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Contribution to the N. Wiener generalized harmonic analysis and its application to the theory of generalized Hilbert transforms

by

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CONTRIBUTION TO THE N.WIENER GENERALIZED HARMONIC ANALYSIS AND ITS APPLICATION TO THE THEORY OF GENERALIZED HILBERT TRANSFORMS

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ABSTRACT

We shall intend to contribute to the theory of Generalized Harmonic Analysis (G.H.A.) in addition to the hypothesis as for existence of the following limit

$$(C_\lambda) \quad c_\lambda = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\lambda x}dx \quad (\forall \text{ real } \lambda)$$

and present here fine and advanced forms and results. This hypothesis is very natural since it correspond to the existence of the Fourier coefficients in the theory of Fourier series and almost periodic functions.

Chaptet 1. We shall intend to prove three Theorems A, B and C. Along to the work of Prof.N.Wiener[1], we shall introduce several classes of functions and Generalized Fourier Transform $(G.F.T.)$ as follows.

Hilbert space W^2: The class of function f that belongs to L^2_{loc} and exists the following integral

$$\int_{-\infty}^{\infty} \frac{|f(x)|^2}{1+x^2}dx < \infty .$$

The Generalized Fourier Transform (G.F.T.) of function $f(x)$ is defined by the following formula

$$s(u;f) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} f(x)\frac{e^{-iux}-1}{-ix}dx + \underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\left[\int_{-A}^{-1} + \int_{1}^{A}\right] f(x)\frac{e^{-iux}}{-ix}dx$$

and we have

$$s(u+\varepsilon;f)-s(u-\varepsilon;f)=\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}f(x)\frac{2\sin\varepsilon x}{x}e^{-iux}dx.$$

Class S_0 : The class of function f that belongs to L^2_{loc} and exists the following integral

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)|^2dx.$$

Class S : The class of function f that belongs to L^2_{loc} and exists the following integral

$$\varphi(x;f)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x+t)\overline{f(t)}dt \qquad (\forall \text{ real } x).$$

The function $\varphi(x;f)$ is called the auto-correlation function of $f(x)$.

Class S' : The class of a function f that belongs to the class S and its auto-correlation function $\varphi(x;f)$ is continuous for all x. It is clear that

$$S'\subset S\subset S_0\subset W^2.$$

Then we shall prove Theorems A, B and C as follows.

1.1.The Relevant Theorems of G.H.A. and Theorem A.

We shall start function $f(x)$ that belongs to the class W^2. Applying the N.Wiener General Tauberian Theorem in this case the so-called Wiener formula one obtains the following theorem (c.f. N.Wiener[1],pp.138~140)

Theorem W_0 Let us suppose that $f(x)$ belongs to the space W^2. Then we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)|^2dx=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2du$$

in the sense that if either side exists, the other side exists and assumes the same value.

Then, we shall prove the necessary and sufficient condition for the hypothesis (C_λ) to be true. That is as follows.

Theorem A Let us suppose that function $f(x)$ belongs to the class S_0. Then we have for each and all real λ

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)e^{-i\lambda x}dx=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}du$$

in the sense that if either side exists, the other side exists and assumes the same value.

Proof of Theorem A. Now let us suppose the hypotheses (C_λ). Then after N.Wiener, we should use the identity

$$4a\bar{b} = |a+b|^2 - |a-b|^2 + i|a+ib|^2 - i|a-ib|^2 = \sum_{\omega\in\Omega} \omega |a+\omega b|^2$$

where $\Omega = \{\pm 1, \pm i\}$. It might be realized the role to represent the inner product as the sum of norms as follows. Then we have

$$f(x)e^{-i\lambda x} = f(x)\overline{e^{i\lambda x}} = \frac{1}{4}\sum_{\omega\in\Omega} \omega |f(x)+\omega e^{i\lambda x}|^2$$

and inversely the term $|f(x)+\omega e^{i\lambda x}|^2$ could be expanded as follows

$$|f(x)+\omega e^{i\lambda x}|^2 = |f(x)|^2 + \bar{\omega} f(x)e^{-i\lambda x} + \omega \overline{f(x)} e^{i\lambda x} + |\omega e^{i\lambda x}|^2.$$

Therefore we have

Lemma A_1 Let us suppose that function $f(x)$ belongs to the class S_0. Then the hypothesis (C_λ) and the statement $f(x)+\omega e^{i\lambda x} \in S_0$ $(\forall \omega \in \Omega)$ are equivalent to each other for each and all real λ.

Next we shall consider the G.F.T. of $f(x)+\omega e^{i\lambda x}$, $(\forall \omega \in \Omega)$. First of all we have by the elementary calculation

$$s(u+\varepsilon; e^{i\lambda x}) - s(u-\varepsilon; e^{i\lambda x}) = \begin{cases} \sqrt{2\pi} & (\lambda-\varepsilon < u < \lambda+\varepsilon) \\ \sqrt{2\pi}/2 & (u = \lambda \pm \varepsilon) \\ 0 & (u < \lambda-\varepsilon,\ \lambda+\varepsilon < u) \end{cases}.$$

Then applying Theorem W_0, we have

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{1}{4}\sum_{\omega\in\Omega} \omega |f(x)+\omega e^{i\lambda x}|^2\, dx \\
&= \frac{1}{4}\sum_{\omega\in\Omega} \omega \lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} |s(u+\varepsilon; f+\omega e^{i\lambda x}) - s(u-\varepsilon; f+\omega e^{i\lambda x})|^2\, du \\
&= \lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} \frac{1}{4}\sum_{\omega\in\Omega} \omega \left| \{s(u+\varepsilon; f) - s(u-\varepsilon; f)\} + \omega\{s(u+\varepsilon; e^{i\lambda x}) - s(u-\varepsilon;; e^{i\lambda x})\} \right|^2 du \\
&= \lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} \{s(u+\varepsilon; f) - s(u-\varepsilon; f\} \overline{\{s(u+\varepsilon; e^{-i\lambda x}) - s(u-\varepsilon; e^{-i\lambda x})\}} du
\end{aligned}$$

$$= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon;f)s(u-\varepsilon;f)\}du.$$

It is easily verified that by the Theorem W_0 ,the estimation of inverse direction is also true. Therefore we have

Lemma A_2 Let us suppose that function $f(x)$ belongs to the class S_0. Then the proposition $f(x)+\omega e^{i\lambda x}\in S_0\ (\forall\omega\in\Omega)$ and the existence of the limit

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}du$$

is equivalent for each and all real λ.

Thus combining two Lemmas A_1 and A_2, we have proved Theorem A.

Next we shall state the N.Wiener theorem as a more fine and advanced forms.

Theorem W_1 Let us suppose that function $f(x)$ belongs to the class S_0. Then we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x+t)\overline{f(t)}dt=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}e^{iux}\,|\,s(u+\varepsilon;f)-s(u-\varepsilon;f)\,|^2\,du$$

in the sense that if either side exists, the other side exists and assumes the same value.

In the first we shall state the following result.

Lemma W_1 Let us suppose that function $f(x)$ belongs to the class S_0. Then the two propositions $f\in S$ and $f(x+t)+\omega f(t)\in S_0\ (\forall\omega\in\Omega,\ \ \forall real\ x)$ are equivalent to each other.

Let us suppose that $f(x+t)+\omega f(t)\in S_0\ (\forall\omega\in\Omega,\ \ \forall real\ x)$. Since we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x+t)\overline{f(t)}dt=\frac{1}{4}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sum_{\omega\in\Omega}\omega\,|\,f(x+t)+\omega f(t)\,|^2dt$$

we shall conclude that $f(x)$ belongs to the class S.

On the other hand let us suppose that $f(x)$ belongs to the class S. Since we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\,f(x+t)+\omega f(t)\,|^2\,dt$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\,f(x+t)\,|^2dt+\overline{\omega}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x+t)\overline{f(t)}dt$$

$$\overline{+\omega \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x+t)\overline{f(t)}dt} + |\omega|^2 \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |f(x)|^2 dt,$$

we shall conclude that $f(x+t)+\omega f(t) \in S_0(\forall \omega \in \Omega)$.

Proof of Theorem W_1. Let us suppose that $f(x)$ belongs to the class S and let us consider the G.F.T. of $f(x+t)+\omega f(t)$. Let us denote after N.Wiener (c.f.[1], p.156)

$$s_x(u;f) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} f(x+t)\frac{e^{-iut}-1}{-it}dt + \underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\left[\int_{-A}^{-1}+\int_{1}^{A}\right] f(x+t)\frac{e^{-iut}}{-it}dt.$$

Then applying Theorem W_0 ,we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |f(x+t)+\omega f(t)|^2 dt$$

$$= \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} |s(u+\varepsilon; f(x+t)+\omega f(t)) - s(u-\varepsilon; f(x+t)+\omega f(t))|^2\, du$$

$$= \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} |(s_x(u+\varepsilon;f)-s_x(u-\varepsilon;f)) + \omega(s(u+\varepsilon;f)-s(u-\varepsilon;f))|^2\, du$$

and we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x+t)\overline{f(t)}dt = \frac{1}{4}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sum_{\omega\in\Omega}\omega\,|f(x+t)+\omega f(t)|^2\, dt$$

$$= \frac{1}{4}\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}\sum_{\omega\in\Omega}\omega\,|(s_x(u+\varepsilon;f)-s_x(u-\varepsilon;f)) + \omega(s(u+\varepsilon;f)-s(u-\varepsilon;f))|^2\, du$$

$$= \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}(s_x(u+\varepsilon;f)-s_x(u-\varepsilon,f))\overline{(s(u+\varepsilon;f)-s(u-\varepsilon;f))}du$$

Now we shall quote the N.Wiener result (c.f. [1], p158). That is if $f(x)$ belongs to the space W^2, then we have

$$\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} |(s_x(u+\varepsilon;f)-s_x(u-\varepsilon;f)) - e^{iux}(s(u+\varepsilon;f)-s(u-\varepsilon;f))|^2\, du = O(\varepsilon^2)$$

as $\varepsilon \to 0$.

Then applying the Minkowski inequality we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}\{s_x(u+\varepsilon;f)-s_x(u-\varepsilon;f)\}\overline{\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}}du$$

$$= \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} e^{iux}\,|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2\, du$$

Therefore we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x+t)\overline{f(t)}dt=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} e^{iux}\,|\,s(u+\varepsilon;f)-s(u-\varepsilon;f)\,|^2\,du$$

On the contrary, let us suppose that a function $f(x)$ belongs to the class S_0 and the following limit

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} e^{iux}\,|\,s(u+\varepsilon;f)-s(u-\varepsilon;f)\,|^2\,du$$

exists for all x.

First of all let us remark that the following identity

$$e^{i\lambda x}=\frac{1}{4}\sum_{\omega\in\Omega}\omega\,|\,e^{i\lambda x}+\omega\,|^2.$$

Then we shall estimate the following formula

$$\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\,e^{iux}+\omega\,|^2|\,s(u+\varepsilon;f)-s(u-\varepsilon;f)\,|^2\,du\ .$$

Here we shall quote one more the same estimation as for $s_x(u+\varepsilon;f)-s_x(u-\varepsilon;f)$ and applying the Minkowski inequality, we have

$$\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\,e^{iux}+\omega\,|^2|\,s(u+\varepsilon;f)-s(u-\varepsilon;f)\,|^2\,du$$

$$=\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\,(e^{iux}+\omega)(s(u+\varepsilon;f)-s(u-\varepsilon;f))\,|^2\,du$$

$$=\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\,s_x(u+\varepsilon;f)-s_x(u-\varepsilon;f)-e^{iux}(s(u+\varepsilon;f)-s(u-\varepsilon;f))+$$

$$+(e^{iux}+\omega)(s(u+\varepsilon;f)-s(u-\varepsilon;f))\,|^2\,du\quad+O(\varepsilon^2)$$

$$=\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\,(s_x(u+\varepsilon;f)-s_x(u-\varepsilon;f))+\omega(s(u+\varepsilon;f)-s(u-\varepsilon;f))\,|^2\,du+O(\varepsilon^2)$$

$$=\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\,(s(u+\varepsilon;f(x+t)+\omega f(t))-s(u-\varepsilon;f(x+t)+\omega f(t))\,|^2\,du+O(\varepsilon^2)$$

as $\varepsilon\to 0$.

Therefore we have by the Theorem W_0

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\,e^{iux}+\omega\,|^2|\,s(u+\varepsilon;f)-s(u-\varepsilon;f)\,|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\,s(u+\varepsilon;f(x+t)+\omega f(t))-s(u-\varepsilon;f(x+t)+\omega f(t))\,|^2\,du$$

$$=\lim_{T\to\in}\frac{1}{2T}\int_{-T}^{T}|\,f(x+t)+\omega f(t)\,|^2\,dt$$

for all x and $\forall\omega\in\Omega$. Then we have by the identity to be stated above

$$\lim_{\varepsilon \to 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} e^{iux} \,|\, s(u+\varepsilon; f) - s(u-\varepsilon; f) \,|^2 \, du$$

$$= \frac{1}{4} \lim_{\varepsilon \to 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} \sum_{\omega \in \Omega} \omega \,|\, e^{iux} + \omega \,|^2 \,|\, s(u+\varepsilon; f) - s(u-\varepsilon; f) \,|^2 \, du$$

$$= \frac{1}{4} \lim_{T \to \in} \frac{1}{2T} \int_{-T}^{T} \sum_{\omega \in \Omega} \omega \,| f(x+t) + \omega f(t) |^2 \, dt = \lim_{T \to \in} \frac{1}{2T} \int_{-T}^{T} f(x+t) \overline{f(t)} dt$$

Thus we have proved the N.Wiener Theorem such as a more fine and advanced form. It should be remarked that the half part is the N.Wiener Theorem (c.f.[1], Theorem 27, p.158) and the remaining half part is due to by the author.

1.2. The Relevant Theorem of G.H.A. and Theorem B.

Let us define the G.F.T. of the auto-correlation function $\varphi(x; f)$ of $f(x)$ as follows

$$\sigma(u;\varphi) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \varphi(x; f) \frac{e^{-iux} - 1}{-ix} dx + \underset{A \to \infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \left[\int_{-A}^{-1} + \int_{1}^{A} \right] \varphi(x; f) \frac{e^{-iux}}{-ix} dx$$

As for the spectral analysis of the N.Wiener class S, we shall need to know the properties of $\sigma(u;\varphi)$. We shall present here the more detailed properties of $\sigma(u;\varphi)$ after the same method of N.Wiener[1] with the assistance of properties of $\sigma_\varepsilon(u;\varphi_\varepsilon)$ $(\varepsilon > 0)$ that is defined by the following formula.

Let us denote

$$\varphi_\varepsilon(x; f) = \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} e^{iux} \,|\, s(u+\varepsilon; f) - s(u-\varepsilon; f) \,|^2 \, du$$

and its G.F.T.

$$\sigma_\varepsilon(u;\varphi_\varepsilon) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \varphi_\varepsilon(x; f) \frac{e^{-iux} - 1}{-ix} dx + \underset{A \to \infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \left[\int_{-A}^{-1} + \int_{1}^{A} \right] \varphi_\varepsilon(x; f) \frac{e^{-iux}}{-ix} dx$$

Then it is clear that the function $\varphi_\varepsilon(u; f)$ is of positive definite in the sense of S.Bochner[3] and it is represented as

$$\varphi_\varepsilon(x; f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iux} d\Lambda_\varepsilon(u)$$

and by the theorem of the Levy inversion formula we have

$$\Lambda_\varepsilon(u) = P.V. \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi_\varepsilon(x) \frac{e^{-iux} - 1}{-ix} dx .$$

On the other hand from the definite formula of $\varphi_\varepsilon(x; f)$ we have directly that

$$\Lambda_\varepsilon(u) = \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{0}^{u} |\, s(v+\varepsilon; f) - s(v-\varepsilon; f) \,|^2 dv$$

Therefore we have

$$P.V.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi_{\varepsilon}(x;f)\frac{e^{-iux}-1}{-ix}dx=\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{0}^{u}|s(v+\varepsilon;f)-s(v-\varepsilon;f)|^{2}dv$$

and

$$(*)\qquad \sigma_{\varepsilon}(u;\varphi_{\varepsilon})=\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{0}^{u}|s(v+\varepsilon;f)-s(v-\varepsilon;f)|^{2}dv-C_{\varepsilon},\qquad a.e.\ u$$

where the constant term C_{ε} in this formula may readily be verified by the limiting value

$$\left[\int_{-A}^{-1}+\int_{1}^{A}\right]\frac{\varphi_{\varepsilon}(x)}{-ix}dx$$

as the A tends to infinity through a sequence $\{A_j\}$. Here we shall quote the same method used in the proof of the F. Riesz-Fischer theorem in the theory of L^2 -space.

Since $\varphi_{\varepsilon}(x;f)$ tends to $\varphi(x;f)$ boundedly as $\varepsilon\to 0$ and $\varphi_{\varepsilon}(x;f)/(-ix)$ tends in the mean to $\varphi(x;f)/(-ix)$ as $\varepsilon\to 0$ over any range of x to be not containing the origin. From these facts we shall conclude that

$$\sigma(u;\varphi)=\underset{\varepsilon\to 0}{l.i.m.}\,\sigma_{\varepsilon}(u;\varphi_{\varepsilon})\quad (L^2)$$

on any finite range of u. Because we have by the Plancherel theorem

$$\int_{-N}^{N}|\sigma_{\varepsilon}(u;\varphi_{\varepsilon})-\sigma(u;\varphi)|^{2}\,du\le\frac{N}{\pi}\left(\int_{-1}^{1}|\varphi_{\varepsilon}(x;f)-\varphi(x;f)|\,dx\right)^{2}+\int_{|x|>1}\frac{|\varphi_{\varepsilon}(x;f)-\varphi(x;f)|^{2}}{x^{2}}dx\to 0$$

as $\varepsilon\to 0$ for any $N>0$.

It is remarked that definition of $\sigma_{\varepsilon}(u;\varphi_{\varepsilon})$ on the set of measure 0 may be permitted to move. Hereafter we shall quote the $\sigma_{\varepsilon}(u;\varphi_{\varepsilon})$ as the above formula$(*)$ for all u.

Now we shall quote the Lemma due to Paley-Wiener (c.f.[2], pp.134~5).

Lemma(Paley-Wiener). If we have a sequence of monotone functions $\{f_n\}$ tending to a function $f(x)$ in the mean, then we have

$$f_n(x)\ \to\ f(x)\ \ a.e.\ \ (n\to\infty)$$

Then applying the Paley-Wiener Lemma to the sequence of $\{\sigma_{\varepsilon}(u;\varphi_{\varepsilon})\}$ we shall conclude that

$$\sigma(u;\varphi) = \lim_{\varepsilon\to 0} \sigma_\varepsilon(u;\varphi_\varepsilon) \qquad a.e.\ u$$

Furthermore we shall intend to consider the more detailed properties of $\sigma(u;\varphi)$ with the assistance of those of $\sigma_\varepsilon(u;\varphi_\varepsilon)$.

Let us denote the set D of u where the sequence $\sigma_\varepsilon(u;\varphi_\varepsilon)$ is convergent and the set E of u where it is not convergent or to be not defined. Then we have $D \cup E = (-\infty, +\infty)$ and $m(E) = 0$.

It is remarkable that hereafter we shall denote $\sigma(u)$ and $\sigma_\varepsilon(u)$ instead of $\sigma(u;\varphi)$ and $\sigma_\varepsilon(u;\varphi_\varepsilon)$ respectively for the sake of simplicity.

We shall also define as follows

$$\underline{\sigma}(u) = \sup_{\substack{v<u \\ v\in D}} \sigma(v), \quad \overline{\sigma}(u) = \inf_{\substack{u<v \\ v\in D}} \sigma(v) \quad \text{and} \quad \sigma(u) = \frac{\underline{\sigma}(u) + \overline{\sigma}(u)}{2}.$$

Then we could define $\sigma(u)$ everywhere and it is a bounded, monotone increasing function of u and first of all we shall prove that it satisfies the following properties

$$\sigma(u-0) = \underline{\sigma}(u) \quad \text{and} \quad \sigma(u+0) = \overline{\sigma}(u)$$

at any point u. Then we shall prove the following results.

(i) On the case of $\overline{\sigma}(u) - \underline{\sigma}(u) = 0$ at a point u.

We have the $\lim_{\varepsilon\to 0} \sigma_\varepsilon(u)$ exist and

$$\sigma(u-0) = \sigma(u+0) = \sigma(u).$$

Then it is continuous there.

(ii) On the case of $\overline{\sigma}(u) - \underline{\sigma}(u) > 0$ at a point u.

We have the $\lim_{\varepsilon\to 0} \sigma_\varepsilon(u)$ does not exist there and

$$\sigma(u+0) - \sigma(u-0) = \overline{\sigma}(u) - \underline{\sigma}(u) > 0.$$

Then we shall conclude that it is discontinuous of the first kind there and has magnitude of jump that states it above.

Now we shall intend to prove the following

Theorm B. Let us suppose that function $f(x)$ belongs to the class S, then the following limit

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{\pm\varepsilon}^{u\pm\varepsilon} |\, s(v+\varepsilon;f) - s(v-\varepsilon;f)\,|^2 dv = \lim_{\varepsilon\to 0} (\sigma_\varepsilon(u\pm\varepsilon) - \sigma_\varepsilon(\pm\varepsilon))$$

exists and equals to

$$\sigma(u \pm 0) - \sigma(\pm 0)$$

for any point u respectively.

We shall start to prove several properties of $\sigma_\varepsilon(u)$ and $\sigma(u)$. We shall prove them by the elementary calculation of Real Analysis. However since it shares of a role of essential part of Theorem C, therefor we shall prove it for the sake of completeness.

Lemma B_1. The $\sigma(u)$ satisfies at any point u

$$\sigma(u-0) = \underline{\sigma}(u) \quad \text{and} \quad \sigma(u+0) = \overline{\sigma}(u)$$

respectively.

Proof. Since $\sigma(u)$ is bounded, monotone increasing function we have for any pair of (u', u'') such as $u' < u''$

$$\overline{\sigma}(u') = \inf_{\substack{u'<v' \\ v'\in D}} \sigma(v') = \inf_{\substack{u'<v'<u'' \\ v'\in D}} \sigma(v') \le \sup_{\substack{u'<v''<u'' \\ v''\in D}} \sigma(v'') = \sup_{\substack{v''<u'' \\ v''\in D}} \sigma(v'') = \underline{\sigma}(u'').$$

Therefore we have proved for any pair of (v,u) such as $v < u$,

$$\underline{\sigma}(v) \le \overline{\sigma}(v) \le \underline{\sigma}(u)$$

and so

$$\sigma(v) = \frac{\underline{\sigma}(v) + \overline{\sigma}(v)}{2} \le \underline{\sigma}(u)$$

Then we shall take the limit such as $v \uparrow u$, we have

$$\sigma(u-0) \le \underline{\sigma}(u)$$

Next we shall intend to prove the inverse inequality. Since $\sigma(u)$ is a bounded monotone increasing function, for any pair (u', u'') such as $u' < u'' < u$, we have

$$\sigma(u') \le \sigma(u'').$$

Then we shall take the limit such as $u'' \uparrow u$, we have

$$\sigma(u') \le \sigma(u-0)$$

and so we have

$$\underline{\sigma}(u) = \sup_{\substack{u'<u \\ u'\in D}} \sigma(u') \le \sigma(u-0).$$

Therefore we have

$$\sigma(u-0) = \underline{\sigma}(u).$$

Similarly we shall prove

$$\sigma(u+0) = \overline{\sigma}(u).$$

Lemma B_2 (i) In the case of $\overline{\sigma}(u)=\underline{\sigma}(u)$ at a point u. The point u belongs to the set D and $\sigma(u)$ is continuous there.

(ii) We have at any point u of the set E

$$\overline{\sigma}(u)<\underline{\sigma}(u).$$

Proof. (i) Since $\sigma_\varepsilon(u)$ is bounded and monotone increasing function of u, we have any pair of $\varepsilon,\varepsilon'>0$

$$\sigma_\varepsilon(u-\varepsilon')\le\sigma_\varepsilon(u)\le\sigma_\varepsilon(u+\varepsilon').$$

Since the measure of the set E is 0 and so there exist a sequence of points $\{u\pm\varepsilon'\}$ such that $u\pm\varepsilon'\in D$ and $\{\varepsilon'\}\downarrow 0$. In the first we shall intend $\{\varepsilon\}\downarrow 0$, then we have

$$\sigma(u-\varepsilon')\le\underline{\lim}_{\varepsilon\to 0}\sigma_\varepsilon(u)\le\overline{\lim}_{\varepsilon\to 0}\sigma_\varepsilon(u)\le\sigma(u+\varepsilon').$$

Next we shall intend $\{\varepsilon'\}\downarrow 0$. Then we have

$$\underline{\sigma}(u)\le\underline{\lim}_{\varepsilon\to 0}\sigma_\varepsilon(u)\le\overline{\lim}_{\varepsilon\to 0}\sigma_\varepsilon(u)\le\overline{\sigma}(u)$$

Since $\sigma(u-0)=\underline{\sigma}(u)$, $\sigma(u+0)=\overline{\sigma}(u)$ by the Lemma B_1, we shall conclude that $\lim_{\varepsilon\to 0}\sigma_\varepsilon(u)=\sigma(u)$ exist and we have

$$\sigma(u-o)=\sigma(u)=\sigma(u+0)$$

and $\sigma(u)$ is continuous there.

(ii) Let us suppose that $\underline{\sigma}(u)=\overline{\sigma}(u)$ at a point u of the set E. Then repeating the same argument as the case (i), we shall conclude that $\lim_{\varepsilon\to 0}\sigma_\varepsilon(u)$ exists. It lead to the contradiction. Therefore
we have

$$\overline{\sigma}(u)-\underline{\sigma}(u)>0\quad(\forall u\in E).$$

Now we have proved

(a) In the case $\underline{\sigma}(u)=\overline{\sigma}(u)$. The limit

$$\sigma(u)=\lim_{\varepsilon\to 0}\sigma_\varepsilon(u)$$

exist and continuous at the point u. The set of point u is subset of D and we shall denote it as D_0.

(b) In the case $\overline{\sigma}(u)>\underline{\sigma}(u)$. The point u belongs to the set E or $D-D_o$ we have

$$\sigma(u+0)-\sigma(u-0)=\overline{\sigma}(u)-\underline{\sigma}(u)>0.$$

Therefore $\sigma(u)$ is discontinuous of the first kind and the set $(D-D_o)\cup E$ is at most countable.

Author suppose that the set $D=D_0$. However this problem is open at present and he leave it to the reader. Therefor hereafter we shall treat our theory in the case $D\cup E=(-\infty,\infty), m(E)=0$ only for the sake of simplicity. Author suppose that this hypothesis does not never lose the essential part of the theory.

Proof of Theorem B. Let us remark that the following formula

$$\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\pm\varepsilon}^{u\pm\varepsilon}|s(v+\varepsilon;f)-s(v-\varepsilon;f)|^2\,dv=\sigma_\varepsilon(u\pm\varepsilon;\varphi_\varepsilon)-\sigma_\varepsilon(\pm\varepsilon;\varphi_\varepsilon)$$

and so without loss of generality we shall prove it at a point $u=0$. That is as follows

$$\lim_{\varepsilon\to 0}\sigma_\varepsilon(\pm\varepsilon;\varphi_\varepsilon)=\sigma(\pm 0;\varphi)$$

(i) In the case of a point $0\in D$. Since the sequence $\{\sigma_\varepsilon(u)\}$ is a bounded monotone increasing function of u, for any pair (o,u) and $\varepsilon>0$ such as $o<\varepsilon<u, u\in D$, we have

$$\sigma_\varepsilon(0)\le\sigma_\varepsilon(+\varepsilon)\le\sigma_\varepsilon(u).$$

Then in the first tending $\varepsilon\downarrow 0$, we have

$$\sigma(0)\le\underline{\lim}_{\varepsilon\to 0}\sigma_\varepsilon(+\varepsilon)\le\overline{\lim}_{\varepsilon\to 0}\sigma_\varepsilon(+\varepsilon)\le\sigma(u)$$

In the second we take the least lower bound of $\sigma(u)$ as for $u\in D$ such as $0<u$ .since we have $\sigma(0)=\sigma(+0)$ and we have

$$\lim_{\varepsilon\to 0}\sigma_\varepsilon(+\varepsilon)=\sigma(+0)$$

We have similarly

$$\lim_{\varepsilon\to 0}\sigma_\varepsilon(-\varepsilon)=\sigma(-0)$$

(ii) In the case a point $0\in E$. We have defined $\sigma(0)$ as follows.

$$\sigma(0)=\frac{\sigma(-0)+\sigma(+0)}{2}.$$

Let us put $\sigma(+0)-\sigma(-0)=d>0$ and define

$$\sigma^*(u)=\sigma(u)-dh(u)$$

where $h(u)$ is the Heaviside operator, that is as follows

$$h(u)=\begin{cases}1 & (u>0)\\ \frac{1}{2} & (u=0)\\ 0 & (u<0).\end{cases}$$

Since we have by the definition

$$\sigma^*(+0)=\sigma(+0)-dh(+0)=\sigma(-0)$$
$$\sigma^*(-0)=\sigma(-0)-dh(-0)=\sigma(-0)$$
$$\sigma^*(0)=\sigma(0)-dh(0)=\sigma(-0)$$

respectively. Then $\sigma^*(u)$ is continuous at $u=0$.

Now let us put

$$\sigma_\varepsilon^*(u)=\sigma_\varepsilon(u)-dh(u)$$

and consider the sequence $\{\sigma_\varepsilon^*(u)\}$ instead of $\{\sigma_\varepsilon(u)\}$ then it is continuous at a point $u\in D$. Because we have

$$\lim_{\varepsilon\to 0}\sigma_\varepsilon^*(u)=\lim_{\varepsilon\to 0}\sigma_\varepsilon(u)-dh(u)=\sigma(u)-dh(u)=\sigma^*(u)$$

and $\sigma^*(u)$ is continuous. Then applying the results of (i) to $\sigma_\varepsilon^*(u)$, we have

$$\lim_{\varepsilon\to 0}\sigma_\varepsilon^*(+\varepsilon)=\sigma^*(+0)$$

where

$$\sigma_\varepsilon^*(+\varepsilon)=\sigma_\varepsilon(+\varepsilon)-dh(+\varepsilon)=\sigma_\varepsilon(+\varepsilon)-d \quad \text{and} \quad \sigma^*(+0)=\sigma(-0)$$

Therefore we have

$$\lim_{\varepsilon\to 0}\sigma_\varepsilon(+\varepsilon)=\lim_{\varepsilon\to 0}\sigma_\varepsilon^*(+\varepsilon)+d$$
$$=\sigma^*(+0)+d=\sigma(-0)+d=\sigma(+0)$$

Similarly we have

$$\lim_{\varepsilon\to 0}\sigma_\varepsilon^*(-\varepsilon)=\sigma^*(-0)$$

where

$$\sigma_\varepsilon^*(-\varepsilon)=\sigma_\varepsilon(-\varepsilon)-dh(-\varepsilon)=\sigma_\varepsilon(-\varepsilon) \quad \text{and} \quad \sigma^*(-0)=\sigma(-0)$$

Therefore we have

$$\lim_{\varepsilon\to 0}\sigma_\varepsilon(-\varepsilon)=\sigma(-0)$$

In general we shall prove by the same argument as above

$$\lim_{\varepsilon\to 0}\sigma_\varepsilon(u\pm\varepsilon)=\sigma(u\pm 0)$$

at any point u respectively. Thus we have proved the Theorem B.

1.3. The Decomposition Theorem on the class S and Theorem C.

Applying Theorem A and Theorem B to function $f(x)$ that belongs to the class S, we have

Theorem C. Let us suppose that function $f(x)$ belongs to the class S and satisfy the hypothesis (C_λ). Then we shall decompose

$$f(x)=g(x)+h(x)$$

with functions g,h that satisfies the following properties.

(i) $g(x)$ is a B^2-almost periodic function in the sense of A.S.Besicovitch. Let us denote its Fourier series expansion such as

$$g(x)\sim\sum_n c_n e^{i\lambda_n x}, \qquad c_n=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x)e^{-i\lambda x}dx \qquad (n=0,1,2,3,\ldots).$$

Then we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(x)|^2dx=\sum_n|c_n|^2\le\frac{\sigma(\infty;\varphi)-\sigma(-\infty;\varphi)}{\sqrt{2\pi}}.$$

(ii) $h(x)$ belongs to the class S and we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|h(x)|^2dx=\sum_n\left\{\frac{\sigma(\lambda_n+0;\varphi)-\sigma(\lambda_n-0;\varphi)}{\sqrt{2\pi}}-|c_n|^2\right\}^2.$$

Moreover let us suppose that the function $f(x)$ belongs to the class S'. Then we have

$$h(x)=0 \quad a.e.\ x \qquad \text{and} \qquad f(x)=g(x) \quad a.e.\ x.$$

Proof of Theorem C. Let us suppose that a function $f(x)$ belongs to the class S and the hypothesis (C_λ) is satisfied. Then we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\varepsilon,f)-s(u-\varepsilon,f)|^2du=\frac{\sigma(\lambda+0,\varphi)-\sigma(\lambda-0,\varphi)}{\sqrt{2\pi}}\qquad(\forall real\ \lambda).$$

by the Theorem B. We have the following formula

$$c_\lambda=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)e^{-i\lambda x}dx=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}(s(u+\varepsilon;f)-s(u-\varepsilon;f))du$$

by hypothesis (C_λ) and Theorem A.

Then we have by the Schwartz inequality

$$|c_\lambda|^2=|\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}(s(u+\varepsilon;f)-s(u-\varepsilon;f))du|^2$$

$$\le\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2du=\frac{\sigma(\lambda+0;\varphi)-\sigma(\lambda-0;\varphi)}{\sqrt{2\pi}}$$

Therefore we have

Step (i). If $u\in D,$ the $\sigma(u)$ is continuous at $u=\lambda$. We have $c_\lambda=0$.

Step (ii). If $u\in E,$ since the $\sigma(u)$ is a bounded and monotone increasing function, the set E is at most countable and we shall present it as follows

$$E=\{\lambda_n\}\quad(n=0,1,2,3,....)\ \text{ and }\ c_{\lambda_n}(=c_n, say),(n=0,1,2,3,....)$$

where $\lambda_0=0$ and $c_0=0$ may be permitted.

We have by the Schwartz inequality

$$|c_\lambda|^2=|\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}du|^2$$

$$\le\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2du=\frac{\sigma(\lambda+0;\varphi)-\sigma(\lambda-0;\varphi)}{\sqrt{2\pi}}.$$

and

$$\sum_n|c_n|^2\le\sum_n\frac{\sigma(\lambda_n+0,\varphi)-\sigma(\lambda_n-0,\varphi)}{\sqrt{2\pi}}\le\frac{\sigma(\infty,\varphi)-\sigma)-\infty,\varphi)}{\sqrt{2\pi}}<\infty.$$

Then there exists the B_2-almost periodic function $g(x)$ and its Fourier series expansion is as follows

$$g(x)\sim\sum_n c_n e^{i\lambda_n x}.$$

(c.f. A.S. Besicovitch [6],pp.91~112). By the hypothesis (C_λ),we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\lambda x}dx=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x)e^{-i\lambda x}dx \qquad (\forall real\ \lambda).$$

(c.f. ibid.V, 2nd ed. p.129).

Step(iii) Then if we put $f(x)-g(x)=h(x)$ say. Then we shall prove that the function $h(x)$ belongs to the class S. Since functions $f(x)$ and $g(x)$ both belong to the class S and we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} h(x+t)\overline{h(t)}dt=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\{f(x+t)-g(x+t)\}\overline{\{f(t)-g(t)\}}dt$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x+t)\overline{f(t)}dt-\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x+t)\overline{g(t)}dt$$

$$-\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x+t)\overline{f(t)}dt+\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x+t)\overline{g(t)}dt$$

and since $g(x)$ is B^2-almost periodic function, we have also

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x+t)\overline{g(t)}dt=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x+t)\overline{f(t)}dt$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x+t)\overline{g(t)}dt=\sum_n |c_n|^2\, e^{i\lambda_n x}$$

(c.f. ibid. IV , pp.105~108). Therefore we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} h(x+t)\overline{h(t)}dt==\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x+t)\overline{f(t)}dt-\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x+t)\overline{g(t)}dt.$$

Thus we have proved that a function $h(x)$ belongs to the class S.

Step (iv) We shall consider auto-correlation functions $\varphi(x;f)$, $\psi(x;g)$, $\chi(x;h)$ of f,g,h and their G.F.T. $\sigma(u;\varphi),\sigma(u;\psi),\sigma(u;\chi)$ of φ,ψ,χ respectively.

Then we have

$$\varphi(x;f)=\psi(x;g)+\chi(x;h) \qquad \text{and} \qquad \sigma(u;\varphi)=\sigma(u;\psi)+\sigma(u;\chi)$$

respectively.

We have already proved that the $\sigma(u;\varphi)$ is bounded monotone increasing function.

The $\sigma(u;\psi)$ is G.F.T. of $\psi(x;g)$ and $\psi(x;g)$ is the auto-correlation function of B_2-almost periodic function $g(x)$, then we have

$$\sigma(u;\psi)=\begin{cases}\sqrt{2\pi}\sum_{\lambda_n<u}|c_n|^2 & (u\neq\lambda_m)\\ \sqrt{2\pi}\left(\sum_{\lambda_n<u}|c_n|^2+\frac{1}{2}|c_m|^2\right) & (u=\lambda_m)\end{cases}$$

and so $\sigma(u;\psi)$ is bounded monotone increasing function.

Since $\sigma(u;\chi)$ is represented as the difference of the two bounded, monotone increasing functions $\sigma(u;\varphi)$ and $\sigma(u;\psi)$, it is a function of bounded variation and we have

$$\frac{1}{2T}\int_{-T}^{T}|\chi(x;h)|^2dx \le \chi(0;h) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|h(x)|^2dx, \quad (\forall\ T)$$

by the N.Wiener theorem[1] (c.f. Theorem 25, p. 154).

Therefore we could apply the N.Wiener theorem[1] (c.f. Theorem 24, pp.146~149) to the function $\chi(x;h)$ and we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\chi(x;h)|^2dx = \sum_n\left\{\frac{\sigma(\lambda_n+0;\varphi)-\sigma(\lambda_n-0;\varphi)}{\sqrt{2\pi}} - |c_n|^2\right\}^2.$$

In particular if $\sigma(u;\varphi)$ is continuous everywhere ,then it lead to

$$|c_n|^2 \le \frac{\sigma(\lambda_n+0;\varphi)-\sigma(\lambda_n-0;\varphi)}{\sqrt{2\pi}} = 0 \qquad (n=0,1,2,3,...)$$

and we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\chi(x;h)|^2dx = 0.$$

Now let us notice that we have

$$\int_{-\infty}^{\infty}\frac{|\chi(x;h)|^2}{1+x^2}dx \le (1+2\pi)\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\chi(x;h)|^2dx$$

by the N.Wiener theorem[1] (c.f. Theorem 20, p.138) and we have $\chi(x;h)=0$, *a.e.* x..

Moreover if a function $f(x)$ belong to the class S', then since we have proved that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}h(x+t)\overline{h(t)}dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x+t)\overline{f(t)}dt - \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}g(x+t)\overline{g(t)}dt$$

and then $\chi(x;h)$ is continuous everywhere and we shall conclude that $\chi(x;h)=0$ for all x. Thus we have proved

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|h(x)|^2dx = \chi(0;h) = 0.$$

Then applying the N.Wiener theorem[1] (c.f. Theorem 20, p.138) to $h(x)$ again ,we have

$$\int_{-\infty}^{\infty}\frac{|h(x)|^2}{1+x^2}dx \le (1+2\pi)\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|h(x)|^2dx$$

Thus we have

$$h(x)=0 \quad a.e.\ \ x \qquad \text{and} \qquad f(x)=g(x) \quad a.e.\ \ x.$$

Thus we have proved Theorem C.

Author suppose that N.Wiener might consider the decomposition theorem of function $f(x)$ on the class S in the research of his first stage. Therefore hereafter Theorem C should be called the Paley-Wiener decomposition on the theory of G.H.A.

Remark (1) Let us suppose that $f(x)$ belongs to the class S and satisfies the hypothesis (C_λ). Then we have for any constant a_λ

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\varepsilon;f)-s(u-\varepsilon;f)-\sqrt{2\pi}a_\lambda|^2\,du$$

$$=\frac{\sigma(\lambda+0;\varphi)-\sigma(\lambda-0;\varphi)}{\sqrt{2\pi}}-|c_\lambda|^2+|c_\lambda-a_\lambda|^2$$

and therefore the value of this integral attains to minimum if and only if $a_\lambda=c_\lambda$ and we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|(s(u+\varepsilon;f)-s(u-\varepsilon;f))-\sqrt{2\pi}c_\lambda|^2\,du=\frac{\sigma(\lambda+0;\varphi)-\sigma(\lambda-0;\varphi)}{\sqrt{2\pi}}-|c_\lambda|^2$$

Since $\sigma(u)$ is bounded and monotone increasing function, there exists the set E of countable points $\lambda=\lambda_n,\ (n=0,1,2,....)$ at which $\sigma(u;\varphi)$ has jump and continuous elsewhere. Thus we have the following results.

(i) If $\lambda\in D,$ then we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2du=\frac{\sigma(\lambda+0;\varphi)-\sigma(\lambda-0;\varphi)}{\sqrt{2\pi}}-|c_\lambda|^2=0.$$

(ii) If $\lambda\in E,$ that is $\lambda=\lambda_n,\ (n=0,1,2,3,...)$, then we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda_n-\varepsilon}^{\lambda_n+\varepsilon}|\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}-\sqrt{2\pi}c_n|^2\,du=\frac{\sigma(\lambda_n+0)-\sigma(\lambda_n-0)}{\sqrt{2\pi}}-|c_n|^2$$

(c.f. ibid.VI, 3rd ed. pp.141~143)

Chapter 2. Generalized Hilbert Transforms

This theory had been constructed by the author (c.f. S.Koizumi[10]) about more than fifty years ago. Let us suppose that a function $f(x)$ belongs to the Hilbert space W^2. Then Generalized Hilbert Transform of order $1(G.H.T.)$ of $f(x)$ is defined by the following formula

$$\tilde{f}_1(x)=P.V.\frac{(x+i)}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}\frac{dt}{x-t}.$$

Here we should be remarked that it has been introduced already for the purpose of the different research by H.Kober (c.f. Research Report I, pp.2~3). Then it can be written as

$$\frac{\widetilde{f}_1(x)}{x+i} = P.V.\int_{-\infty}^{\infty} \frac{f(t)}{t+i}\frac{dt}{x-t}$$

and then $\widetilde{f}_1(x)$ belongs to the space W^2 and so G.F.T. of $\widetilde{f}_1(x)$ is well defined.

Let us rewrite the multiplier $x+i=(t+i)+(x-t)$, then it can be written formally as follows

$$\widetilde{f}_1(x) = P.V.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{x-t}dt + P.V.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}dt = \widetilde{f}(x)+A(f),$$

where $\widetilde{f}$ is the ordinary Hilbert transform of f.

2.1 Relevant theorems of G.H.A. and advanced results.

About fifty years ago, the author established the following theorem (c.f. [10] Theorem 49, pp.201~205 ; see also [12] Theorem A)

Theorem K_0. Let us suppose that $f(x)$ belongs to the class W^2. Then we have for any positive number ε

(i) if $|u|>\varepsilon$ then

$$s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1) = (-i\,sign\,u)\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}$$

000000nd

(ii) If $|u|\le\varepsilon$ then

$$s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1) = i\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}+2r_1(u+\varepsilon;f)+2r_2(u+\varepsilon;f)$$

where

$$r_1(u;f) = \underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(t)}{t+i}\frac{e^{-iut}-1}{-it}dt$$

and

$$r_2(u;f) = \underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(t)}{t+i}e^{-iut}dt.$$

We shall intend to estimate remainder terms $r_1(u;f)$ and $r_2(u;f)$ in Theorem K_0.

For $r_1(u;f)$, if $f(x)$ belongs to the space W^2, then

$$(R_1) \qquad \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} | r_1(u+\varepsilon; f)|^2 \, du = O(\varepsilon), \quad (\varepsilon \to 0).$$

(c.f. Research Report I, p.19).

As for $r_2(u;f)$, we shall set the following condition (R_2): There exist a constant $a(f)$ such that

$$(R_2) \qquad \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} | r_2(u+\varepsilon; f) - \sqrt{\frac{\pi}{2}} a(f)|^2 \, du \to 0, \quad (\varepsilon \to 0).$$

This condition should be indispensable to reconstruct the theory of G.H.T. under the assistance of hypothesis (C_λ). Then we shall obtain many rich results as follows.

Theorem 1. Let us suppose that $f \in S_0$ and the hypothesis (C_λ) and the condition (R_2) are satisfied. Then we have $\tilde{f}_1 \in S_0$ and the following equality

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\widetilde{f}_1(x)|^2 \, dx = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(x)|^2 \, dx + |a(f) + ic_0|^2 - |c_0|^2 .$$

(c.f. ibid. VI, 3rd ed. Theorem B_1^*, pp.148~9)

Proof. We shall intend to estimate the following formula

$$I = \frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} | s(u+\varepsilon; \tilde{f}_1) - s(u-\varepsilon; \tilde{f}_1)|^2 \, du .$$

Then by the Theorem K_0, we have

$$I = \frac{1}{4\pi\varepsilon}\int_{|u|\ge\varepsilon} |(-i)\, sign\, u)\, \{s(u+\varepsilon; f) - s(u-\varepsilon; f)\}|^2 \, du$$

$$+ \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} | i\{s(u+\varepsilon; f) - s(u-\varepsilon; f)\} + 2r_1(u+\varepsilon; f) + 2r_2(u+\varepsilon; f)|^2 \, du$$

$= I_1 + I_2$, say. As for I_1, we have

$$I_1 = \frac{1}{4\pi\varepsilon}\int_{|u|\ge\varepsilon} | s(u+\varepsilon; f) - s(u-\varepsilon; f)|^2 du .$$

As for I_2, we have by the Minkovski inequality ,

$$I_2 = \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} | i\{s(u+\varepsilon; f) - s(u-\varepsilon; f)\} + 2r_2(u+\varepsilon; f)|^2 du \; + \; o(1) \quad (\varepsilon \to 0)$$

and by the use of the condition (R_2) and applying the Minkovski inequality again, we have

$$I_2 = \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} | i\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}+\sqrt{2\pi}a(f)|^2\,du \;+\; o(1)\;\;(\varepsilon\to 0).$$

Moreover we have

$$I_2 = \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} | s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2 du + i\frac{\overline{a(f)}}{2\varepsilon\sqrt{2\pi}}\int_{|u|\le\varepsilon}\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}\,du$$

$$-i\frac{a(f)}{2\varepsilon\sqrt{2\pi}}\overline{\int_{|u|\le\varepsilon}\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}\,du} + |a(f)|^2 \;+\; o(1)\quad(\varepsilon\to 0).$$

Here we shall notice that

$$c_0 = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)dx = \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}\,du$$

by Theorem A , then we have

$$I_2 = \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} | s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2 du + |a(f)+ic_0|^2 - |c_0|^2 \;+\; o(1)\;\;(\varepsilon\to 0).$$

Therefore we have that

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} | s(u+\varepsilon;\tilde{f}_1)-s(u-\varepsilon;\tilde{f}_1)|^2\,du$$

$$= \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} | s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2\,du + |a(f)+ic_0|^2 - |c_0|^2 .$$

Since $f(x)$ belongs to the class S_0 , then if we apply Theorem W_0 to the above formula, we have proved that a function $\tilde{f}_1(x)$ belongs to the class S_0 and we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |\tilde{f}_1(x)|^2\,dx = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |f(x)|^2\,dx + |a(f)+ic_0|^2 - |c_0|^2 .$$

If we apply Theorem A to the results of Theorem 1, we have obtained directly the following Corollary.

Corollary 1.1. Let us suppose that f belongs to the class S_0 and satisfies hypothesis (C_λ) and condition (R_2) . Then we shall prove that $\tilde{f}_1$ satisfies the hypothesis

$$(\tilde{C}_\lambda)\qquad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{f}_1(x)e^{-i\lambda x}dx = \tilde{c}_\lambda \quad (\forall real\ \lambda)$$

Proof. The existence of the following limit

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\left\{s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)\right\}du \qquad (\forall real\ \lambda)$$

is derived as follows.

We shall apply Theorem K_0 to this problem. If $\lambda\neq 0$, we have

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\left\{s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)\right\}du=(-isign\lambda)\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\left\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\right\}du$$

and if $\lambda=0$, we have

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\left\{s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)\right\}du=\lim_{\varepsilon\to 0}\frac{i}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\left\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\right\}du+a(f)$$

Therefore if we apply Theorem A to $\tilde{f}_1(x)$ then we have proved that hypothesis $(\tilde{C}_\lambda)$ is derived and we have

$$\tilde{c}_\lambda=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{f}_1(x)e^{-i\lambda x}dx=\begin{cases}(-isign\lambda)c_\lambda & (\lambda\neq 0)\\ \\ a(f)+ic_0 & (\lambda=0)\end{cases}$$

respectively.

Theorem 2. Let us suppose that $f\in S$ and hypothesis (C_λ) and condition (R_2) are satisfied. Then we have $\tilde{f}_1\in S$ and the following equality

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}e^{iux}\,|\,s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)\,|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}e^{iux}\,|\,s(u+\varepsilon,f)-s(u-\varepsilon,f)\,|^2\,du-|\,c_0\,|^2+|\,\tilde{c}_0\,|^2$$

where $\widetilde{c_0}=ic_0+a(f)$. Moreover we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{f}_1(x+t)\overline{\tilde{f}_1(t)}dt=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x+t)\overline{f(t)}dt-|\,c_0\,|^2+|\,\tilde{c}_0\,|^2.$$

That is to say, we have

$$\varphi(u;\tilde{f}_1)=\varphi(u;f)+|\,\tilde{c}_0\,|^2-|\,c_0\,|^2$$

(c.f. ibid. II, Theorem W_1 pp.25~29).

In particular result of Theorem 1 could be rewritten as follows

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\widetilde{f}_1(x)|^2\,dx=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)|^2\,dx-|c_0|^2+|\widetilde{c_0}|^2\,.$$

Furthermore we have

Theorem 3. Let us suppose that $f\in S'$ and the hypothesis (C_λ) and the condition (R_2) are satisfied. Then we have that $\widetilde{f}_1\in S'$ and we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{f}_1(x+t)\overline{\tilde{f}_1(t)}dt=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x+t)\overline{f(t)}dt-|c_0|^2+|\tilde{c}_0|^2$$

where $\widetilde{c_0}=ic_0+a(f)$.

Proof. By the Theorem 2, we have $\widetilde{f}_1(x)$ to belong to the class S. Then applying the theorem of N.Wiener[1] (c.f. Theorem 28, p. 160) and Theorem K_0, we have

$$\lim_{A\to\infty}\overline{\lim_{\varepsilon\to 0}}\frac{1}{4\pi\varepsilon}\left[\int_{-\infty}^{-A}+\int_{A}^{\infty}\right]|s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)|^2\,du$$

$$=\lim_{A\to\infty}\overline{\lim_{\varepsilon\to 0}}\frac{1}{4\pi\varepsilon}\left[\int_{-\infty}^{-A}+\int_{A}^{\infty}\right]|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2\,du\;=0.$$

Thus we have proved that $\widetilde{f}_1(x)$ belongs to the class S'.

We have also

Theorem 4. Let us suppose that $f(x)$ is a B^2-almost periodic function and satisfies condition (R_2). Let us write its Fourier series expansion as follows

$$f(x)\sim\sum_n c_n e^{i\lambda_n x}.$$

Then its G.H.T. $\tilde{f}_1(x)$ is also a function of B^2-almost periodic and has its Fourier series expansion as follows

$$\tilde{f}_1(x)\sim\sum_n \tilde{c}_n e^{i\lambda_n x},$$

where

$$\tilde{c}_n=\begin{cases}(-i\,sign\lambda_n)c_n & (n=1,2,3,\ldots)\\[2ex] ic_0+a(f) & (n=0).\end{cases}$$

(c.f. ibid. VI, 3rd ed. Theorem C^* , p.151).

Remark (2) Let us suppose that $f(x)$ belongs to the class S_0 and satisfies the hypothesis (C_λ) and the condition (R_2). Then we have by the Corollary of Theorem 1, its G.H.T. $\widetilde{f}_1(x)$ belongs to the class S_0 and satisfies hypothesis $(\widetilde{C}_\lambda)$.

In the case $\lambda = 0$. Then we have for any constant a_0

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|(s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1))-\sqrt{2\pi}\,\widetilde{a_0}\,|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|(s(u+\varepsilon;f)-s(u-\varepsilon;f))-\sqrt{2\pi}\,a_0\,|^2\,du$$

$$=\frac{\sigma(+0;\varphi)-\sigma(-0;\varphi)}{\sqrt{2\pi}}-|c_0|^2+|c_0-a_0|^2,$$

where $\widetilde{a_0}=ia_0+a(f)$. Therefore the value of the integral attains to minimum if and only if $a_0=c_0$, i.e. $\widetilde{a_0}=\widetilde{c_0}$ and $\widetilde{c_0}=ic_0+a(f)$. Then we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|(s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1))-\sqrt{2\pi}\,\widetilde{c_0}\,|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|(s(u+\varepsilon;f)-s(u-\varepsilon;f))-\sqrt{2\pi}\,c_0\,|^2\,du$$

$$=\frac{\sigma(+0;\varphi)-\sigma(-0;\varphi)}{\sqrt{2\pi}}-|c_0|^2.$$

In the case $\lambda \neq 0$. Then we have for any constant a_λ

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|(s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1))-\sqrt{2\pi}\,\widetilde{a}_\lambda\,|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|(s(u+\varepsilon;f)-s(u-\varepsilon;f))-\sqrt{2\pi}\,a_\lambda\,|^2\,du$$

$$=\frac{\sigma(\lambda+0;\varphi)-\sigma(\lambda-0;\varphi)}{\sqrt{2\pi}}-|c_\lambda|^2+|c_\lambda-a_\lambda|^2,$$

where $\tilde{a}_\lambda=(-isign\lambda)a_\lambda$. Therefore the value of the integral attains to minimum if and only if $a_\lambda=c_\lambda$ and we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|(s(u+\varepsilon;\tilde{f}_1)-s(u-\varepsilon;\tilde{f}_1))-\sqrt{2\pi}\tilde{c}_\lambda|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|(s(u+\varepsilon;f)-s(u-\varepsilon;f))-\sqrt{2\pi}c_\lambda|^2\,du$$

$$=\frac{\sigma(\lambda+0;\varphi)-\sigma(\lambda-0;\varphi)}{\sqrt{2\pi}}-|c_\lambda|^2.$$

where $\tilde{c}_\lambda=(-isign\lambda)c_\lambda$. Therefore we shall conclude that

(i) $\lambda\in D$. Then we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\varepsilon;\tilde{f}_1)-s(u-\varepsilon;\tilde{f}_1)|^2du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2du=\frac{\sigma(\lambda+0;\varphi)-\sigma(\lambda-0;\varphi)}{\sqrt{2\pi}}-|c_\lambda|^2=0$$

(ii) $\lambda\in E$, if $\lambda=\lambda_n,(n=1,2,3,...)$. Then we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda_n-\varepsilon}^{\lambda_n+\varepsilon}|(s(u+\varepsilon;\tilde{f}_1)-s(u-\varepsilon;\tilde{f}_1))-\sqrt{2\pi}\widetilde{c_n}|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda_n-\varepsilon}^{\lambda_n+\varepsilon}|(s(u+\varepsilon;f)-s(u-\varepsilon;f))-\sqrt{2\pi}c_n|^2\,du$$

$$=\frac{\sigma(\lambda_n+0;\varphi)-\sigma(\lambda_n-0;\varphi)}{\sqrt{2\pi}}-|c_n|^2$$

where $\widetilde{c_n}=(-isign\lambda_n)c_n$.

If $\lambda=\lambda_0\ \ (n=0)$. Then we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|(s(u+\varepsilon;\tilde{f}_1)-s(u-\varepsilon;\tilde{f}_1))-\sqrt{2\pi}\widetilde{c_0}|^2\,du$$

$$= \frac{\sigma(+0;\varphi)-\sigma(-0;\varphi)}{\sqrt{2\pi}} + |\widetilde{c_0}|^2 - |c_0|^2$$

where $\widetilde{c_0} = ic_0 + a(f)$. (c.f. ibid. VI, 3rd ed pp.143~145)

2.2 The Decomposition Theorem of G.H.T. $\widetilde{f_1}(x)$ on the class S.

Let us suppose that $f(x)$ belongs to the class S and satisfies the hypothesis (C_λ) and the condition (R_2). Then by the Theorem 1, Corollary 1 and Theorem 2, we proved that its G.H.T. $\widetilde{f_1}(x)$ belongs to the class S and satisfies hypothesis $(\widetilde{C}_\lambda)$. Then with the assistance of Theorem C, we have the decomposition of G.H.T. $\widetilde{f_1}(x)$ as follows.

Let us denote

$$f(x) = g(x) + h(x)$$

as the decomposition of $f(x)$ in Theorem C ,then as for G.H.T. of them we have

$$\widetilde{f_1}(x) = \widetilde{g_1}(x) + \widetilde{h_1}(x)$$

and it gives the required decomposition of $\widetilde{f_1}(x)$.Therefore we have

Theorem 5. Let us suppose that $f(x)$ belongs to the class S and satisfies hypothesis (C_λ) and condition (R_2). Let us set the decomposition of $f(x)$ as in the Theorem C

$$f(x) = g(x) + h(x).$$

Then we have the decomposition

$$\widetilde{f_1}(x) = \widetilde{g_1}(x) + \widetilde{h_1}(x)$$

and it satisfies the same properties of $f(x)$ as in the Theorem C.

Proof. The proof can be done by the same argument as Theorem C. By Theorem 4 the $\widetilde{g_1}(x)$ is B^2 - almost periodic with Fourier series expansion

$$\widetilde{g_1}(x) \sim \sum_n \widetilde{c_n} e^{i\lambda_n x}$$

where

$$\widetilde{c_n} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \widetilde{f_1}(x) e^{-i\lambda_n x} dx = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \widetilde{g_1}(x) e^{-i\lambda_n x} dx \qquad (n = 0,1,2,3,\dots)$$

and

$$\widetilde{c_n} = \begin{cases} (-i\,sign\lambda_n)c_n & (n=1,2,3,\ldots) \\ \\ ic_0 + a(f) & (n=0) \end{cases}.$$

Then we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\widetilde{g_1}(x)|^2\,dx = \sum_n |\widetilde{c_n}|^2$$
$$\le \sum_n \frac{\sigma(\lambda_n+0;\widetilde{\varphi})-\sigma(\lambda_n-0;\widetilde{\varphi})}{\sqrt{2\pi}} = \frac{\sigma(\infty;\widetilde{\varphi})-\sigma(-\infty;\widetilde{\varphi})}{\sqrt{2\pi}} < \infty$$

where

$$\frac{\sigma(\lambda_n+0;\widetilde{\varphi})-\sigma(\lambda_n-0;\widetilde{\varphi})}{\sqrt{2\pi}} = \begin{cases} \dfrac{\sigma(\lambda_n+0;\varphi)-\sigma(\lambda_n-0;\varphi)}{\sqrt{2\pi}} & (n=1,2,3,\ldots) \\ \\ \dfrac{\sigma(+0;\varphi)-\sigma(-0;\varphi)}{\sqrt{2\pi}} + |\widetilde{c_0}|^2 - |c_0|^2 & (n=0). \end{cases}$$

Therefore since $\widetilde{f_1}(x)$ belongs to the N,Wiener class S, then $\widetilde{h_1}(x)$ does too by the same argument as Theorem C.

Now let us denotes $\tilde{\varphi}(x) = \varphi(x,\tilde{f}_1), \tilde{\psi}(x) = \psi(x,\widetilde{g_1})$ and $\tilde{\chi}(x) = \chi(x,\widetilde{h_1})$ as the auto-correlation function of $\tilde{f}_1$, $\widetilde{g_1}$ and $\widetilde{h_1}$ respectively and $\sigma(u,\tilde{\varphi}), \sigma(u,\tilde{\psi})$ and $\sigma(u,\tilde{\chi})$ as their G.F.T. respectively. Then we have

$$\varphi(x,\widetilde{f}_1) = \psi(x,\widetilde{g_1}) + \chi(x,\widetilde{h_1}) \quad \text{and} \quad \sigma(u,\tilde{\varphi}) = \sigma(u,\tilde{\psi}) + \sigma(u,\tilde{\chi}).$$

In particular we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\widetilde{f_1}(x)|^2\,dx = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\widetilde{g_1}(x)|^2\,dx + \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\widetilde{h_1}(x)|^2\,dx.$$

Let us remark that $\widetilde{h_1}(x)$ belongs to the class S, $\sigma(u;\widetilde{\varphi})$ and $\sigma(u;\widetilde{\psi})$ are both bounded increasing as a function of u. Then $\sigma(u;\widetilde{\chi}) = \sigma(u;\widetilde{\varphi}) - \sigma(u;\widetilde{\psi})$ is a function of bounded variation and its magnitude of jumps are as follows

(i) If $\lambda \in D$, we have

$$\sigma(\lambda+0;\tilde{\varphi})-\sigma(\lambda-0;\tilde{\varphi})=0 \quad \text{and} \quad \sigma(\lambda+0;\tilde{\psi})-\sigma(\lambda-0;\tilde{\psi})=0$$

then we have

$$\sigma(\lambda+0;\tilde{\chi})-\sigma(\lambda-0;\tilde{\chi})=0$$

(ii) If $\lambda \in E=\{\lambda_n\}\ (n=0,1,2,3,\ldots)$

We have in the case $n=1,2,3,\ldots$

$$\sigma(\lambda_n+0;\tilde{\varphi})-\sigma(\lambda_n-0;\tilde{\varphi})=\sigma(\lambda_n+0;\varphi)-\sigma(\lambda_n-0;\varphi)$$

$$\sigma(\lambda_n+0;\tilde{\psi})-\sigma(\lambda_n-0;\tilde{\psi})=\sqrt{2\pi}\,|\,\tilde{c_n}\,|^2=\sqrt{2\pi}\,|\,c_n\,|^2$$

and

$$\sigma(\lambda_n+0;\tilde{\chi})-\sigma(\lambda_n-0;\tilde{\chi})$$

$$=\left\{\sigma(\lambda_n+0;\tilde{\varphi})-\sigma(\lambda_n-0;\tilde{\varphi})\right\}-\left\{\sigma(\lambda_n+0;\tilde{\psi})-\sigma(\lambda_n-0;\tilde{\psi})\right\}$$

$$=\left\{\sigma(\lambda_n+0;\varphi)-\sigma(\lambda_n-0;\varphi)\right\}-\sqrt{2\pi}\,|\,c_n\,|^2.$$

We have in the case $n=0$

$$\sigma(+0;\tilde{\varphi})-\sigma(-0;\tilde{\varphi})=\sigma(+0;\varphi)-\sigma(-0;\varphi)+\sqrt{2\pi}(|\,\tilde{c_0}\,|^2-|\,c_0\,|^2)$$

$$\sigma(+0;\tilde{\psi})-\sigma(-0;\tilde{\psi})=\sqrt{2\pi}\,|\,\tilde{c_0}\,|^2$$

and

$$\sigma(+0;\tilde{\chi})-\sigma(-0;\tilde{\chi})$$

$$=\left\{\sigma(+0;\tilde{\varphi})-\sigma(-0;\tilde{\varphi})\right\}-\left\{\sigma(+0;\tilde{\psi})-\sigma(-0;\tilde{\psi})\right\}$$

$$=\left\{\sigma(+0;\varphi)-\sigma(-0;\varphi)\right\}-\sqrt{2\pi}\,|\,c_0\,|^2$$

Therefore we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\,\chi(x;\tilde{h_1})\,|^2dx=\sum_n\left\{\frac{\sigma(\lambda_n+0;\tilde{\varphi})-\sigma(\lambda_n-0;\tilde{\varphi})}{\sqrt{2\pi}}-|\,\tilde{c_n}\,|^2\right\}^2$$

$$= \sum_n \left\{ \frac{\sigma(\lambda_n + 0; \varphi) - \sigma(\lambda_n - 0; \varphi)}{\sqrt{2\pi}} - |c_n|^2 \right\}^2$$

by the N.Wiener theorem[1] (c.f. Theorem 24, pp146~149)

In particular, if $\sigma(u;\varphi)$ is continuous everywhere, then we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\chi(x;\widetilde{h}_1)|^2 \, dx = 0 .$$

and moreover we have

$$\int_{-\infty}^{\infty} \frac{|\chi(x;\widetilde{h}_1)|^2}{1+x^2} dx \le (1+2\pi) \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\chi(x;\widetilde{h}_1)|^2 \, dx = 0$$

by the N.Wiener theorem[1](c.f. Theorem20,p.138). Thus we have

$$\chi(x,\widetilde{h}_1) = 0 \quad a.e.\ x .$$

In addition, if $f(x)$ belongs to the class S', then since $\widetilde{f}_1(x)$ and $\widetilde{g}_1(x)$ are both belong to the class S', $\widetilde{h}_1(x)$ does too. Then since $\chi(x;\widetilde{h}_1)$ is continuous everywhere we have $\chi(x,\widetilde{h}_1)$=0 for all x and in particular we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\widetilde{h}_1(x)|^2 \, dx = \chi(0;\widetilde{h}_1) = 0$$

and repeating the same arguments as above we have by the N.Wiener Theorem[1](c.f. Theorem20, p.138)

$$\widetilde{h}_1(x) = 0 \quad a.e.\ x .$$

and we have

$$\widetilde{f}_1(x) = \widetilde{g}_1(x) \qquad a.e.\ x$$

Thus Theorem5 has proved .

Chapter 3. Generalized Harmonic Analysis on the upper-half plane.

We shall reconstruct the theory of Generalized Hardy Space(G.H.S.) in the upper - half plane as another application of G.H.A. As for the Theory of ordinary Hardy Space, it should be refered to E.C.Titchmarsh [3].

Generalized Hardy space H_1^2 and Relevant Theorems.

We shall denote it by H_1^2 and it is defined by the set of a function $F(z)$, $(z = x + iy)$ that is analytic in the upper half-plane $y > 0$ and the integral

$$\int_{-\infty}^{\infty} \frac{|F(x+iy)|^2}{1+x^2} dx < \infty$$

exists uniformly in $y > 0$.

Generalized Cauchy Integral $(G.H.T.)$. We shall denote it by $C_1(z;F)$ and it is defined for a function $F(x)$ that belongs to the space W^2 by the following formular

$$C_1(z;F) = \frac{z+i}{2\pi i} \int_{-\infty}^{\infty} \frac{F(t)}{t+i} \frac{dt}{t-z}$$

where $z = x + iy$, $y > 0$.

Then along just the same arguments as Hardy space H^2 (c.f. E.C.Titchmarsh [3],Chap. V, pp.119~132) we had been proved the several theorems.

Theorem K_1. Let $F(z)$ be analytic in the upper half-plane $y > 0$ and belongs to the class H_1^2. Then we shall find out the boundary function at $y = 0$ to be denoted it by $F(x)$. Then we have

(i) We have

$$\lim_{y \to 0} F(x+iy) = F(x), \quad a.e. \quad x$$

where if we write $f(x)$ as the *Real Part of* $F(x)$, then *Imaginary Part of* $F(x)$ is $\tilde{f}_1(x)$ and we have $F(x) = f(x) + i\tilde{f}_1(x)$.

(ii) It belongs to the space W^2 and we have

$$\lim_{y \to 0} \int_{-\infty}^{\infty} \frac{|F(x+iy) - F(x)|^2}{1+x^2} dx = 0.$$

(iii) The $F(z)$ is represented as the G.C.I. of $F(x)$, that is

$$F(z) = C_1(z;F) = \frac{z+i}{2\pi i} \int_{-\infty}^{\infty} \frac{F(t)}{t+i} \frac{dt}{t-z}$$

where $z = x + iy$, $y > 0$.

We shall prove that the inverse theorem is also true as follows.

Theorem K_2. Let us suppose that $f(x)$ belongs to the space W^2 and let us define $F(x) = f(x) + i\tilde{f}_1(x)$ and $F(z) = C_1(z;F)$. Then we have

(i) The $F(x)$ belongs to the space W^2 and $F(z)$ belongs to the class H_1^2.

(ii) The $F(x)$ is the boundary function of $F(z)$ in the following sense

$$\lim_{y\to 0} F(x+iy) = F(x), \quad a.e. \quad x$$

and

$$\lim_{y\to 0} \int_{-\infty}^{\infty} \frac{|F(x+iy)-F(x)|^2}{1+x^2} dx = 0.$$

Theorem K_3. Let $F(z),(z=x+iy)$ be analytic in the upper –half plane $y>0$ and belongs to the class H_1^2. Let us denote by $F(x)$ its boundary function at $y=0$. Then we have for any given positive number ε

(i) if $|u| \geq \varepsilon$, then

$$s(u+\varepsilon;F(z)) - s(u-\varepsilon;F(z))$$
$$= \frac{(1+signu)}{2} e^{-yu} \{(s(u+\varepsilon;F) - s(u-\varepsilon;F)) + r_0(u,y,\varepsilon;F)\}$$

where

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{|u|>\varepsilon} |r_0(u,y,\varepsilon;F)|^2 \, du = 0$$

for every $y>0$,

(ii) if $|u| \leq \varepsilon$, then

$$s(u+\varepsilon;F(z)) - s(u-\varepsilon;F(z)) = ir_1(u+\varepsilon;F) + ir_2(u+\varepsilon;F) + r_3(u+\varepsilon,y;F)$$

where

$$ir_1(u+\varepsilon;F) + ir_2(u+\varepsilon;F) = s(u+\varepsilon;F) - s(u-\varepsilon;F)$$

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{|u|<\varepsilon} |r_1(u+\varepsilon;F)|^2 \, du = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \int_{|u|<\varepsilon} |r_3(u+\varepsilon,y,:F)|^2 \, du = 0$$

for every $y>0$.

(c.f. S.Koizumi[10] and Research Report III, pp. 46~52).

Let us suppose that a function $F(z)$ is analytic in the upper-half plane $y>0$ and belongs to the class H_1^2. Then by the Theorem K_1, there exists the boundary function at $y=0$ and we shall denote it by $F(x)$. Then it is represented as follows

$$F(x) = f(x) + i\tilde{f}_1(x)$$

where $f(x)$ is the real part of $F(x)$ and $\tilde{f}_1(x)$ is the imaginary part of $F(x)$ and G.H.T. of $f(x)$.

Furthermore we have

$$F(z) = C_1(z;F) = \frac{z+i}{2\pi i} \int_{-\infty}^{\infty} \frac{F(t)}{t+i} \frac{dt}{z-t} \qquad (z=x+iy, \quad y>0).$$

Here we shall quote the skew reciprocal formula of G.H.T.

$$\frac{\tilde{f}_1(x)}{x+i} = P.V.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}\frac{dt}{x-t} \quad \text{and} \quad P.V.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\tilde{f}_1(t)}{t+i}\frac{dt}{x-t} = -\frac{f(x)}{x+i}$$

and it is derived from that of the Ordinary Hilbert Transform.(c.f.E.C.Tichmarsh[3], Chap.V, Theorem91, pp.122~123), (c.f. S.Koizumi[11],Theorems36,37,pp.192~3) and (c.f. Research Report III,pp. 50~51 and 63~64).

Then we have

$$F(z) = 2C_1(z;f)$$

with $f(x)$ the real part of $F(x)$.

Now we shall denote

$$f_1(z) = C_1(z;f).$$

Then we should remark that $F(z) = 2f_1(z)$ and this enable us to argue with $f_1(z)$ instead of $F(z)$ and to set hypothesis (C_λ) and condition (R_2) on $f(x)$.

Now we shall going to construct the theory of spectral analysis and synthesis of G.H.S. and the Decomposition Theorem of $F(z)$. We shall state them step by step steadily for the sake of completeness.

Theorem 6 Let us suppose that $F(z)$ belongs to the class H_1^2. Let us denote $F(x)$ as its boundary function at $y=0$. Let us suppose that $f(x)$ the real part of $F(x)$ belongs to the class S and satisfies hypothesis (C_λ) and condition (R_2). Then we have

If $|u| > \varepsilon$, then we have

$$s(u+\varepsilon;F(z)) - s(u-\varepsilon;F(z)) = \frac{(1+signu)}{2}e^{-yu}\{2(s(u+\varepsilon;f) - s(u-\varepsilon;f)) + r_0(u,y,\varepsilon;F)\}$$

where

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|>\varepsilon}|r_0(u,y,\varepsilon;F)|^2du = 0.$$

If $|u| < \varepsilon$, then we have

$$s(u+\varepsilon;F(z)) - s(u-\varepsilon;F(z)) = 2ir_1(u+\varepsilon;f) + 2ir_2(u+\varepsilon;f) + r_3(u+\varepsilon,y;F)$$

where

$$ir_1(u+\varepsilon;f) + ir_2(u+\varepsilon;f) = s(u+\varepsilon;f) - s(u-\varepsilon;f)$$

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|<\varepsilon}|r_1(u+\varepsilon;f)|^2\,du = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|<\varepsilon}|r_3(u+\varepsilon,y;F)|^2du = 0$$

and there exist constant $a(f)$ such as

$$(R_2) \qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|<\varepsilon} | r_2(u+\varepsilon; f) - \sqrt{\frac{\pi}{2}}a(f)|^2\, du = 0$$

Proof. We can apply Theorem K_0 and Theorem K_3, then we shall obtain the required results. We shall omit the detailed proof.

Theorem 7 Under the same hypotheses and condition as Theorem 6 , the function $F(z)$ belongs to the class S' for all $y>0$.

Proof. In the first, we shall intend to prove that $F(z)$ belongs to the class S as function of x for all $y>0$. This can be done by the application of Theorem W_1. We shall estimate it by the integration by parts and apply Theorem B.

Let us estimate the following integral

$$\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} e^{iux} | s(u+\varepsilon; F) - s(u-\varepsilon; F)|^2\, du$$

$$= \frac{1}{4\pi\varepsilon}\int_{|u|\geq\varepsilon} e^{iux} |\,''\,|^2\, du + \frac{1}{4\pi\varepsilon}\int_{|u|\leq\varepsilon} e^{iux} |\,''\,|^2\, du = I_1 + I_2, \quad say.$$

We have

$$I_1 = \frac{1}{4\pi\varepsilon}\int_{|u|>\varepsilon} e^{iux} | \frac{(1+signu)}{2} e^{-yu} \{2(s(u+\varepsilon; f) - s(u-\varepsilon; f)) + r_0(u, y, \varepsilon; F)\} |^2\, du$$

$$= \frac{1}{\pi\varepsilon}\int_{\varepsilon}^{\infty} e^{iux} e^{-2yu} | s(u+\varepsilon; f) - s(u-\varepsilon; f)|^2\, du + o(1) \quad (\varepsilon\to 0)$$

by the Minkowski inequality and (R_1) we have

$$\frac{1}{\pi\varepsilon}\int_{\varepsilon}^{\infty} e^{iux} e^{-2yu} | s(u+\varepsilon; f) - s(u-\varepsilon; f)|^2\, du = \left[\frac{e^{(ix-2y)}}{\pi\varepsilon}\int_{\varepsilon}^{u} | s(v+\varepsilon; f) - s(v-\varepsilon; f)|^2\, dv\right]_{u=\varepsilon}^{u=\infty}$$

$$-\frac{(ix-2y)}{\pi\varepsilon}\int_{\varepsilon}^{\infty} e^{(ix-2y)u}\left(\int_{\varepsilon}^{u} | s(v+\varepsilon; f) - s(v-\varepsilon; f)|^2\, dv\right) du$$

by the integration by parts.

Now we have by the Theorem B

$$\frac{1}{4\pi\varepsilon}\int_{\varepsilon}^{\infty} | s(v+\varepsilon; f) - s(v-\varepsilon; f)|^2 dv \to \frac{\sigma(\infty) - \sigma(0+)}{\sqrt{2\pi}} \quad (\varepsilon\to 0)$$

and

$$\frac{1}{4\pi\varepsilon}\int_{\varepsilon}^{u}|s(v+\varepsilon;f)-s(v-\varepsilon;f)|^2dv\to\frac{\sigma(u)-\sigma(0+)}{\sqrt{2\pi}}\quad a.e.\ u\quad(\varepsilon\to 0)$$

boundedly. Therefore we have

$$I_1=-4(ix-2y))\int_{\varepsilon}^{\infty}e^{(ix-2y)u}\frac{\sigma(u)-\sigma(0+)}{\sqrt{2\pi}}du+o(1)\quad(\varepsilon\to 0)$$

Next we have

$$I_2=\frac{1}{4\pi\varepsilon}\int_{|u|<\varepsilon}e^{iux}|s(u+\varepsilon;F(z))-s(u-\varepsilon;F(z))|^2\,du$$

$$=\frac{1}{4\pi\varepsilon}\int_{|u|<\varepsilon}e^{iux}|2ir_1(u+\varepsilon;f)+2ir_2(u+\varepsilon;f)+r_3(u,y,\varepsilon;F)|^2\,du$$

where let us remark the following properties

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|\le\varepsilon}|r_1(u+\varepsilon,f)|^2\,du=0\quad\text{and}\quad\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|<\varepsilon}|r_3(u,y,\varepsilon;F)|^2\,du=0$$

and the condition (R_2). Then we have by the Minkowski inequality

$$I_2=\frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon}|r_2(u+\varepsilon,f)-\sqrt{\frac{\pi}{2}}a(f)|^2\,du+|a(f)|^2+o(1)=|a(f)|^2+o(1)\quad(\varepsilon\to 0).$$

Therefore we have proved

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}e^{iux}|s(u+\varepsilon;F(z))-s(u-\varepsilon;F(z))|^2\,du$$

$$=-\frac{4(ix-2y)}{\sqrt{2\pi}}\int_{0}^{\infty}e^{(ix-2y)u}(\sigma(u)-\sigma(0+))du+|a(f)|^2.$$

Thus we have proved that $F(z)$ belongs to the class S by Theorem W_1.

Next we shall prove $F(z)$ belongs to the class S'. Applying Theorem of N.Wiener(c.f. [1], Theorem 28, p.160) with the assistance of Theorem6, we have

$$\lim_{A\to\infty}\overline{\lim_{\varepsilon\to 0}}\frac{1}{4\pi\varepsilon}\left[\int_{-\infty}^{-A}+\int_{A}^{\infty}\right]|s(u+\varepsilon;F(z))-s(u-\varepsilon;F(z))|^2\,du$$

$$=\lim_{A\to\infty}\overline{\lim_{\varepsilon\to 0}}\frac{1}{4\pi\varepsilon}\left[\int_{-\infty}^{-A}+\int_{A}^{\infty}\right]|\frac{(1+sign u)}{2}e^{-yu}\{2(s(u+\varepsilon;f)-s(u-\varepsilon;f))+r_0(u,y,\varepsilon;F\}|^2\,du$$

$$= \lim_{A\to\infty} \overline{\lim_{\varepsilon\to 0}} \frac{1}{\pi\varepsilon} \int_A^\infty e^{-2yu} \mid s(u+\varepsilon; f) - s(u-\varepsilon; f) \mid^2 du$$

$$\le \lim_{A\to\infty} e^{-2yA} \lim_{\varepsilon\to 0} \frac{1}{\pi\varepsilon} \int_{-\infty}^\infty \mid s(u+\varepsilon; f) - s(u-\varepsilon; f) \mid^2 du = 0 .$$

Therefore we obtain that $F(z)$ belongs to the class S' as a function of x for all $y > 0$ and we have

$$\varphi(x; F(z)) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(x+t, y)\overline{F(t,y)}dt$$

$$= \lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^\infty e^{iux} \mid s(u+\varepsilon; F(z)) - s(u-\varepsilon; F(z)) \mid^2 du \quad (z = x+iy, \ \ y > 0) .$$

Theorem 8 Under the same hypotheses as the Theorem 6, $F(z)$ satisfies the hypothesis (C_λ) $(\forall\ real\ \lambda)$ as function of x for all $y > 0$.

Proof For this purpose we shall need the support of Theorem A. There we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(z) e^{-i\lambda x} dx = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \{s(u+\varepsilon; F(z)) - s(u-\varepsilon; F(z))\} du$$

where $z = x+iy$, $y > 0$. Then we shall intend to estimate the formula in the right hand side with the assistace of Theorem 6.

(i) The case $\lambda \ne 0$. We have by Theorem 6

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \{s(u+\varepsilon; F(z)) - s(u-\varepsilon; F(z))\} du$$

$$= \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \frac{(1+signu)}{2} e^{-yu} \{2(s(u+\varepsilon; f) - s(u-\varepsilon; f)) + r_0(u, y, \varepsilon; F\} du$$

$$= (1+sign\lambda) e^{-y\lambda} \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} (s(u+\varepsilon; f) - s(u-\varepsilon; f)) du = (1+sign\lambda) e^{-y\lambda} c_\lambda$$

Thus we have proved

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(z) e^{-i\lambda x} dx = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \{s(u+\varepsilon; F(z)) - s(u-\varepsilon; F(z))\} du = (1+sign\lambda) e^{-y\lambda} c_\lambda$$

(ii) The case $\lambda = 0$. We have by Theorem 6

$$s(u+\varepsilon; F(z)) - s(u-\varepsilon; F(z)) = 2ir_1(u+\varepsilon; f) + 2ir_2(u+\varepsilon; f) + r_3(u+\varepsilon, y; F)$$

where we shall notice the following properties

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|<\varepsilon}|r_1(u+\varepsilon;f)|^2\,du=0 \quad \text{and} \quad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|<\varepsilon}|r_3(u+\varepsilon,y;F)|^2du=0$$

and the condition

$$(R_2) \qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|\le\varepsilon}|r_2(u+\varepsilon;f)-\sqrt{\frac{\pi}{2}}a(f)|^2\,du=0$$

Therefore we have

$$\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{|u|<\varepsilon}\{s(u+\varepsilon;F(z))-s(u-\varepsilon;F(z))\}du$$

$$=\frac{2i}{2\varepsilon\sqrt{2\pi}}\int_{|u|<\varepsilon}(r_2(u+\varepsilon;f)-\sqrt{\frac{\pi}{2}}a(f))du+ia(f)+o(1)=ia(f)+o(1) \quad (\varepsilon\to 0).$$

Thus we have proved

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}F(z)dx=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{|u|<\varepsilon}\{s(u+\varepsilon;F(z))-s(u-\varepsilon;F(z))\}du=ia(f).$$

Thus we have proved

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}F(z)e^{-i\lambda x}dx== \begin{cases} 0, & (\lambda\in D) \\ (1+sign\lambda)e^{-y\lambda}c_\lambda, & (\lambda\in E,\lambda\ne 0) \\ ia(f), & (\lambda\in E,\lambda=0) \end{cases}$$

where

$$c_n=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)e^{-i\lambda_n x}dx \quad \text{and} \quad \lambda_n\in E \quad (n=0,1,2,3,\ldots)$$

Now let us suppose that $F(z)$ belongs to the G.H.S. H_1^2 on the upper-half plane. Let us denote $F(x)$ the boundary function at $y=0$ and $f(x)$ the real part of $F(x)$. Let us suppose that $f(x)$ belongs to the N.Wiener class S and satisfies hypotheses (C_λ) and condition (R_2). Let $f(x)=g(x)+h(x)$ be the decomposition in Theorem C and we shall denote G.C.I. of each function as follows

$$f_1(z)=C_1(z;f), \quad g_1(z)=C_1(z;g) \quad \text{and} \quad h_1(z;h)=C_1(z;h)$$

respectively.

Then $g_1(z)$ is B^2-almost periodic function and we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f_1(z)e^{-i\lambda x}dx=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}g_1(z)e^{-i\lambda x}dx, \quad (z=x+iy,\ y>0).$$

Let us denote the Fourier series expansion of $g(x)$ as follows

$$g(x) \sim \sum_n c_n e^{i\lambda_n x}.$$

Then we have the Fourier series expansion of $g_1(z)$

$$g_1(z) \sim \frac{1}{2} ia(f) + \sum_{n\neq 0} \frac{(1+sign\lambda_n)}{2} e^{-\lambda_n y} c_n e^{i\lambda_n x}, \qquad (z = x+iy, \quad y>0).$$

This is proved as follows. Applying Theorem A ,Theorem C and Theorem K_3 ,we have
(i) In the case $\lambda \neq 0$. We have by Theorem A and part (i) of Theorem K_3

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1(z) e^{-i\lambda x} dx = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \{s(u+\varepsilon; f_1(z)) - s(u-\varepsilon; f_1(z))\} dx$$

$$= \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \frac{(1+signu)}{2} e^{-yu} \{(s(u+\varepsilon; f) - s(u-\varepsilon; f)) + r_0(u,y,\varepsilon; f)\} du$$

$$= \frac{(1+sign\lambda)}{2} e^{-y\lambda} \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \{s(u+\varepsilon; f) - s(u-\varepsilon; f)\} du$$

$$= \frac{(1+sign\lambda)}{2} e^{-y\lambda} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x) e^{-i\lambda x} dx$$

On the other hand we have by the same argument as before

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g_1(z) e^{-i\lambda x} dx = \frac{(1+sign\lambda)}{2} e^{-y\lambda} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(x) e^{-i\lambda x} dx$$

and

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x) e^{-i\lambda x} dx = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(x) e^{-i\lambda x} dx$$

by Theorem C. Therefore we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1(z) e^{-i\lambda x} dx = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g_1(z) e^{-i\lambda x} dx, \qquad (z = x+iy, \quad y>0).$$

(ii) In the case $\lambda = 0$. We have by Theorem A and part (ii) of Theorem K_3

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1(z) dx = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{|u|<\varepsilon} \{s(u+\varepsilon; f_1(z)) - s(u-\varepsilon; f_1(z))\} du$$

$$= \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{|u|<\varepsilon} \{ir_1(u+\varepsilon; f) + ir_2(u+\varepsilon; f) + r_3(u+\varepsilon, y; f)\} du$$

$$= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{|u|<\varepsilon}\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}du = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)dx$$

Similarly we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g_1(z)dx = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x)dx .$$

Therefore we have by Theorem C

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_1(z)dx = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g_1(z)dx$$

On the other hand we have by the part (ii) of Theorem K_3 and condition (R_2)

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_1(z)dx = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{|u|<\varepsilon} ir_2(u+\varepsilon;f)du$$

$$= \lim_{\varepsilon\to 0}\frac{i}{2\varepsilon\sqrt{2\pi}}\int_{|u|<\varepsilon}(r_2(u+\varepsilon;f)-\sqrt{\frac{\pi}{2}}a(f))du + \frac{1}{2}ia(f) = \frac{1}{2}ia(f)$$

Then we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g_1(z)e^{-i\lambda_n x}dx = \begin{cases} \frac{1}{2}ia(f) & (n=0) \\ \frac{(1+sign\lambda_n)}{2}c_n e^{-\lambda_n y}e^{i\lambda_n x} & (n=1,2,3,\dots) \end{cases}$$

Therefore we have

$$g_1(z) \sim \frac{1}{2}ia(f) + \sum_{n\neq 0}\frac{(1+sign\lambda_n)}{2}c_n e^{-\lambda_n y}e^{i\lambda_n x} \quad (z=x+iy,\ \ y>0).$$

Next we have by the same argument as Theorem C

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} h_1(x+t+iy)\overline{h_1(t+iy)}dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_1(x+t+iy)\overline{f_1(t+iy)}dt - \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g_1(x+t+iy)\overline{g_1(t+iy)}dt$$

Let us denotes $\varphi_1(x,y;f_1(z))$, $\psi_1(x,y;g_1(z))$ and $\chi_1(x,y;h_1(z))$ as auto-correlation function of $f_1(z)$, $g_1(z)$ and $h_1(z)$ respectively. Let us denotes $\sigma(u,y;\varphi_1)$, $\sigma(u,y;\psi_1)$ and $\sigma(u,y;\chi_1)$ as G.F.T. of $\varphi_1(x,y;f_1(z))$, $\psi_1(x,y;g_1(z))$ and $\chi_1(x,y;h_1(z))$ respectively.

Then we have

$$\varphi_1(x,y;f_1(z)) = \psi_1(x,y;g_1(z)) + \chi_1(x,y;h_1(z))$$

and

$$\sigma(u,y;\varphi_1) = \sigma(u,y;\psi_1) + \sigma(u,y;\chi_1).$$

Since the $\sigma(u,y;\chi_1)$ is represented as a difference of two bounded and monotone increasing function $\sigma(u,y;\varphi_1)$ and $\sigma(u,y;\psi_1)$, it is a function of bounded variation and we can apply to it the N.Wiener Theorem (c.f. [1] Theorem 24, pp.146~9).

Let us estimate the magnitude of each jumps. We have by Theorem B

$$\frac{\sigma(\lambda+0,y;\varphi_1)-\sigma(\lambda-0,y;\varphi_1)}{\sqrt{2\pi}} = \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-0}^{\lambda+0} |s(u+\varepsilon,y;f_1(z))-s(u-\varepsilon,y;f_1(z))|^2\,du$$

We shall obtain also by Theorem A and Theorem K_3 (c.f. ibid. III, Theorem D_3 p.47) the following estimations

(i) In the case $\lambda \neq 0$. We have by the part (i) of Theorem K_3

$$\frac{\sigma(\lambda+0,y;\varphi_1)-\sigma(\lambda-0,y;\varphi_1)}{\sqrt{2\pi}} = \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} |s(u+\varepsilon;f_1(z))-s(u-\varepsilon;f_1(z))|^2\,du$$

$$= \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \left|\frac{(1+sign u)}{2}e^{-yu}\left\{(s(u+\varepsilon;f)-s(u-\varepsilon;f)) + r_0(u,y,\varepsilon;f)\right\}\right|^2 du$$

$$= \left(\frac{1+sign\lambda}{2}\right)^2 e^{-2y\lambda}\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} |s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2\,du$$

$$= \left(\frac{1+sign\lambda}{2}\right)^2 e^{-2y\lambda}\,\frac{\sigma(\lambda+0;\varphi)-\sigma(\lambda-0;\varphi)}{\sqrt{2\pi}}\,..$$

(ii) The case $\lambda = 0$. We have also by the part (ii) of Theorem K_3, hypothesis (R_2) and the Minkowski inequality

$$\frac{\sigma(+0,y;\varphi_1)-\sigma(-0,y;\varphi_1)}{\sqrt{2\pi}}$$

$$= \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{|u|<\varepsilon} |ir_1(u+\varepsilon;f)+ir_2(u+\varepsilon;f)+r_3(u+\varepsilon,y;f)|^2\,du$$

$$= \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{|u|<\varepsilon} | i(r_1(u+\varepsilon;f) - \sqrt{\frac{\pi}{2}}a(f)|^2\, du + |\frac{1}{2}ia(f)|^2 = |\frac{1}{2}ia(f)|^2 .$$

On the other hand we have by the Theorem 9

$$\frac{\sigma(\lambda+0,y;\psi_1)-\sigma(\lambda-0,y;\psi_1)}{\sqrt{2\pi}} = \begin{cases} |\frac{1}{2}ia(f)|^2 & (n=0) \\ |\frac{(1+sign\lambda_n)}{2}e^{-\lambda_n y}c_n|^2 & (n=1,2,3,\ldots) \end{cases}$$

Therefore we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |\chi_1(x,y,;h_1(z))|^2\, dx = \sum_n \left\{\frac{\sigma(\lambda_n+0,y;\chi_1)-\sigma(\lambda_n-0,y;\chi_1)}{\sqrt{2\pi}}\right\}^2$$

$$= \sum_n \left\{\frac{\sigma(\lambda_n+0,y;\varphi_1)-\sigma(\lambda_n-0,y;\varphi_1)}{\sqrt{2\pi}} - \frac{\sigma(\lambda_n+0,y;\psi_1)-\sigma(\lambda_n-0,y;\psi_1)}{\sqrt{2\pi}}\right\}^2$$

$$= \sum_{n\neq 0}\left(\frac{1+sign\lambda_n}{2}\right)^2 e^{-2\lambda_n y}\left\{\frac{\sigma(\lambda_n+0;\varphi)-\sigma(\lambda_n-0;\varphi)}{\sqrt{2\pi}} - |c_n|^2\right\}^2 .$$

Now we shall prove the decomposition theorem of $F(z)$.

Theorem 9. Let us suppose that $F(z)$, $(z=x+iy,\ y>0)$ belongs to space H_1^2. Let us denote $F(x)$ the boundary function at $y=0$ and $f(x)$ the real part of $F(x)$. Let us suppose that $f(x)$ satisfy hypotheses (C_λ) and condition (R_2).

Then we have the decomposition

$$F(z) = G(z) + H(z)$$

where

$$F(z) = 2C_1(z;f),\quad G(z) = 2C_1(z;g)\quad \text{and}\quad H(z) = 2C_1(z;h)$$

respectively and satisfies the following properties

The $G(z)$ is B^2-almost periodic function and its Fourier series expansion is as follows

$$G(z) = ia(f) + \sum_{n\neq 0}(1+sign\lambda_n)c_n e^{i\lambda_n z}$$

where $G(z)$ and $H(z)$ both belong to the space H_1^2.

The $F(z), G(z)$ and $H(z)$ belong to the N.Wiener class S' and we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\chi(x,y;H(z))|^2\,dx$$

$$=\sum_{n\neq 0}(1+sign\lambda_n)^2 e^{-2\lambda_n y}\left\{\frac{\sigma(\lambda_n+0;\varphi)-\sigma(\lambda_n-0;\varphi)}{\sqrt{2\pi}}-|c_n|^2\right\}^2$$

In particular if the $\sigma(u;\varphi)$ is continuous everywhere as for $u>0$, then we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\chi(x,y;H(z)|^2dx=0 \qquad (\forall\ y>0).$$

and we have

$$F(x+iy)=G(x+iy) \quad a.e.\ x \quad (\forall y>0).$$

Proof Since $F(z)$ and $G(z)$ are both belong to the class S', and therefore $H(z)$ does too. Then applying the N.Wiener Theorem[1](c.f. Theorem 20,p.138), we have

$$\chi(x,y;H(z))=0 \quad (\forall x,\ \ \forall y>0)$$

and in particular

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|H(x+iy)|^2dx=0 \quad (\forall y>0).$$

Then applying the N.Wiener Theorem(c.f. [1] Theorem 20,p.138) to $H(x+iy)$ again, we have

$$H(x+iy)=0 \qquad (\forall x,\ \ \forall y>0)$$

and therefore we have

$$F(x+iy)=G(x+iy) \quad (\forall x,\ \ \forall y>0).$$

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