\frac{1}{2}-(l+m)}.$$

Therefore from (4.4)

$$\begin{aligned} \sum_{|r_j| \leq T} e^{-r_j/T} X^{ir_j} &\ll T^{3+\varepsilon} N^{-\frac{1}{2}} + N^{\frac{1}{2}+\varepsilon} T^{\frac{1}{2}+l+m+\varepsilon} X^{\frac{1}{2}-(l+m)} \\ &\ll T^{\frac{7}{4}+\frac{l+m}{2}+\varepsilon} X^{\frac{1}{4}-\frac{l+m}{2}+\varepsilon} \end{aligned}$$

by choosing $N = \frac{T^{\frac{5}{2}-(l+m)}}{X^{\frac{1}{2}-(l+m)}}$. Then we obtain

$$\Psi_\Gamma(X) = \frac{X^2}{2} + O(X^{\frac{11+2(l+m)}{7+2(l+m)}+\epsilon}) \tag{4.11}$$

by choosing $T = X^{\frac{3+2(l+m)}{7+2(l+m)}}$. The error term in (4.11) is biggest when $l = m = 0$. The standard argument leads to Theorem 1.1. □

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