

Research Report

KSTS/RR-17/006

October 16, 2017

**Bell's inequality is violated in classical systems
as well as quantum systems**

by

Shiro Ishikawa

Shiro Ishikawa
Department of Mathematics
Keio University

Department of Mathematics
Faculty of Science and Technology
Keio University

©2017 KSTS
3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522 Japan

Bell's inequality is violated in classical systems as well as quantum systems

Shiro ISHIKAWA

Department of Mathematics, Faculty of Science and Technology, Keio University,
 3-14-1, Hiyoshi, Kouhoku-ku Yokohama, Japan. E-mail: ishikawa@math.keio.ac.jp

Abstract

Bell's inequality itself is usually considered to belong to mathematics and not quantum mechanics. We think that this is making our understanding of Bell' theory be confused. Thus in this paper, contrary to Bell's spirit (which inherits Einstein's spirit), we try to discuss Bell's inequality in the framework of quantum theory with the linguistic Copenhagen interpretation. And we clarify that the violation of Bell's inequality (i.e., whether or not Bell's inequality holds) does not depend on whether classical systems or quantum systems, but depend on whether a kind of simultaneous measurements exist or not. And further we conclude that our argument (based on the linguistic Copenhagen interpretation) should be regarded as a scientific representation of Bell's philosophical argument (based on Einstein's spirit).

Key phrases: Bohr-Einstein debates, Bell's inequality, Combined observable, Linguistic Copenhagen interpretation, Quantum Language

1 Review: Quantum language (= Measurement theory (=MT))

1.1 Introduction

Recently (*cf.* refs. [6, 7, 8, 9, 10, 11, 12, 13, 14]), we proposed “quantum language”, which was not only characterized as the metaphysical and linguistic turn of quantum mechanics but also the linguistic turn of Descartes=Kant epistemology. And further we believe that quantum language is the only scientifically successful theory in dualistic idealism. That is, we think that the location of quantum language in the history of world-description is as follows.

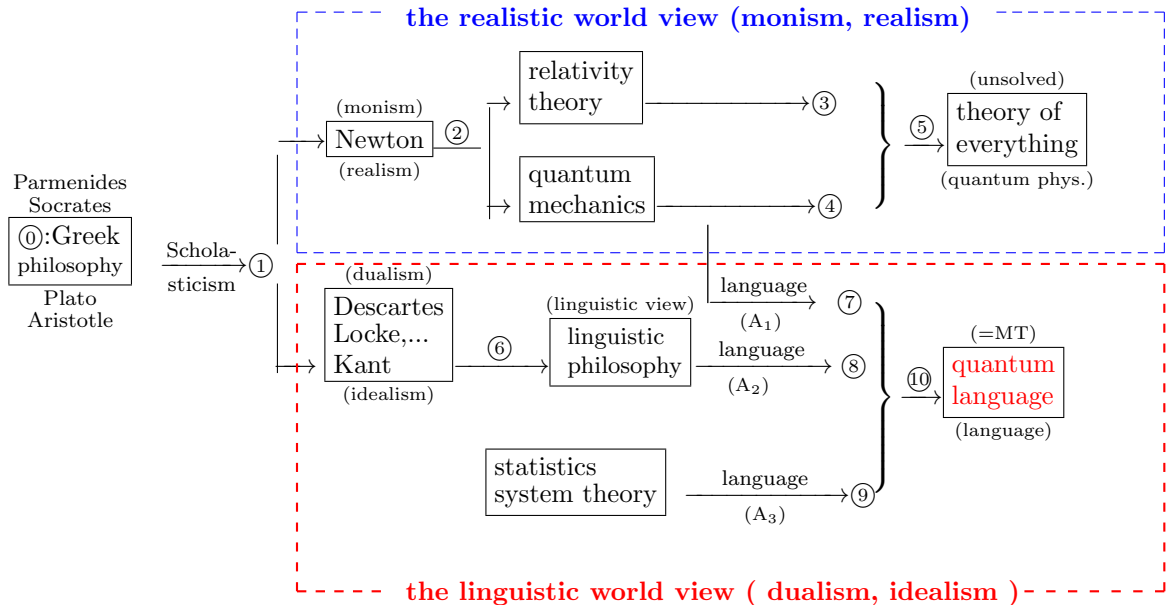


Figure 1: The history of the world-description

And in Figure 1, we think that the following four are equivalent (refs. [6, 13]):

(A₀) to propose quantum language (cf. ⑩ in Figure 1, ref.[13])

(A₁) to clarify the Copenhagen interpretation of quantum mechanics (cf. ⑦ in Figure 1, refs.[7, 12])

(A₂) to clarify the final goal of the dualistic idealism (cf. ⑧ in Figure 1, refs.[8, 14])

(A₃) to reconstruct statistics in the dualistic idealism (cf. ⑨ in Figure 1, refs.[9, 10, 11])

In Bohr-Einstein debates (refs. [2, 5]), Einstein's standing-point is on the side of the realistic view in Figure 1. On the other hand, we think that Bohr's standing point is on the side of the linguistic view in Figure 1 (though N. Bohr might believe that the Copenhagen interpretation (proposed by his school) belongs to physics). In this paper, contrary to Bell's spirit (which inherits Einstein's spirit), we try to discuss Bell's inequality (refs. [1, 17]) in quantum language (i.e., quantum theory with the linguistic Copenhagen interpretation). And we clarify that whether or not Bell's inequality holds does not depend on whether classical systems or quantum systems (in Section 3), but depend on whether a kind of simultaneous measurements exist or not (in Section 2). And further we assert that our argument (based on the linguistic Copenhagen interpretation) should be regarded as a scientific representation of Bell's philosophical argument (based on Einstein's spirit).

1.2 Quantum language (=measurement theory); Mathematical preparations

Although quantum language has two formulations (i.e., C^* -algebraic formulation and C^* -algebraic formulation), in this paper we devote ourselves to the C^* -algebraic formulation (cf. Remark 1 later).

As a mathematical generalization of quantum mechanics, quantum language is constructed in a certain C^* -algebra \mathcal{A} (i.e., a norm closed subalgebra in $B(H)$ (i.e., an operator algebra composed of all bounded linear operators on a Hilbert space H with the norm $\|F\|_{B(H)} = \sup_{\|u\|_H=1} \|Fu\|_H$), cf. [16]) as follows:

$$(B) \quad \boxed{\text{Quantum language}}_{\text{(language)}} \stackrel{(\text{=measurement theory})}{=} \boxed{\text{Measurement}}_{\text{(Axiom 1)}} + \boxed{\text{Causality}}_{\text{(Axiom 2)}} + \boxed{\text{Linguistic (Copenhagen) interpretation}}_{\text{(how to use Axioms 1 and 2)}}$$

Note that this theory (B) is not physics but a kind of language based on “the mechanical world view”.

When $\mathcal{A} = B_c(H)$, the C^* -algebra composed of all compact operators on a Hilbert space H , the (B) is called quantum measurement theory (or, quantum system theory), which can be regarded as the linguistic turn of quantum mechanics. Also, when \mathcal{A} is commutative (that is, when \mathcal{A} is characterized by $C_0(\Omega)$, the C^* -algebra composed of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space Ω (cf. [16])), the (B) is called classical measurement theory. Thus, we have the following classification:

$$(C) \quad \text{measurement theory} \stackrel{(\text{= quantum language})}{=} \begin{cases} \text{quantum measurement theory} \\ \quad \text{(when } \mathcal{A} = B_c(H)) \\ \text{classical measurement theory} \\ \quad \text{(when } \mathcal{A} = C_0(\Omega)) \end{cases}$$

That is, this theory covers several conventional system theories (i.e., statistics, dynamical system theory, quantum system theory).

Now we shall explain the measurement theory (B). Let $\mathcal{A}(\subseteq B(H))$ be a C^* -algebra, and let \mathcal{A}^* be the dual Banach space of \mathcal{A} . That is, $\mathcal{A}^* = \{\rho \mid \rho \text{ is a continuous linear functional on } \mathcal{A}\}$, and the norm $\|\rho\|_{\mathcal{A}^*}$ is defined by $\sup\{|\rho(F)| \mid F \in \mathcal{A} \text{ such that } \|F\|_{\mathcal{A}} (= \|F\|_{B(H)}) \leq 1\}$. Define the *mixed state* $\rho \in \mathcal{A}^*$ such that $\|\rho\|_{\mathcal{A}^*} = 1$ and $\rho(F) \geq 0$ for all $F \in \mathcal{A}$ such that $F \geq 0$. And define the mixed state space $\mathfrak{S}^m(\mathcal{A}^*)$ such that

$$\mathfrak{S}^m(\mathcal{A}^*) = \{\rho \in \mathcal{A}^* \mid \rho \text{ is a mixed state}\}.$$

A mixed state $\rho \in \mathfrak{S}^m(\mathcal{A}^*)$ is called a *pure state* if it satisfies that “ $\rho = \theta\rho_1 + (1 - \theta)\rho_2$ for some $\rho_1, \rho_2 \in \mathfrak{S}^m(\mathcal{A}^*)$ and $0 < \theta < 1$ ” implies “ $\rho = \rho_1 = \rho_2$ ”. Put

$$\mathfrak{S}^p(\mathcal{A}^*) = \{\rho \in \mathfrak{S}^m(\mathcal{A}^*) \mid \rho \text{ is a pure state}\}, \quad (1)$$

which is called a *state space*. It is well known (cf. [16]) that $\mathfrak{S}^p(B_c(H)^*) = \{|u\rangle\langle u| \text{ (i.e., the Dirac notation) } | \|u\|_H = 1\}$, and $\mathfrak{S}^p(C_0(\Omega)^*) = \{\delta_{\omega_0} \mid \delta_{\omega_0} \text{ is a point measure at } \omega_0 \in \Omega\}$, where $\int_{\Omega} f(\omega)\delta_{\omega_0}(d\omega) = f(\omega_0)$ ($\forall f \in C_0(\Omega)$). The latter implies that $\mathfrak{S}^p(C_0(\Omega)^*)$ can be also identified with Ω such as $\mathfrak{S}^p(C_0(\Omega)^*) \ni \delta_{\omega_0} \leftrightarrow \omega_0 \in \Omega$.

Define the C^* -algebra $\widehat{\mathcal{A}}$ by the smallest C^* -algebra such that $\mathcal{A} \cup \{I\} \subseteq \widehat{\mathcal{A}} \subseteq B(H)$, where I is the identity in $B(H)$. Note that $\mathcal{A} = \widehat{\mathcal{A}}$ holds if $I \in \mathcal{A}$.

According to the noted idea (cf. [4]) in quantum mechanics, an *observable* $\mathcal{O} := (X, \mathcal{F}, F)$ in \mathcal{A} (or precisely, $\widehat{\mathcal{A}}$) is defined as follows:

- (i) [Field] X is a set, $\mathcal{F}(\subseteq \mathcal{P}(X))$, the power set of X is a field of X , that is, “ $\Xi_1, \Xi_2 \in \mathcal{F} \Rightarrow \Xi_1 \cup \Xi_2 \in \mathcal{F}$ ”, “ $\Xi \in \mathcal{F} \Rightarrow X \setminus \Xi \in \mathcal{F}$ ”.
- (ii) [Finite additivity] F is a mapping from \mathcal{F} to $\widehat{\mathcal{A}}$ satisfying: (a): for every $\Xi \in \mathcal{F}$, $F(\Xi)$ is a non-negative element in \mathcal{A} such that $0 \leq F(\Xi) \leq I$, (b): $F(\emptyset) = 0$ and $F(X) = I$, where 0 and I is the 0-element and the identity in $\widehat{\mathcal{A}}$ respectively. (c): for any finite decomposition $\{\Xi_1, \Xi_2, \dots, \Xi_n, \dots, \Xi_N\}$ of Ξ (i.e., $\Xi, \Xi_n \in \mathcal{F}$ ($n = 1, 2, 3, \dots, N$), $\cup_{n=1}^N \Xi_n = \Xi$, $\Xi_i \cap \Xi_j = \emptyset$ ($i \neq j$)), it holds that $F(\Xi) = \sum_{n=1}^N F(\Xi_n)$.

Remark 1 Quantum language has two formulations (i.e., the C^* -algebraic formulation and the W^* -algebraic formulation). In the W^* -algebraic formulation, the W^* -algebra $\overline{\mathcal{A}}$ such that $\mathcal{A} \subseteq \overline{\mathcal{A}} \subseteq B(H)$ plays an important role. From the mathematical point of view, the W^* -algebraic formulation may be superiority to the C^* -algebraic formulation. That is, in the above (ii), the countable additivity (i.e., $F(\Xi) = \lim_{N \rightarrow \infty} \sum_{n=1}^N F(\Xi_n)$) is naturally discussed in the W^* -algebraic formulation. However, in this preprint, we devote ourselves to the C^* -algebraic formulation, which is handy. For the W^* -algebraic version of this preprint (i.e. Bell’s inequality), see ref. [15], in which the mathematical exactness is sufficiently satisfied.

1.3 Axiom 1(Measurement), Axiom 2(Causality)

With any *system* S , a C^* -algebra $\mathcal{A}(\subseteq B(H))$ can be associated in which the measurement theory (B) of that system can be formulated. A *state* of the system S is represented by an element $\rho(\in \mathfrak{S}^p(\mathcal{A}^*))$ and an *observable* is represented by an observable $\mathcal{O} := (X, \mathcal{F}, F)$ in $\widehat{\mathcal{A}}$. Also, the *measurement of the observable* \mathcal{O} for the system S with the state ρ is denoted by $M_{\mathcal{A}}(\mathcal{O}, S_{[\rho]})$ (or more precisely, $M_{\mathcal{A}}(\mathcal{O} := (X, \mathcal{F}, F), S_{[\rho]})$). An observer can obtain a measured value $x (\in X)$ by the measurement $M_{\mathcal{A}}(\mathcal{O}, S_{[\rho]})$.

The Axiom 1 presented below is a kind of mathematical generalization of Born’s probabilistic interpretation of quantum mechanics (A). And thus, it is a statement without reality.

Axiom 1 [Measurement] *The probability that a measured value $x (\in X)$ obtained by the measurement $M_{\mathcal{A}}(\mathcal{O} := (X, \mathcal{F}, F), S_{[\rho_0]})$ belongs to a set $\Xi(\in \mathcal{F})$ is given by $\rho_0(F(\Xi))$.*

Next, we explain Axiom 2 in (B). Let $\mathcal{A}_1(\subseteq B(H_1))$ and $\mathcal{A}_2(\subseteq B(H_2))$ be C^* -algebras. A continuous linear operator $\Phi_{1,2} : \mathcal{A}_2 \rightarrow \mathcal{A}_1$ (and, $\widehat{\mathcal{A}}_2 \rightarrow \widehat{\mathcal{A}}_1$) is called a *Markov operator*, if it satisfies that (i): $\Phi_{1,2}(F_2) \geq 0$ for any non-negative element F_2 in \mathcal{A}_2 , (ii): $\Phi_{1,2}(I_2) = I_1$, where I_k is the identity in $\widehat{\mathcal{A}}_k$, ($k = 1, 2$). Here note that, for any observable $\mathcal{O}_2 := (X, \mathcal{F}, F_2)$ in $\widehat{\mathcal{A}}_2$, the $(X, \mathcal{F}, \Phi_{1,2}F_2)$ is an observable in $\widehat{\mathcal{A}}_1$, which is denoted by $\Phi_{1,2}\mathcal{O}_2$. Also, the dual operator $\Phi_{1,2}^* : \mathcal{A}_1^* \rightarrow \mathcal{A}_2^*$ clearly satisfies that $\Phi_{1,2}^*(\mathfrak{S}^p(\mathcal{A}_1^*)) \subseteq \mathfrak{S}^m(\mathcal{A}_2^*)$.

Now Axiom 2 in the measurement theory (B) is presented as follows:

Axiom 2 [Causality] *Let $t_1 \leq t_2$. The causality is represented by a Markov relation $\Phi_{t_1, t_2} : \mathcal{A}_{t_2} \rightarrow \mathcal{A}_{t_1}$.*

1.4 The linguistic interpretation (= the manual to use Axioms 1 and 2)

In the above, Axioms 1 and 2 are kinds of spells, (i.e., incantation, magic words, metaphysical statements), and thus, it is nonsense to verify them experimentally. Therefore, what we should do is not “to understand” but “to use”. After learning Axioms 1 and 2 by rote, we have to improve how to use them through trial and error.

We can do well even if we do not know the linguistic interpretation. However, it is better to know the linguistic interpretation (= the manual to use Axioms 1 and 2), if we would like to make progress quantum language early.

The essence of the manual is as follows:

(D) **Only one measurement is permitted.** And thus, the state after a measurement is meaningless since it can not be measured any longer. Thus, the collapse of the wavefunction is prohibited (cf. [12]). We are not concerned with anything after measurement. That is, any statement including the phrase “*after the measurement*” is wrong. Also, the causality should be assumed only in the side of system, however, a state never moves. Thus, the Heisenberg picture should be adopted, and thus, the Schrödinger picture should be prohibited.

and so on. For details, see [13].

1.5 Simultaneous measurement, parallel measurement

Definition 1. [Simultaneous observable, Simultaneous measurement] *Let $\mathcal{A}(\subseteq B(H))$ be a C^* -algebra. Consider observables $\mathbf{O}_k = (X_k, \mathcal{F}_k, F_k)$ ($k = 1, 2, \dots, K$) in $\hat{\mathcal{A}}$. Let $(\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k)$ be the product measurable space, i.e., the product space $\times_{k=1}^K X_k$ and the product field $\boxtimes_{k=1}^K \mathcal{F}_k$, which is defined by the smallest field that contains a family $\{\times_{k=1}^K \Xi_k \mid \Xi_k \in \mathcal{F}_k, k = 1, 2, \dots, K\}$. An observable $\mathbf{O} = \times_{k=1}^K \mathbf{O}_k = (\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k, F)$ in $\hat{\mathcal{A}}$ is called the simultaneous observable of \mathbf{O}_k ($k = 1, 2, \dots, K$), if it holds that*

$$\times_{k=1}^K F_k(\Xi_k) = F(\times_{k=1}^K \Xi_k) \quad (\forall \Xi_k \in \mathcal{F}_k) \quad (2)$$

Also, the measurement $M_{\mathcal{A}}(\mathbf{O}, S_{[\rho_0]})$ is called a simultaneous measurement of measurements $M_{\mathcal{A}}(\mathbf{O}_k, S_{[\rho_0]})$ ($k = 1, 2, \dots, K$). Note that a simultaneous observable $\mathbf{O} = (\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k, F)$ in $\hat{\mathcal{A}}$ always exists if observables \mathbf{O}_k ($k = 1, 2, \dots, K$) commute, i.e.,

$$F_k(\Xi_k)F_l(\Xi_l) = F_l(\Xi_l)F_k(\Xi_k) \quad (\forall \Xi_k \in \mathcal{F}_k, \forall \Xi_l \in \mathcal{F}_l, k \neq l) \quad (3)$$

Definition 2. [Parallel observable, Parallel measurement] *For each $k = 1, 2, \dots, K$, consider a measurement $M_{\mathcal{A}_k}(\mathbf{O}_k := (X_k, \mathcal{F}_k, F_k), S_{[\rho_k]})$. We consider the spatial tensor C^* -algebra $\otimes_{k=1}^K \hat{\mathcal{A}}_k(\subseteq B(\otimes_{k=1}^K H_k))$, and consider the product measurable space $(\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k)$. Consider the observable $\otimes_{k=1}^K \mathbf{O}_k = (\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k, \tilde{F})$ in $\otimes_{k=1}^K \hat{\mathcal{A}}_k$ such that*

$$\tilde{F}(\times_{k=1}^K \Xi_k) = \otimes_{k=1}^K F_k(\Xi_k) \quad (\forall \Xi_k \in \mathcal{F}_k, k = 1, 2, \dots, K).$$

which is called the parallel observable of $\mathbf{O}_k := (X_k, \mathcal{F}_k, F_k)$ ($k = 1, 2, \dots, K$). And let $\otimes_{k=1}^K \rho_k \in \mathfrak{S}^p((\otimes_{k=1}^K \mathcal{A}_k)^)$. Then the measurement $M_{\otimes_{k=1}^K \mathcal{A}_k}(\otimes_{k=1}^K \mathbf{O}_k = (\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k, \otimes_{k=1}^K F_k), S_{[\otimes_{k=1}^K \rho_k]})$ (which is also denoted by $\otimes_{k=1}^K M_{\mathcal{A}_k}(\mathbf{O}_k, S_{[\rho_k]})$) is called a parallel measurement of $M_{\mathcal{A}_k}(\mathbf{O}_k = (X_k, \mathcal{F}_k, F_k), S_{[\rho_k]})$ ($k = 1, 2, \dots, K$). Note that the parallel measurement always exists uniquely.*

2 Bell's inequality always holds in classical and quantum systems

2.1 Our assertion about Bell's inequality

In this paper, I assert that Bell's inequality should be studied in the framework of quantum theory (i.e., quantum theory with the linguistic Copenhagen interpretation). Let us start from the following definition, which is a slight modification of the simultaneous observable in Definition 1.

Definition 3. [Combined observable] *Let $\mathcal{A}(\subseteq B(H))$ be a C^* -algebra. Put $X = \{-1, 1\}$. Consider four observables: $\mathbf{O}_{13} = (X^2, \mathcal{P}(X^2), F_{13})$, $\mathbf{O}_{14} = (X^2, \mathcal{P}(X^2), F_{14})$, $\mathbf{O}_{23} = (X^2, \mathcal{P}(X^2), F_{23})$, $\mathbf{O}_{24} = (X^2, \mathcal{P}(X^2), F_{24})$ in $\hat{\mathcal{A}}$. The four observables are said to be combinable if there exists an observable $\mathbf{O} = (X^4, \mathcal{P}(X^4), F)$ in $\hat{\mathcal{A}}$ such that*

$$\begin{aligned} F_{13}(\{(x_1, x_3)\}) &= F(\{x_1\} \times X \times \{x_3\} \times X), & F_{14}(\{(x_1, x_4)\}) &= F(\{x_1\} \times X \times X \times \{x_4\}) \\ F_{23}(\{(x_2, x_3)\}) &= F(X \times \{x_2\} \times \{x_3\} \times X), & F_{24}(\{(x_2, x_4)\}) &= F(X \times \{x_2\} \times X \times \{x_4\}) \end{aligned} \quad (4)$$

for any $(x_1, x_2, x_3, x_4) \in X^4$. The observable \mathbf{O} is said to be a combined observable of \mathbf{O}_{ij} ($i = 1, 2, j = 3, 4$). Note that the \mathbf{O} is regarded as a kind of simultaneous observable of \mathbf{O}_{ij} ($i = 1, 2, j = 3, 4$). Also, the measurement $\mathbf{M}_{\mathcal{A}}(\mathbf{O} = (X^4, \mathcal{P}(X^4), F), S_{[\rho_0]})$ is called the combined measurement of $\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{13}, S_{[\rho_0]})$, $\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{14}, S_{[\rho_0]})$, $\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{23}, S_{[\rho_0]})$ and $\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{24}, S_{[\rho_0]})$.

The following theorem is all of our insistence concerning Bell's inequality. We assert that this is the true Bell's inequality.

Theorem 4. [Bell's inequality in quantum language] *Let \mathcal{A} be a C^* -algebra. Put $X = \{-1, 1\}$. Fix the pure state $\rho_0 (\in \mathfrak{S}^p(\mathcal{A}^*))$. And consider the four measurements $\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{13} = (X^2, \mathcal{P}(X^2), F_{13}), S_{[\rho_0]})$, $\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{14} = (X^2, \mathcal{P}(X^2), F_{14}), S_{[\rho_0]})$, $\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{23} = (X^2, \mathcal{P}(X^2), F_{23}), S_{[\rho_0]})$ and $\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{24} = (X^2, \mathcal{P}(X^2), F_{24}), S_{[\rho_0]})$. Or equivalently, consider the parallel measurement $\otimes_{i=1,2,j=3,4} \mathbf{M}_{\mathcal{A}}(\mathbf{O}_{ij} = (X^2, \mathcal{P}(X^2), F_{ij}), S_{[\rho_0]})$. Define four correlation functions ($i = 1, 2, j = 3, 4$),*

$$R_{ij} = \sum_{(u,v) \in X \times X} u \cdot v \rho_0(F_{ij}(\{(u,v)\}))$$

Assume that four observables $\mathbf{O}_{13} = (X^2, \mathcal{P}(X^2), F_{13})$, $\mathbf{O}_{14} = (X^2, \mathcal{P}(X^2), F_{14})$, $\mathbf{O}_{23} = (X^2, \mathcal{P}(X^2), F_{23})$ and $\mathbf{O}_{24} = (X^2, \mathcal{P}(X^2), F_{24})$ are combinable, that is, we have the combined observable $\mathbf{O} = (X^4, \mathcal{P}(X^4), F)$ in \mathcal{A} such that it satisfies (4). Then we have a kind of simultaneous measurement $\mathbf{M}_{\mathcal{A}}(\mathbf{O} = (X^4, \mathcal{P}(X^4), F), S_{[\rho_0]})$ of $\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{13}, S_{[\rho_0]})$, $\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{14}, S_{[\rho_0]})$, $\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{23}, S_{[\rho_0]})$ and $\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{24}, S_{[\rho_0]})$. And further, we have Bell's inequality in quantum language as follows.

$$|R_{13} - R_{14}| + |R_{23} + R_{24}| \leq 2 \quad (5)$$

Proof. Clearly we see, $i = 1, 2, j = 3, 4$,

$$R_{ij} = \sum_{(x_1, x_2, x_3, x_4) \in X \times X \times X \times X} x_i \cdot x_j \rho_0(F(\{(x_1, x_2, x_3, x_4)\})) \quad (6)$$

(for example, $R_{13} = \sum_{(x_1, x_2, x_3, x_4) \in X \times X \times X \times X} x_1 \cdot x_3 \rho_0(F(\{(x_1, x_2, x_3, x_4)\}))$). Therefore, we see that

$$\begin{aligned} & |R_{13} - R_{14}| + |R_{23} + R_{24}| \\ &= \sum_{(x_1, x_2, x_3, x_4) \in X \times X \times X \times X} \left[|x_1 \cdot x_3 - x_1 \cdot x_4| + |x_2 \cdot x_3 + x_2 \cdot x_4| \right] \rho_0(F(\{(x_1, x_2, x_3, x_4)\})) \\ &= \sum_{(x_1, x_2, x_3, x_4) \in X \times X \times X \times X} \left[|x_3 - x_4| + |x_3 + x_4| \right] \rho_0(F(\{(x_1, x_2, x_3, x_4)\})) \leq 2 \end{aligned}$$

This completes the proof. □

As the corollary of this theorem, we have the followings:

Corollary 5. *Consider the parallel measurement $\otimes_{i=1,2,j=3,4} \mathbf{M}_{\mathcal{A}}(\mathbf{O}_{ij} = (X^2, \mathcal{P}(X^2), F_{ij}), S_{[\rho_0]})$ as in Theorem 4. Let*

$$\begin{aligned} x &= ((x_{13}^1, x_{13}^2), (x_{14}^1, x_{14}^2), (x_{23}^1, x_{23}^2), (x_{24}^1, x_{24}^2)) \\ &\in \times_{i,j=1,2} X^2 (\equiv \{-1, 1\}^8) \end{aligned}$$

be a measured value of the parallel measurement $\otimes_{i=1,2,j=3,4} \mathbf{M}_{\mathcal{A}}(\mathbf{O}_{ij} = (X^2, \mathcal{P}(X^2), F_{ij}), S_{[\rho_0]})$. Let N be sufficiently large natural number. Consider N -parallel measurement $\bigotimes_{n=1}^N [\otimes_{i=1,2,j=2,3} \mathbf{M}_{\mathcal{A}}(\mathbf{O}_{ij} := (X^2, \mathcal{P}(X^2), F_{ij}), S_{[\rho_0]})]$. Let $\{x^n\}_{n=1}^N$ be the measured value. That is,

$$\{x^n\}_{n=1}^N = \begin{bmatrix} ((x_{13}^{1,1}, x_{13}^{2,1}), (x_{14}^{1,1}, x_{14}^{2,1}), (x_{23}^{1,1}, x_{23}^{2,1}), (x_{24}^{1,1}, x_{24}^{2,1})) \\ ((x_{13}^{1,2}, x_{13}^{2,2}), (x_{14}^{1,2}, x_{14}^{2,2}), (x_{23}^{1,2}, x_{23}^{2,2}), (x_{24}^{1,2}, x_{24}^{2,2})) \\ \vdots \\ ((x_{13}^{1,N}, x_{13}^{2,N}), (x_{14}^{1,N}, x_{14}^{2,N}), (x_{23}^{1,N}, x_{23}^{2,N}), (x_{24}^{1,N}, x_{24}^{2,N})) \end{bmatrix} \in (X^8)^N$$

Here, note that the law of large numbers says:

$$R_{ij} \approx \frac{1}{N} \sum_{n=1}^N x_{ij}^{1,n} x_{ij}^{2,n} \quad (i = 1, 2, j = 3, 4)$$

Then, it holds, by the formula (5), that

$$\left| \sum_{n=1}^N \frac{x_{13}^{1,n} x_{13}^{2,n}}{N} - \sum_{n=1}^N \frac{x_{14}^{1,n} x_{14}^{2,n}}{N} \right| + \left| \sum_{n=1}^N \frac{x_{23}^{1,n} x_{23}^{2,n}}{N} + \sum_{n=1}^N \frac{x_{24}^{1,n} x_{24}^{2,n}}{N} \right| \leq 2, \quad (7)$$

which is also called Bell's inequality in quantum language.

Remark 2 [The conventional Bell's inequality (cf. [17])] The mathematical Bell's inequality is as follows: Let (Θ, \mathcal{B}, P) be a probability space. Let $(f_1, f_2, f_3, f_4) : \Theta \rightarrow X^4 (\equiv \{-1, 1\}^4)$ be a measurable functions. Define the correlation functions $\tilde{R}_{ij} (i = 1, 2, j = 3, 4)$ by $\int_{\Theta} f_i(\theta) f_j(\theta) P(d\theta)$. Then, the following mathematical Bell's inequality holds:

$$|\tilde{R}_{13} - \tilde{R}_{14}| + |\tilde{R}_{23} - \tilde{R}_{24}| \leq 2 \quad (8)$$

(E) This is easily proved as follows.

$$\text{"the left-hand side of the above (8)} \leq \int_{\Theta} |f_3(\theta) + f_4(\theta)| P(d\theta) + \int_{\Theta} |f_3(\theta) - f_4(\theta)| P(d\theta) \leq 2$$

This completes the proof.

Recall Theorem 4 (Bell's inequality in quantum language), in which we have, by the combinable condition, the probability space $(X^4, \mathcal{P}(X^4), \rho_0(F(\cdot)))$. Therefore the proof of Theorem 4 and the above proof (the conventional Bell's inequality) are, from the mathematical point of view, the same.

3 "Bell's inequality" is violated in classical systems as well as quantum systems

In the previous section, we show that

(F₁) Under the combinable condition (cf. Definition 3), Bell's inequality (5) (or, (7)) holds in both classical systems and quantum systems.

Or, equivalently,

(F₂) If Bell's inequality (5) (or (7)) is violated, then the combined observable does not exist, and thus, we cannot obtain the measured value (by the measurement of the combined observable).

This makes us expect that

(G) Bell's inequality (5) (or (7)) is violated in classical systems as well as quantum systems without the combinable condition.

This (G) was already shown in my previous paper [7]. However, I got a lot of questions concerning (G) from the readers. Thus, in this section, we again explain the (G) precisely.

3.1 Bell's thought experiment

In order to show the (G), three steps ([Step:I] ~[Step:III]) are prepared in what follows.

[Step: I]. Put $X = \{-1, 1\}$. Define complex numbers $a_k (= \alpha_k + \beta_k \sqrt{-1} \in \mathbb{C} : \text{the complex field})$ ($k = 1, 2, 3, 4$) such that $|a_k| = 1$. Define the probability space $(X^2, \mathcal{P}(X^2), \nu_{a_i a_j})$ such that ($i = 1, 2, j = 3, 4$)

$$\begin{aligned} \nu_{a_i a_j}(\{(1, 1)\}) &= \nu_{a_i a_j}(\{(-1, -1)\}) = (1 - \alpha_i \alpha_j - \beta_i \beta_j)/4 \\ \nu_{a_i a_j}(\{(-1, 1)\}) &= \nu_{a_i a_j}(\{(1, -1)\}) = (1 + \alpha_i \alpha_j + \beta_i \beta_j)/4 \end{aligned} \quad (9)$$

The correlation $R(a_i, a_j)$ ($i = 1, 2, j = 3, 4$) is defined as follows:

$$R(a_i, a_j) \equiv \sum_{(x_1, x_2) \in X \times X} x_1 \cdot x_2 \nu_{a_i a_j}(\{(x_1, x_2)\}) = -\alpha_i \alpha_j - \beta_i \beta_j \quad (10)$$

Now we have the following problem:

(H) Find a measurement $\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{a_i a_j} := (X^2, \mathcal{P}(X^2), F_{a_i a_j}), S_{[\rho_0]})$ ($i = 1, 2, j = 3, 4$) in a C^* -algebra \mathcal{A} such that

$$\rho_0(F_{a_i a_j}(\Xi)) = \nu_{a_i a_j}(\Xi) \quad (\forall \Xi \in \mathcal{P}(X^2)) \quad (11)$$

and

$$\begin{aligned} F_{a_1 a_3}(\{x_1\} \times X) &= F_{a_1 a_4}(\{x_1\} \times X) & F_{a_1 a_3}(X \times \{x_3\}) &= F_{a_2 a_3}(X \times \{x_3\}) \\ F_{a_2 a_3}(\{x_2\} \times X) &= F_{a_2 a_4}(\{x_2\} \times X) & F_{a_1 a_4}(X \times \{x_4\}) &= F_{a_2 a_4}(X \times \{x_4\}) \\ & & (\forall x_k \in X (\equiv \{-1, 1\}), k = 1, 2, 3, 4) & \end{aligned}$$

[Step: II].

Let us answer this problem (H) in the two cases (i.e., classical case and quantum case), that is,

$$\bullet \left\{ \begin{array}{l} \text{(i):the case of quantum systems: } [\mathcal{A} = B(\mathbb{C}^2 \otimes \mathbb{C}^2)] \\ \text{(ii):the case of classical systems: } [\mathcal{A} = C_0(\Omega \times \Omega)] \end{array} \right.$$

(i):the case of quantum system: $[\mathcal{A} = B(\mathbb{C}^2) \otimes B(\mathbb{C}^2) = B(\mathbb{C}^2 \otimes \mathbb{C}^2)]$

Put

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (\in \mathbb{C}^2).$$

For each a_k ($k = 1, 2, 3, 4$), define the observable $O_{a_k} \equiv (X, \mathcal{P}(X), G_{a_k})$ in $B(\mathbb{C}^2)$ such that

$$G_{a_k}(\{1\}) = \frac{1}{2} \begin{bmatrix} 1 & \bar{a}_k \\ a_k & 1 \end{bmatrix}, \quad G_{a_k}(\{-1\}) = \frac{1}{2} \begin{bmatrix} 1 & -\bar{a}_k \\ -a_k & 1 \end{bmatrix}.$$

where $\bar{a}_k = \alpha_k - \beta_k \sqrt{-1}$. Then, we have four observable:

$$\widehat{O}_{a_i} = (X, \mathcal{P}(X), G_{a_i} \otimes I), \quad \widehat{O}_{a_j} = (X, \mathcal{P}(X), I \otimes G_{a_j}) \quad (i = 1, 2, j = 3, 4) \quad (12)$$

and further,

$$O_{a_i a_j} = (X^2, \mathcal{P}(X^2), F_{a_i a_j} := G_{a_i} \otimes G_{a_j}) \quad (i = 1, 2, j = 3, 4) \quad (13)$$

in $B(\mathbb{C}^2 \otimes \mathbb{C}^2)$, where it should be noted that $F_{a_i a_j}$ is separated by G_{a_i} and G_{a_j} .

Further define the singlet state $\rho_0 = |\psi_s\rangle\langle\psi_s|$ ($\in \mathfrak{S}^p(B(\mathbb{C}^2 \otimes \mathbb{C}^2)^*)$), where

$$\psi_s = (e_1 \otimes e_2 - e_2 \otimes e_1) / \sqrt{2}$$

Thus we have the measurement $M_{B(\mathbb{C}^2 \otimes \mathbb{C}^2)}(O_{a_i a_j}, S_{[\rho_0]})$ in $B(\mathbb{C}^2 \otimes \mathbb{C}^2)$ ($i = 1, 2, j = 3, 4$). The followings are clear: for each $(x_1, x_2) \in X^2 (\equiv \{-1, 1\}^2)$,

$$\rho_0(F_{a_i a_j}(\{(x_1, x_2)\})) = \langle\psi_s, (G_{a_i}(\{x_1\}) \otimes G_{a_j}(\{x_2\}))\psi_s\rangle = \nu_{a_i a_j}(\{(x_1, x_2)\}) \quad (i = 1, 2, j = 3, 4) \quad (14)$$

For example, we easily see:

$$\begin{aligned} \rho_0(F_{a_i b_j}(\{(1, 1)\})) &= \langle\psi_s, (G_{a_i}(\{1\}) \otimes G_{a_j}(\{1\}))\psi_s\rangle \\ &= \frac{1}{8} \langle (e_1 \otimes e_2 - e_2 \otimes e_1), \left(\begin{bmatrix} 1 & \bar{a}_i \\ a_i & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \bar{a}_j \\ a_j & 1 \end{bmatrix} \right) (e_1 \otimes e_2 - e_2 \otimes e_1) \rangle \\ &= \frac{1}{8} \langle \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} 1 & \bar{a}_i \\ a_i & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \bar{a}_j \\ a_j & 1 \end{bmatrix} \right) \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \rangle \\ &= \frac{1}{8} \langle \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} 1 \\ a_i \end{bmatrix} \otimes \begin{bmatrix} \bar{a}_j \\ 1 \end{bmatrix} - \begin{bmatrix} \bar{a}_i \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ a_j \end{bmatrix} \right) \rangle \\ &= \frac{1}{8} (2 - a\bar{a}_j - \bar{a}_i a_j) = (1 - \alpha_i \alpha_j - \beta_i \beta_j) / 4 = \nu_{a_i a_j}(\{(1, 1)\}). \end{aligned}$$

Therefore, the measurement $M_{B(\mathbb{C}^2 \otimes \mathbb{C}^2)}(O_{a_i a_j}, S_{[\rho_0]})$ satisfies the condition (H).

(ii): the case of classical systems: $[\mathcal{A} = C_0(\Omega) \otimes C_0(\Omega) = C_0(\Omega \times \Omega)]$

Put $\omega_0 (= (\omega'_0, \omega''_0)) \in \Omega \times \Omega$, $\rho_0 = \delta_{\omega_0}$ ($\in \mathfrak{S}^p(C_0(\Omega \times \Omega)^*)$, i.e., the point measure at ω_0). Define the observable $O_{a_i a_j} := (X^2, \mathcal{P}(X^2), F_{a_i a_j})$ in $\widehat{C}_0(\Omega \times \Omega)$ such that

$$[F_{a_i a_j}(\{(x_1, x_2)\})](\omega) = \nu_{a_i a_j}(\{(x_1, x_2)\}) \quad (\forall (x_1, x_2) \in X^2, i = 1, 2, j = 3, 4, \forall \omega \in \Omega \times \Omega)$$

Thus, we have four observables

$$O_{a_i a_j} = (X^2, \mathcal{P}(X^2), F_{a_i a_j}) \quad (i = 1, 2, j = 3, 4) \quad (15)$$

in $\widehat{C}_0(\Omega \times \Omega)$ (though the variables are not separable (cf. the formula (13)). Then, it is clear that the measurement $M_{C_0(\Omega \times \Omega)}(O_{a_i a_j}, S_{[\delta_{\omega_0}]})$ satisfies the condition (H).

(ii)': the case of classical systems: $[\mathcal{A} = C_0(\Omega) \otimes C_0(\Omega) = C_0(\Omega \times \Omega)]$

It is easy to show a lot of different answers from the above (ii). For example, as a slight generalization of (9), define the probability measure $\nu_{a_i a_j}^t$ ($0 \leq t \leq 1$) such that

$$\begin{aligned}\nu_{a_i a_j}^t(\{(1, 1)\}) &= \nu_{a_i a_j}^t(\{(-1, -1)\}) = (1 - t(\alpha_i \alpha_j + \beta_i \beta_j))/4 \\ \nu_{a_i a_j}^t(\{(-1, 1)\}) &= \nu_{a_i a_j}^t(\{(1, -1)\}) = (1 + t(\alpha_i \alpha_j + \beta_i \beta_j))/4\end{aligned}$$

And consider the real-valued continuous function $t(\in C_0(\Omega \times \Omega))$ such that $0 \leq t(\omega', \omega'') \leq 1$ ($\forall \omega = (\omega', \omega'') \in \Omega \times \Omega$). And assume that $t(\omega_0) = 1$ for some $\omega_0 = (\omega'_0, \omega''_0) \in \Omega \times \Omega$, $\rho_0 = \delta_{\omega_0}$ ($\in \mathfrak{S}^p(C_0(\Omega \times \Omega))^*$), i.e., the point measure at ω_0). Define the observable $\mathbf{O}_{a_i a_j} := (X^2, \mathcal{P}(X^2), F_{a_i a_j})$ in $\widehat{C}_0(\Omega \times \Omega)$ such that

$$[F_{a_i a_j}(\{(x_1, x_2)\})](\omega) = \nu_{a_i a_j}^{t(\omega)}(\{(x_1, x_2)\}) \quad (\forall (x_1, x_2) \in X^2, i = 1, 2, j = 3, 4, \forall \omega \in \Omega \times \Omega)$$

Thus, we have four observables

$$\mathbf{O}_{a_i a_j} = (X^2, \mathcal{P}(X^2), F_{a_i a_j}) \quad (i = 1, 2, j = 3, 4) \quad (16)$$

in $\widehat{C}_0(\Omega \times \Omega)$ (though the variables are not separable (cf. the formula (13)). Then, it is clear that the measurement $\mathbf{M}_{C_0(\Omega \times \Omega)}(\mathbf{O}_{a_i a_j}, S_{[\delta_{\omega_0}]})$ satisfies the condition (H).

[Step: III].

As defined by (9), consider four complex numbers $a_k (= \alpha_k + \beta_k \sqrt{-1}; k = 1, 2, 3, 4)$ such that $|a_k| = 1$. Thus we have four observables

$$\begin{aligned}\mathbf{O}_{a_1 a_3} &:= (X^2, \mathcal{P}(X^2), F_{a_1 a_3}), & \mathbf{O}_{a_1 a_4} &:= (X^2, \mathcal{P}(X^2), F_{a_1 a_4}), \\ \mathbf{O}_{a_2 a_3} &:= (X^2, \mathcal{P}(X^2), F_{a_2 a_3}), & \mathbf{O}_{a_2 a_4} &:= (X^2, \mathcal{P}(X^2), F_{a_2 a_4}),\end{aligned}$$

in $\widehat{\mathcal{A}}$. Thus, we have the parallel measurement $\otimes_{i=1,2,j=3,4} \mathbf{M}_{\mathcal{A}}(\mathbf{O}_{a_i a_j} := (X^2, \mathcal{P}(X^2), F_{a_i a_j}), S_{[\rho_0]})$ in $\otimes_{i=1,2,j=3,4} \mathcal{A}$.

Thus, putting

$$a_1 = \sqrt{-1}, \quad a_2 = 1, \quad a_3 = \frac{1 + \sqrt{-1}}{\sqrt{2}}, \quad a_4 = \frac{1 - \sqrt{-1}}{\sqrt{2}},$$

we see, by (10), that

$$|R(a_1, a_3) - R(a_1, a_4)| + |R(a_2, a_3) + R(a_2, a_4)| = 2\sqrt{2} \quad (17)$$

Further, assume that the measured value is $x(\in X^8)$. That is,

$$x = ((x_{13}^1, x_{13}^2), (x_{14}^1, x_{14}^2), (x_{23}^1, x_{23}^2), (x_{24}^1, x_{24}^2)) \in \prod_{i,j=1,2} X^2 (\equiv \{-1, 1\}^8)$$

Let N be sufficiently large natural number. Consider N -parallel measurement $\otimes_{n=1}^N [\otimes_{i=1,2,j=3,4} \mathbf{M}_{\mathcal{A}}(\mathbf{O}_{a_i a_j} := (X^2, \mathcal{P}(X^2), F_{a_i a_j}), S_{[\rho_0]})]$. Assume that its measured value is $\{x^n\}_{n=1}^N$. That is,

$$\{x^n\}_{n=1}^N = \begin{bmatrix} ((x_{13}^{1,1}, x_{13}^{2,1}), (x_{14}^{1,1}, x_{14}^{2,1}), (x_{23}^{1,1}, x_{23}^{2,1}), (x_{24}^{1,1}, x_{24}^{2,1})) \\ ((x_{13}^{1,2}, x_{13}^{2,2}), (x_{14}^{1,2}, x_{14}^{2,2}), (x_{23}^{1,2}, x_{23}^{2,2}), (x_{24}^{1,2}, x_{24}^{2,2})) \\ \vdots \\ ((x_{13}^{1,N}, x_{13}^{2,N}), (x_{14}^{1,N}, x_{14}^{2,N}), (x_{23}^{1,N}, x_{23}^{2,N}), (x_{24}^{1,N}, x_{24}^{2,N})) \end{bmatrix} \in \left(\prod_{i=1,2,j=3,4} X^2 \right)^N (\equiv \{-1, 1\}^{8N})$$

Then, the law of large numbers says that

$$R(a_i, a_j) \approx \frac{1}{N} \sum_{n=1}^N x_{ij}^{1,n} x_{ij}^{2,n} \quad (i = 1, 2, j = 3, 4)$$

This and the formula (17) say that

$$\left| \sum_{n=1}^N \frac{x_{13}^{1,n} x_{13}^{2,n}}{N} - \sum_{n=1}^N \frac{x_{14}^{1,n} x_{14}^{2,n}}{N} \right| + \left| \sum_{n=1}^N \frac{x_{23}^{1,n} x_{23}^{2,n}}{N} + \sum_{n=1}^N \frac{x_{24}^{1,n} x_{24}^{2,n}}{N} \right| \approx 2\sqrt{2} \quad (18)$$

Therefore, Bell's inequality (5) (or (7)) is violated in classical systems as well as quantum systems.

4 Conclusion

In this paper, contrary to Bell's spirit (which inherits Einstein's spirit), we try to discuss Bell's inequality in the framework of quantum theory with the linguistic Copenhagen interpretation of quantum mechanics. And we show Theorem 4 (Bell's inequality in quantum language), which says the statement (F₂), that is,

(I₁) (\equiv (F₂)): If Bell's inequality (5) (or (7)) is violated, then the combined observable does not exist, and thus, we cannot obtain the measured value (by the measurement of the combined observable).

Also, recall that Bell's original argument (based on Einstein's spirit) says, roughly speaking, that

(I₂) If the mathematical Bell's inequality (8) is violated in Bell's thought experiment (the quantum case of Section 3.1), then hidden variables do not exist.

It should be note that the (I₂) is a philosophical statement in Einstein's spirit, on the other hand, the (I₁) is a statement in scientific theory (i.e., quantum theory with the linguistic Copenhagen interpretation). It is sure that Bell's answer (I₂) is attractive philosophically, however, we believe in the scientific superiority of our answer (I₁). That is, we conclude that our (I₁) is a scientific representation of the philosophical (I₂). If so, we can, for the first time, understand Bell's inequality in science. That is, Theorem 4 is the true Bell's inequality.

We hope that our proposal will be examined from various points of view¹.

References

- [1] Bell, J.S. *On the Einstein-Podolsky-Rosen Paradox*, Physics 1, 195–200 (1966)
- [2] Bohr, N. *Can quantum-mechanical description of physical reality be considered complete ?*, Phys. Rev. (48) 696-702 (1935)
- [3] Born, M. *Zur Quantenmechanik der Stoßprozesse (Vorläufige Mitteilung)*, Z. Phys. (37) 863–867 1926
- [4] Davies, E. B. *Quantum Theory of Open Systems*, Academic Press, 1976
- [5] Einstein, A., Podolsky, B. and Rosen, N. *Can quantum-mechanical description of reality be considered completely ?* Phys. Rev. (47) 777–780 (1935)
- [6] Ishikawa,S., *Mathematical Foundations of Measurement Theory*, Keio University Press Inc. 335pages, 2006. (<http://www.keio-up.co.jp/kup/mfomt/>)
- [7] Ishikawa,S., *A New Interpretation of Quantum Mechanics*, Journal of quantum information science, Vol. 1, No. 2, 35-42, 2011 doi: 10.4236/jqis.2011.12005 (<http://www.scirp.org/journal/PaperInformation.aspx?paperID=7610>)

¹

For the further information of quantum language, see my home page: <http://www.math.keio.ac.jp/~ishikawa/indexe.html>

- [8] Ishikawa,S., *Quantum Mechanics and the Philosophy of Language: Reconsideration of traditional philosophies*, Journal of quantum information science, Vol. 2, No. 1, 2-9, 2012 doi: 10.4236/jqis.2012.21002 (<http://www.scirp.org/journal/PaperInformation.aspx?paperID=18194>)
- [9] Ishikawa,S., *A Measurement Theoretical Foundation of Statistics*, Applied Mathematics, Vol. 3, No. 3, 283-292, 2012 doi: 10.4236/am.2012.33044 (<http://www.scirp.org/journal/PaperInformation.aspx?paperID=18109>)
- [10] Ishikawa,S., *Monty Hall Problem and the Principle of Equal Probability in Measurement Theory*, Applied Mathematics, Vol. 3, No. 7, 788-794, 2012 doi:10.4236/am.2012.37117 (<http://www.scirp.org/journal/PaperInformation.aspx?PaperID=19884>)
- [11] Ishikawa,S., *Ergodic Hypothesis and Equilibrium Statistical Mechanics in the Quantum Mechanical World View*, World Journal of Mechanics, Vol. 2, No. 2, 125-130, 2012 doi:10.4236/wjm.2012.22014 (<http://file.scirp.org/Html/18861.html>)
- [12] Ishikawa,S., *Linguistic interpretation of quantum mechanics ; Projection Postulate*, Journal of quantum information science, Vol. 5, No.4 , 150-155, 2015, DOI: 10.4236/jqis.2015.54017 (<http://www.scirp.org/Journal/PaperInformation.aspx?PaperID=62464%20>)
- [13] Ishikawa,S., *Linguistic interpretation of quantum mechanics: Quantum language* Version 2, Research Report (Department of mathematics, Keio university), (KSTS-RR-16/001, 2016, 426 pages) (http://www.math.keio.ac.jp/academic/research_pdf/report/2016/16001.pdf)
This preprint is a draft of my book "*Linguistic Interpretation of Quantum Mechanics –Towards World-Description in Quantum Language –*" Shiho-Shuppan Publisher,(2016)
- [14] Ishikawa,S., *Final solution to mind-body problem by quantum language*, Journal of quantum information science, Vol. 7, No.2 , 48-56, 2017, DOI: 10.4236/jqis.2017.72005 (<http://www.scirp.org/Journal/PaperInformation.aspx?PaperID=76391>)
- [15] Ishikawa,S., *Bell's inequality is violated in classical systems as well as quantum systems*, (The W^* -algebraic version)(2017)
Preprint (<http://viXra.org/abs/1710.0067>)
- [16] Sakai, S., *C^* -algebras and W^* -algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete (Band 60), Springer-Verlag, (1971)
- [17] Selleri, F. *Die Debatte um die Quantentheorie*, Friedr. Vieweg&Sohn Verlagsgesellschaft MBH, Braunschweig (1983)
- [18] J. von Neumann, *Mathematical foundations of quantum mechanics*, Springer Verlag, Berlin (1932)
- [19] Yosida, K., "*Functional Analysis*," Springer-Verlag, 6th edition, 1980.

Department of Mathematics
Faculty of Science and Technology
Keio University

Research Report

2016

- [16/001] Shiro Ishikawa,
Linguistic interpretation of quantum mechanics: Quantum Language [Ver. 2],
KSTS/RR-16/001, January 8, 2016
- [16/002] Yuka Hashimoto, Takashi Nodera,
Inexact shift-invert Arnoldi method for evolution equations,
KSTS/RR-16/002, May 6, 2016
- [16/003] Yuka Hashimoto, Takashi Nodera,
A Note on Inexact Rational Krylov Method for Evolution Equations,
KSTS/RR-16/003, November 9, 2016
- [16/004] Sumiyuki Koizumi,
*On the theory of generalized Hilbert transforms (Chapter V: The spectre analysis
and synthesis on the N .Wiener class S)*,
KSTS/RR-16/004, November 25, 2016 (Second edition, August 24, 2017)
- [16/005] Shiro Ishikawa,
History of Western Philosophy from the quantum theoretical point of view,
KSTS/RR-16/005, December 6, 2016

2017

- [17/001] Yuka Hashimoto, Takashi Nodera,
Inexact Shift-invert Rational Krylov Method for Evolution Equations,
KSTS/RR-17/001, January 27, 2017 (Revised July 24, 2017)
- [17/002] Dai Togashi, Takashi Nodera,
Convergence analysis of the GKB-GCV algorithm,
KSTS/RR-17/002, March 27, 2017
- [17/003] Shiro Ishikawa,
Linguistic solution to the mind-body problem,
KSTS/RR-17/003, April 3, 2017
- [17/004] Shiro Ishikawa,
History of Western Philosophy from the quantum theoretical point of view; Version 2,
KSTS/RR-17/004, May 12, 2017
- [17/005] Sumiyuki Koizumi,
*On the theory of generalized Hilbert transforms (Chapter VI: The spectre analysis
and synthesis on the N .Wiener class S (2))*,
KSTS/RR-17/005, June 8, 2017 (Second edition, September 1, 2017)
- [17/006] Shiro Ishikawa,
Bell's inequality is violated in classical systems as well as quantum systems,
KSTS/RR-17/006, October 16, 2017