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On the theory of generalized Hilbert transforms
Chapter VI
The spectre analysis and synthesis on the N.Wiener class S (2)

by

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ABSTRACT

We shall continue the problem of spectrum of function of the N.Wiener class S after the preceding section 14 in this research report V and we shall prove that in the Theorem E, we need not always the hypothesis (D_λ) and present it as the Theorem E^*. We shall also treat the same problems as for Generalized Hilbert Transforms.

15 The Spectral Analysis and Synthesis on the N.Wiener class S.

We shall explain these circumstances for the sake of completeness as follows

15.1 Let us suppose that for function f of the class S_0, there exist the following limit

$$(C_\lambda) \qquad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\lambda x}dx = c_\lambda \quad (\forall real\ \lambda).$$

Then we have by the one-sided Wiener formula

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\lambda x}dx = \lim_{\varepsilon\to 0}\frac{1}{2\sqrt{2\pi}\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}du \qquad (\forall real\ \lambda),$$

We shall begine to define the class S_0 after Prof. N.Wiener[1].

Definition of the class S_0. In case f is measurable over $(-\infty,\infty)$ and integrable of its square modulus locally and exist the following limit

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)|^2dx,$$

we shall say that f belongs to the class S_0.

Let us introduce the generalized Fourier transforms (G.F.T.) after Prof. N.Wiener

$$s(u,f)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}f(x)\frac{e^{-iux}-1}{-ix}dx+\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\left[\int_{-A}^{-1}+\int_{1}^{A}\right]f(x)\frac{e^{-iux}}{-ix}dx.$$

Then we have

$$s(u+\varepsilon,f)-s(u-\varepsilon,f)=\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}f(x)\frac{2\sin x}{x}e^{-iux}dx$$

and

$$\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}du=\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\left(\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}f(x)\frac{2\sin x}{x}e^{-iux}dx\right)du.$$

Let us define the following formulas

$$F_A(u)=\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}f(x)\frac{2\sin x}{x}e^{-iux}dx$$

and

$$F(u)=\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}f(x)\frac{2\sin x}{x}e^{-iux}dx$$

respectively, then we have by the Plancherel theorem

$$\|F_A(u)-F(u)\|_{L^2}\to 0 \quad (A\to\infty). \quad (\forall real\lambda)$$

Since the strong convergence implies the weak convergence, we have

$$\int_{-\infty}^{\infty}F_A(u)\chi_{\lambda,\varepsilon}(u)du\to\int_{-\infty}^{\infty}F(u)\chi_{\lambda,\varepsilon}(u)du \quad (A\to\infty), \quad (\forall real\lambda)$$

where the $\chi_{\lambda,\varepsilon}(u)$ denotes the characteristic function of interval $(\lambda-\varepsilon,\lambda+\varepsilon)$ and this formula is written as follows

$$\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}F_A(u)du\to\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}F(u)du \quad (A\to\infty), \quad (\forall real\lambda).$$

Let us remark that this formula is also proved by the Schwartz inequality directly.

Now we have by the theorem of Fubini

$$\frac{1}{\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\left(\int_{-A}^{A}f(x)\frac{2\sin\varepsilon x}{x}e^{-iux}dx\right)du=\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}f(x)\frac{2\sin\varepsilon x}{x}\left(\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}e^{-iux}du\right)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\left(f(x)e^{-i\lambda x}\right)\left(\frac{2\sin\varepsilon x}{x}\right)^2dx\to\frac{4}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(f(x)e^{-i\lambda x}\right)\frac{\sin^2\varepsilon x}{x^2}dx$$

$(A\to\infty), \quad (\forall real\lambda).$

Therefore we have

$$\lim_{\varepsilon \to 0} \frac{1}{4\pi\varepsilon} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \{s(u+\varepsilon, f) - s(u-\varepsilon, f)\} du = \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \to 0} \frac{1}{\pi\varepsilon} \int_{-\infty}^{\infty} (f(x)e^{-i\lambda x}) \frac{\sin^2 \varepsilon x}{x^2} dx.$$

The one-sided Wiener formula: Let us suppose that $f(x)$ is measurable and integrable locally and $\frac{1}{2T}\int_{-T}^{T} \backslash f(x) | dx$ is bounded as for $T \to \infty$. Then we have

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x)e^{-i\lambda x} dx = \lim_{\varepsilon \to 0} \frac{1}{\pi\varepsilon} \int_{-\infty}^{\infty} (f(x)e^{-i\lambda x}) \frac{\sin^2 \varepsilon x}{x^2} dx$$

in the sense that if the limit of left hand side exist then the limit of right hand side also exist and their limiting values are equal

Let us remark that if $f(x)$ belongs to the class S_0, then the presupposed conditions of the one-sided Wiener formula are all satisfied. Then applying the one-sided Wiener formula we have

$$\mathbf{c}_\lambda = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x)e^{-i\lambda x} dx = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \{s(u+\varepsilon, f) - s(u-\varepsilon, f)\} du$$

$(\forall real \lambda)$.

15.2 On the Lemma E.

We have

Lemma E^* Let us suppose that $f(x)$ belongs to the class S and satisfies the hypothesis (C_λ) $(\forall real\ \lambda)$. Then we have

$$\lim_{\varepsilon \to 0} \frac{1}{4\pi\varepsilon} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} | s(u+\varepsilon, f) - s(u-\varepsilon, f) |^2 du = \frac{\sigma(\lambda+0,\varphi) - \sigma(\lambda-0,\varphi)}{\sqrt{2\pi}} \qquad (\forall real\ \lambda).$$

Proof. Let us define $\varphi(x)$ as auto-correlation function of $f(x)$

$$\varphi(x) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x+t)\overline{f(t)} dt$$

and $\sigma(u) = \sigma(u,\varphi)$ as its G.F.T. Then applying just the same argument as Lemma E(c.f. ibid.V,p.125) we have the above formula

$$\lim_{\varepsilon \to 0} \frac{1}{4\pi\varepsilon} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} | s(u+\varepsilon, f) - s(u-\varepsilon, f) |^2 du = \frac{\sigma(\lambda+0,\varphi) - \sigma(\lambda-0,\varphi)}{\sqrt{2\pi}}$$

15.3 On the Theorem E.

We have

Theorem E^*. Let us suppose that $f(x)$ belongs to the class S and satisfies (C_λ) and

(R_2). Then we have the same conclusions of the Theorem E without the Hypothesis (D_λ).

Proof. We shall prove Theorem E^* step by step as follows.

Step(i) We have by the Schwartz inequality

$$|c_\lambda|^2 = |\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}\,du|^2$$

$$\leq \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\varepsilon,f)-s(u-\varepsilon,f)|^2\,du = \frac{\sigma(\lambda+0,\varphi)-\sigma(u-0,\varphi)}{\sqrt{2\pi}}.$$

Therefore we shall conclude that if $\sigma(u)$ is continuous at $u=\lambda$, then we have $c_\lambda = 0$.

Step(ii) Since $\sigma(u)$ is a bounded and monotone increasing function, there exists the set of at most countable points Λ and satisfies properties as follows.

Let us denote $\Lambda = \{\lambda_n\}$ $(n = 0,1,2,3,....)$ and $c_{\lambda_n} = c_n$ $(n = 0,1,2,3,....)$ where $\lambda_0 = 0$ and $c_0 = 0$ may be permitted.

Then we have

(i) If $\lambda \notin \Lambda$, the we have

$$\sigma(\lambda+0,\varphi)-\sigma(u-0,\varphi) = c_\lambda = 0.$$

(ii) If $\lambda_n \in \Lambda$ $(n = 0,1,2,3,....)$. Then we have

$$|c_n|^2 \leq \frac{\sigma(\lambda_n+0,\varphi)-\sigma(\lambda_n-0,\varphi)}{\sqrt{2\pi}} \qquad (n = 0,1,2,3,....)$$

and

$$\sum_n |c_n|^2 \leq \sum_n \frac{\sigma(\lambda_n+0,\varphi)-\sigma(\lambda_n-0,\varphi)}{\sqrt{2\pi}} \leq \frac{\sigma(\infty,\varphi)-\sigma)-\infty,\varphi)}{\sqrt{2\pi}} < \infty.$$

Then there exists the B_2-almost periodic function $g(x)$ of which Fourier series is as follows

$$g(x) \sim \sum_n c_n e^{i\lambda_n x}.$$

By the hypothesis (C_λ), we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\lambda x}dx = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x)e^{-i\lambda x}dx \qquad (\forall real\ \lambda).$$

(c.f. V ibid. p.129).

Step(iii) Then if we put $f(x)-g(x) = h(x)$ say. Then we shall prove that the function $h(x)$ belongs to the class S. Since $f(x)$ and $g(x)$ both belong to the class S and we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} h(x+t)\overline{h(t)}dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\{f(x+t)-g(x+t)\}\overline{\{f(t)-g(t)\}}dt$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x+t)\overline{f(t)}dt-\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x+t)\overline{g(t)}dt-\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x+t)\overline{f(t)}dt+\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x+t)\overline{g(t)}dt$$

and we have also

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x+t)\overline{g(t)}dt=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x+t)\overline{f(t)}dt=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x+t)\overline{g(t)}dt$$

(c.f. IV ibid. pp.105~108).

Therefore we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} h(x+t)\overline{h(t)}dt = = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x+t)\overline{f(t)}dt - \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x+t)\overline{g(t)}dt\,.$$

Thus we shall prove that $h(x)$ belongs to the class S.

Step (iv) We shall consider auto-correlation functions $\varphi(x;f)$, $\psi(x;g)$ and $\chi(x;h)$ of f,g,h ; their G.F.T. $\sigma(u;\varphi),\sigma(u;\psi)$ and $\sigma(u;\chi)$ of φ,ψ,χ respectively.

Then we shall prove

$$\varphi(x;f)=\psi(x;g)+\chi(x;h) \qquad \text{and} \qquad \sigma(u;\varphi)=\sigma(u;\psi)+\sigma(u;\chi)$$

respectively.

Step (v) Since $\sigma(u;\varphi)$ is continuous on the se Λ^c and discontinuous of the first kind with jump on the set Λ, we have

$$|c_n|^2 \le \frac{\sigma(\lambda_n+0;\varphi)-\sigma(\lambda_n-0;\varphi)}{\sqrt{2\pi}}$$

$$=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda_n-\varepsilon}^{\lambda_n+\varepsilon}\{\sigma(u+\varepsilon;\varphi)-\sigma(u-\varepsilon;\varphi)\}du \qquad (n=0,1,2,\cdots\cdots)$$

On the other hand, since $\sigma(u;\psi)$ is G.F.T. of $\psi(x;g)$ and $\psi(x;g)$ is the auto-correlation function of B_2-almost periodic function $g(x)$, we have

$$\sigma(u;\psi)=\begin{cases}\sqrt{2\pi}\sum_{\lambda_n<u}|c_n|^2 & (u\neq\lambda_m)\\[2ex] \sqrt{2\pi}\left(\sum_{\lambda_n<u}|c_n|^2+\frac{1}{2}|c_m|^2\right) & (u=\lambda_m)\end{cases}$$

and so we have

$$|c_n|^2 = \frac{\sigma(\lambda_n+0,\psi)-\sigma(\lambda_n-0,\psi)}{\sqrt{2\pi}} \qquad (n=0,1,2,\cdots\cdots)$$

on the set Λ and

$$c_\lambda = \frac{\sigma(\lambda+0,\psi)-\sigma(\lambda-0,\psi)}{\sqrt{2\pi}} = 0$$

on the set Λ^c..

Step (vi) Therefore we have proved that $\sigma(u;\chi)$ is bounded, monotone increasing function. Because $\sigma(u;\chi)$ is G.F.T. of $\chi(x;h)$ and $\chi(x;h)$ is the auto-correlation function of $h(x)$, we have by the N.Wiener Theorem[1](Theorem 24,pp. 146~149)

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|h(x)|^2dx = \sum_n\left\{\frac{\sigma(\lambda_n+0,\varphi)-\sigma(\lambda_n-0,\varphi)}{\sqrt{2\pi}} - |c_n|^2\right\}$$

In particular, if the $\sigma(u,\varphi)$ is continuous everywhere then it is satisfied

$$\sigma(\lambda+0,\varphi)-\sigma(u-0,\varphi) = c_\lambda = 0 \qquad (\forall\ real\ \lambda)$$

Therefore we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|h(x)|^2dx = 0.$$

Thus we have constructed the theory of spectral analysis and synthesis on the class S without the hypothesis (D_λ) completely.

15.4 In the last of this section we shall prove

Theorem F_1. Let us suppose that $f(x)\in S_0$. Then the necessary and sufficient condition for the hypotheses (C_λ) are satisfied for all real λ, is the following conditions

$$f(x)+\omega e^{-i\lambda x}\in S_0 \qquad (\omega=\pm 1,\ \pm i)$$

are satisfied for all real λ.

Lemma F. We have the following formula

$$s(u+\varepsilon,e^{i\lambda x})-s(u-\varepsilon,e^{i\lambda x}) = \begin{cases} \sqrt{2\pi} & (\lambda-\varepsilon<u<\lambda+\varepsilon) \\ \dfrac{\sqrt{2\pi}}{2} & (u=\lambda\pm\varepsilon) \\ 0 & (u<\lambda-\varepsilon, \lambda+\varepsilon<u). \end{cases}$$

Proof of the Lemma F. Let us start to calculations of the G.F.T. of $e^{i\lambda x}$ .We have by the definition of G.F.T.

$$s(u+\varepsilon,e^{i\lambda x})-s(u-\varepsilon,e^{i\lambda x})$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}e^{i\lambda x}\frac{e^{-i(u+\varepsilon)x}-1}{-ix}dx+\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\left[\int_{-A}^{-1}+\int_{1}^{A}\right]e^{i\lambda x}\frac{e^{-i(u+\varepsilon)x}}{-ix}dx$$

$$-\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}e^{i\lambda x}\frac{e^{-i(u-\varepsilon)x}-1}{-ix}dx-\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\left[\int_{-A}^{-1}+\int_{1}^{A}\right]e^{i\lambda x}\frac{e^{-i(u-\varepsilon)x}}{-ix}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\frac{e^{-i(u-\lambda+\varepsilon)x}-e^{-i(u-\lambda-\varepsilon)x}}{-ix}dx+\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\left[\int_{-A}^{-1}+\int_{1}^{A}\right]\frac{e^{-i(u-\lambda+\varepsilon)x}-e^{-i(u-\lambda-\varepsilon)x}}{-ix}dx$$

$$=P.V.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-i(u-\lambda+\varepsilon)x}-e^{-i(u-\lambda-\varepsilon)x}}{-ix}dx$$

Therefore we have

$$s(u+\varepsilon,e^{i\lambda x})-s(u-\varepsilon,e^{i\lambda x})=\frac{\sqrt{2\pi}}{2}\{sign(u-\lambda+\varepsilon)-sign(u-\lambda-\varepsilon\}$$

$$=P.V.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{sin(u-\lambda+\varepsilon)x}{x}dx-P.V.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{sin(u-\lambda-\varepsilon)x}{x}dx$$

where we have

$$P.V.\int_{-\infty}^{\infty}\frac{sin\eta x}{x}dx=(sign\eta)\pi \qquad ,$$

and then we have

$$s(u+\varepsilon,e^{i\lambda x})-s(u-\varepsilon,e^{i\lambda x})=\begin{cases}\sqrt{2\pi} & (\lambda-\varepsilon<u<\lambda+\varepsilon)\\ \frac{\sqrt{2\pi}}{2} & (u=\lambda\pm\varepsilon)\\ 0 & (u<\lambda-\varepsilon,\lambda+\varepsilon<u).\end{cases}$$

Proof of Theorem F_1. (The necessity of condition): Let us suppose that $f(x)$ belongs to the class S_0 and satisfies the condition (C_λ).

First of all, we shall remark the following identities

$$|f(x)+\omega e^{-i\lambda x}|^2=|f(x)|^2+\overline{\omega}f(x)e^{i\lambda x}+\omega\overline{f(x)}e^{-i\lambda x}+|\omega|^2 \qquad (\omega=\pm1,\ \pm i).$$

Then applying the condition (C_λ), the existence of limit of following formula

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)+\omega e^{-i\lambda x}|^2\,dx \qquad (\omega=\pm1,\ \pm i) \qquad (\forall real\ \lambda)$$

is guaranteed and therefore we have

$$f(x)+\omega e^{-i\lambda x}\in S_0 \qquad (\omega=\pm1,\ \pm i) \qquad (\forall real\ \lambda).$$

(The sufficiency of condition): First of all, we shall remark also the identities

$$f(x)e^{-i\lambda x}=f(x)\overline{e^{i\lambda x}}$$

$$=\frac{1}{4}\left\{|f(x)+e^{i\lambda x}|^2-|f(x)-e^{i\lambda x}|^2+i|f(x)+ie^{i\lambda x}|^2-i|f(x)-ie^{i\lambda x}|^2\right\}.$$

Then applying the condition $f(x)+\omega e^{-i\lambda x}\in S_0$ $(\omega=\pm1,\ \pm i)$, $(\forall real\ \lambda)$,the existence of following limit

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)e^{-i\lambda x}dx \qquad (\forall real\ \lambda)$$

is guaranteed and the condition(C_λ) is satisfied.

Theorem F_2 Let us suppose that $f(x)$ belongs to the class S_0. Let us suppose that the following limit

$$\lim_{\varepsilon\to0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}du \qquad (\forall real\ \lambda)$$

exist. Then we shall conclude that the following limit

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)+\omega e^{i\lambda x}|^2\,dx \qquad (\omega=\pm1,\pm i) \qquad (\forall real\ \lambda)$$

exist.

Proof. This is obtained by the expansion of the required formula as follows.

$$\lim_{\varepsilon\to0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon,f+\omega e^{-i\lambda x})-s(u-\varepsilon,f+\omega e^{-i\lambda x})|^2du$$

$$=\lim_{\varepsilon\to0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\left\{s(u+\varepsilon,f)-s(u-\varepsilon,f)+\omega(s(u+\varepsilon,e^{-i\lambda x})-s(u-\varepsilon,e^{-i\lambda x}))\right\}|^2\,du$$

$$=\lim_{\varepsilon\to0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}\ |s(u+\varepsilon,f)-s(u-\varepsilon,f)|^2\,du$$

$$-\lim_{\varepsilon\to0}\frac{\overline{\omega}}{4\pi\varepsilon}\int_{-\infty}^{\infty}\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}\overline{\left\{s(u+\varepsilon,e^{-i\lambda x})-s(u-\varepsilon,e^{-i\lambda x})\right\}}du$$

$$-\lim_{\varepsilon\to 0}\frac{\omega}{4\pi\varepsilon}\int_{-\infty}^{\infty}\overline{\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}}\{s(u+\varepsilon,e^{-i\lambda x})-s(u-\varepsilon,e^{-i\lambda x})\}\,du$$

$$+\lim_{\varepsilon\to 0}\frac{|\omega|^2}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon.e^{-i\lambda x})-s(u-\varepsilon,e^{-i\lambda x})|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon,f)-s(u-\varepsilon,f)|^2\,du$$

$$-\lim_{\varepsilon\to 0}\frac{\overline{\omega}}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}du \quad -\lim_{\varepsilon\to 0}\frac{\omega}{4\pi\varepsilon}\int_{-\infty}^{\infty}\overline{\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}du}$$

$$+\lim_{\varepsilon\to 0}\frac{|\omega|^2}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}du$$

Therefore we can conclude that the following limit

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon,f+\omega e^{-i\lambda x})-s(u-\varepsilon,f+\omega e^{-i\lambda x})|^2du\,.$$

exist and we have by the Theorem W_0 (c.f. I, ibid. p.2)

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)+\omega e^{i\lambda x}|^2\,dx$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon,f+\omega e^{-i\lambda x})-s(u-\varepsilon,f+\omega e^{-i\lambda x})|^2du\,.$$

Thus we have proved that $f(x)+\omega e^{i\lambda x}\in S_0$, $(\omega=\pm 1,\pm i)$, $(\forall real\ \lambda)$.

Now we shall attain the desired consequence by combining the result of two Theorems F_1 and F_2. We have proved

Theorem F_3 Let us suppose that $f(x)$ belongs to the class S_0. Then the following formula

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)e^{-i\lambda x}dx \;=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}du$$

is true for each λ in the sense that either of the limit exists, then the other limit exists and assume the same value.

16. The Spectral Analysis and Synthesis of the G.H.T. $\tilde{f}_1(x)$

16.1 Remark (1). On the hypothesis (R_λ).

Let us suppose that $f(x)$ belongs to the class S and satisfies the condition (C_λ). Then applying Lemma E^* ,we have for any constant a_λ

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}-\sqrt{2\pi}a_\lambda|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}|^2\,du-\lim_{\varepsilon\to 0}\frac{\sqrt{2\pi}\,\overline{a_\lambda}}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}du$$

$$-\lim_{\varepsilon\to 0}\frac{\sqrt{2\pi}a_\lambda}{4\pi\varepsilon}\overline{\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}du}+\lim_{\varepsilon\to 0}\frac{2\pi\,|a_\lambda|^2}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}du$$

and we shall notice the following formulas

$$c_\lambda=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}du\qquad(\forall real\ \lambda)$$

by the hypotheses (C_λ). Then we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}-\sqrt{2\pi}a_\lambda|^2\,du$$

$$=\frac{\sigma(\lambda+0)-\sigma(\lambda-0)}{\sqrt{2\pi}}-c_\lambda\overline{a}_\lambda-\overline{c}_\lambda a_\lambda+|a_\lambda|^2$$

$$=\frac{\sigma(\lambda+0)-\sigma(\lambda-0)}{\sqrt{2\pi}}-|c_\lambda|^2+|c_\lambda-a_\lambda|^2.$$

and therefore the value of this formula attains to minimum if and only if $a_\lambda=c_\lambda$ and we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}-\sqrt{2\pi}c_\lambda|^2\,du=\frac{\sigma(\lambda+0)-\sigma(\lambda-0)}{\sqrt{2\pi}}-|c_\lambda|^2.$$

and we have

$$|c_\lambda|^2\le\frac{\sigma(\lambda+0)-\sigma(\lambda-0)}{\sqrt{2\pi}}\qquad(\forall real\ \lambda).$$

Since $\sigma(u)$ is bounded and monotone increasing function, there exists the set Λ of countable points $\lambda=\lambda_n,\ (n=0,1,2,....)$ at which $\sigma(u)$ has jump and continuous elsewhere. Thus we have the following results.

(i) If $\lambda\notin\Lambda$, then we have

$$\sigma(\lambda+0)-\sigma(\lambda-0)=c_\lambda=0$$

and

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\varepsilon,f)-s(u-\varepsilon,f)|^2\,du=0.$$

(ii) If $\lambda\in\Lambda$, that is $\lambda=\lambda_n$, $(n=0,1,2,....)$, then we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda_n-\varepsilon}^{\lambda_n+\varepsilon}|\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}-\sqrt{2\pi}c_n|^2\,du=\frac{\sigma(\lambda_n+0)-\sigma(\lambda_n-0)}{\sqrt{2\pi}}-|c_n|^2$$

16.2 Remark (2). On the hypothesis $(\tilde{R}_\lambda)$.

Let us introduce the generalized Hilbert transform

$$\tilde{f}_1(x)=P.V.\frac{x+i}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}\frac{dt}{x-t}.$$

On the case $\lambda=0$. If $|u|\le\varepsilon$, we have by the Theorem A(c.f.ibid. I, p.4 , p.19) as for G.F.T. of $\tilde{f}_1(x)$

$$s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)$$

$$=i\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}+2r_1(u+\varepsilon,f)+2r_2(u+\varepsilon,f)$$

where it is satisfied that

$$(R_1)\qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|r_1(u+\varepsilon,f)|^2\,du=0$$

and we shall assume that there exist a constant $a(f)$ such as

$$(R_2)\qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|r_2(u+\varepsilon,f)-\sqrt{\frac{\pi}{2}}a(f)|^2\,du=0.$$

Now let us suppose that $\tilde{f}_1(x)$ belongs to the class S and the condition (R_2) is satisfied. Then we have for any constant $\tilde{a}_0=ia_0+a(f)$

$$\left\{s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)-\sqrt{2\pi}\tilde{a}_0\right\}$$

$$=i\left\{s(u+\varepsilon,f)-s(u-\varepsilon,f)-\sqrt{2\pi}a_0\right\}+2r_1(u+\varepsilon,f)+2\left\{r_2(u+\varepsilon,f)-\sqrt{\frac{\pi}{2}}a(f)\right\}$$

and we have by the Minkowski inequality

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\left\{s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)-\sqrt{2\pi}\tilde{a}_0\right\}|^2$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}-\sqrt{2\pi}a_0|^2\,du$$

$$=\frac{\sigma(+0)-\sigma(-0)}{\sqrt{2\pi}}-|c_0|^2+|c_0-a_0|^2$$

Therefore the value of this integral attain to minimum if and only if $a_0=c_0$ i.e. $\tilde{a}_0=\tilde{c}_0$, and $\tilde{c}_0=ic_0+a(f)$ and we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\left\{s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)-\sqrt{2\pi}\tilde{c}_0\right\}|^2\,du$$

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}-\sqrt{2\pi}c_0|^2\,du$$

$$=\frac{\sigma(0+)-\sigma(0-)}{\sqrt{2\pi}}-|c_0|^2.$$

In particular, if $\sigma(u,\varphi)$ is continuous at $u=0$, then $c_0=0$, $\tilde{c}_0=a(f)$ and we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\left\{s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)\right\}-\sqrt{2\pi}a(f)|^2\,du=0$$

In the case $\lambda\neq 0$. If $|u|<\varepsilon$ and $|u\pm\varepsilon|>0$ for sufficiently small ε, we have by the Theorem A (c.f. ibid. I, p.4)

$$s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)=(-isignu)\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}.$$

Then we have by the same arguments as Remark(1) for any constant $\tilde{a}_\lambda$

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|\left\{s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)\right\}-\sqrt{2\pi}\tilde{a}_\lambda|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|\left\{s(u+\varepsilon,f)-s(u-\varepsilon,f)-\sqrt{2\pi}a_\lambda\right\}|^2\,du$$

$$=\frac{\sigma(\lambda+0)-\sigma(\lambda-0)}{\sqrt{2\pi}}-|c_\lambda|^2+|c_\lambda-a_\lambda|^2$$

where $\tilde{a}_\lambda=(-isign\lambda)a_\lambda$. Therefore the value of this integral attains to minimum if and only if $a_\lambda = c_\lambda$ and we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|\left\{s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)\right\}-\sqrt{2\pi}\tilde{c}_\lambda|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|\left\{s(u+\varepsilon,f)-s(u-\varepsilon,f)-\sqrt{2\pi}c_\lambda\right\}|^2\,du$$

$$=\frac{\sigma(\lambda+0)-\sigma(\lambda-0)}{\sqrt{2\pi}}-|c_\lambda|^2$$

where $\tilde{c}_\lambda=(-isign\lambda)c_\lambda$. Therefore we shall conclude that

(i) If $\lambda\in\Lambda$. Then we have

$$\sigma(\lambda+0)-\sigma(\lambda-0)=c_\lambda=0$$

and

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)|^2\,du=0$$

(ii) If $\lambda\in\Lambda$, that is $\lambda=\lambda_n,(n=1,2,3,\ldots)$. Then we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda_n-\varepsilon}^{\lambda_n+\varepsilon}|\left\{s((u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)\right\}-\sqrt{2\pi}\tilde{c}_n|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda_n-\varepsilon}^{\lambda_n+\varepsilon}|\left\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\right\}-\sqrt{2\pi}c_n|^2\,du$$

$$=\frac{\sigma(\lambda_n+0)-\sigma(\lambda_n-0)}{\sqrt{2\pi}}-|c_n|^2$$

where $\tilde{c}_n=(-isign\lambda_n)c_n$.

If $\lambda=\lambda_0$ (=0). Then we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\left\{s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)-\sqrt{2\pi}\tilde{c}_0\right\}|^2\,du$$

$$=\frac{\sigma(0+)-\sigma(0-)}{\sqrt{2\pi}}-|c_0|^2,$$

where $\tilde{c}_0=ic_0+a(f)$.

We have seen that the conditions (R_0) and $(\tilde{R}_0)$ are destroyed (c.f. I ibid. p.23) and so we could not necessarily apply the Minkowski inequality to estimations of remainder terms. We should correct these conditions and instead of them we should state here properties $(R_0)^*$ and $(\tilde{R}_0)^*$ of which we can prove respectively.

$$(R_0)^* \quad \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} |\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}-\sqrt{2\pi}c_0|^2\,du=\frac{\sigma(0+)-\sigma(0-)}{\sqrt{2\pi}}-|c_0|^2$$

and

$$(\tilde{R}_0)^* \quad \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} |\{s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)\}-\sqrt{2\pi}\tilde{c}_0|^2\,du=\frac{\tilde{\sigma}(0+)-\tilde{\sigma}(0-)}{\sqrt{2\pi}}-|\tilde{c}_0|^2$$

where $\tilde{c}_0=ic_0+a(f)$.

We shall remark that applying the Theorem E^*, we have

$$\frac{\tilde{\sigma}(0+)-\tilde{\sigma}(0-)}{\sqrt{2\pi}}=\frac{\sigma(0+)-\sigma(0-)}{\sqrt{2\pi}}-|c_0|^2+|\tilde{c}_0|^2.$$

Similarly we have proved in the case $\lambda\ne 0$. $(R_\lambda)^*$:

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} |\left\{s(u+\varepsilon,f)-s(u-\varepsilon,f)-\sqrt{2\pi}c_\lambda\right\}|^2\,du=\frac{\sigma(\lambda+0)-\sigma(\lambda-0)}{\sqrt{2\pi}}-|c_\lambda|^2$$

and $(\tilde{R}_\lambda)^*$:

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} |\left\{s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)-\sqrt{2\pi}\tilde{c}_\lambda\right\}|^2\,du=\frac{\tilde{\sigma}(\lambda+0)-\tilde{\sigma}(\lambda-0)}{\sqrt{2\pi}}-|\tilde{c}_\lambda|^2$$

respectively.

We shall also remark that applying the Theorem E^*, the relation $\tilde{c}_\lambda=(-i\,sign\lambda)c_\lambda$ and $|\tilde{c}_\lambda|=|c_\lambda|$ we have

$$\frac{\tilde{\sigma}(\lambda+0)-\tilde{\sigma}(\lambda-0)}{\sqrt{2\pi}}=\frac{\sigma(\lambda+0,\varphi)-\sigma(\lambda-0,\varphi)}{\sqrt{2\pi}}$$

16.3 On the hypothesis $(\tilde{C}_\lambda)$. In the preceding sections, if it is required we are going to assume the existence of the following limits

$$(\tilde{C}_\lambda) \qquad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{f}_1(x)e^{-i\lambda x}dx=\tilde{c}_\lambda \qquad (\forall real\ \lambda)$$

However we shall conclude that hypothesis $(\tilde{C}_\lambda)$ could be derived by the hypothesis

(C_λ) and condition (R_2) as follows

Theorem F_4 Let us suppose that $f(x)$ belongs to the class S_0 and satisfies the hypothesis

$$(C_\lambda) \qquad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\lambda x}dx = c_\lambda \qquad (\forall real\ \lambda)$$

and the condition

$$(R_2) \qquad \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} | r_2(u+\varepsilon, f) - \sqrt{\frac{\pi}{2}}a(f)|^2\, du \to 0 \qquad (\varepsilon \to 0).$$

Then we have

$$(\tilde{C}_\lambda) \qquad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \tilde{f}_1(x)e^{-i\lambda x}dx = \tilde{c}_\lambda \qquad (\forall real\ \lambda)$$

and

$$\tilde{c}_\lambda = \begin{cases} (-isign\lambda)c_\lambda & (\lambda \neq 0) \\ \tilde{c}_0 = ic_0 + a(f) & (\lambda = 0). \end{cases}$$

Proof. By the Theorem F_3 we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \tilde{f}_1(x)e^{i\lambda x}dx = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\left\{s(u+\varepsilon,\tilde{f}_1) - s(u-\varepsilon,\tilde{f}_1)\right\}du$$

in the sense that if either side exists, the other side exists and assumes the same value.

By the Theorem A and the condition (R_2) we shall prove existence of the limit of right hand side of the above formula.

(i) If $\lambda \neq 0$ By the Theorem A, we have

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\left\{s(u+\varepsilon,\tilde{f}_1) - s(u-\varepsilon,\tilde{f}_1)\right\}du = (-isign\lambda)\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\left\{s(u+\varepsilon,f) - s(u-\varepsilon,f)\right\}du$$

(ii) If $\lambda = 0$ By the Theorem A and (R_2),we have

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\left\{s(u+\varepsilon,\tilde{f}_1) - s(u-\varepsilon,\tilde{f}_1)\right\}du = \lim_{\varepsilon\to 0}\frac{i}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\left\{s(u+\varepsilon,f) - s(u-\varepsilon,f)\right\}du + a(f)$$

Thus applying the hypothesis (C_λ) we have proved limit of the following formula

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\left\{s(u+\varepsilon,\tilde{f}_1) - s(u-\varepsilon,\tilde{f}_1)\right\}du \qquad (\forall real\ \lambda)$$

exists.

The remaining part of the theorem are obvious by the Theorem F_3.

16.4 As we have pointed out that the conditions (R_0) and $(\tilde{R}_0)$ are destroyed, we should correct the Theorem B_1, Theorem B_2 and Theorem C.

(i) On the case Theorem B_1. We have

Theorem B_1^* Let us suppose that $f \in S_0$ and the hypothesis (C_λ) and the condition (R_2) are satisfied. Then we have that $\tilde{f}_1 \in S_0$ and the following equality

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\tilde{f}_1(x)|^2\, dx = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(x)|^2\, dx - |c_0|^2 + |\tilde{c}_0|^2$$

Proof. We shall prove the following equality

$$\lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} |s(u+\varepsilon,\tilde{f}_1) - s(u-\varepsilon,\tilde{f}_1)|^2\, dx$$

$$= \lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} |s(u+\varepsilon,f) - s(u-\varepsilon,f)|^2\, dx - |c_0|^2 + |\tilde{c}_0|^2 .$$

and apply the N.Wiener theorem (c.f.N.Wiener[ ],Theorem22,p.140).

For this purpose we shall divide the integral of left-hand side into two parts

$$\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} |s(u+\varepsilon,\tilde{f}_1) - s(u-\varepsilon,\tilde{f}_1)|^2\, dx = \frac{1}{4\pi\varepsilon}\int_{|u|\ge\varepsilon} |('')|^2\, du + \frac{1}{4\pi\varepsilon}\int_{u\le\varepsilon} |('')|^2\, du$$

$$= I_1 + I_2, say.$$

Then by the part (i) of the Theorem A (c.f. ibid. I, p.4),we have

$$I_1 = \frac{1}{4\pi\varepsilon}\int_{|u|\ge\varepsilon} |s(u+\varepsilon,\tilde{f}_1) - s(u-\varepsilon,\tilde{f}_1)|^2\, du = \frac{1}{4\pi\varepsilon}\int_{|u|\ge\varepsilon} |(-isignu)\{s(u+\varepsilon,f) - s(u-\varepsilon,f)\}|^2\, du$$

$$= \frac{1}{4\pi\varepsilon}\int_{|u|\ge\varepsilon} |s(u+\varepsilon,f) - s(u-\varepsilon,f)|^2\, du$$

and by the part (ii) of the Theorem A, we have

$$I_2 = \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} |s(u+\varepsilon,\tilde{f}_1) - s(u-\varepsilon,\tilde{f}_1)|^2\, du$$

$$= \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} |i\{s(u+\varepsilon,f) - s(u-\varepsilon,f)\} + 2r_1(u+\varepsilon,f) + 2r_2(u+\varepsilon,f)|^2{}_1)|^2\, du$$

$$= \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} |i\{s(u+\varepsilon,f) - s(u-\varepsilon,f) - i\sqrt{2\pi}a(f)\} + 2r_1(u+\varepsilon,f) + 2(r_2(u+\varepsilon,f) - \sqrt{\frac{\pi}{2}}a(f))|^2\, du$$

Here we can apply the Minkowski inequality, and we have by the use of condition (R_2) (c.f. ibid. I, p.19) and hypothesis (C_λ) (c.f. ibid. VI, p.133)

$$I_2 = \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} |\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\} - i\sqrt{2\pi}a(f)|^2\,du + 0(1) \qquad (\varepsilon\to 0)$$

$$= \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} |s(u+\varepsilon,f)-s(u-\varepsilon,f)|^2 du$$

$$+\frac{i\sqrt{2\pi}\,a(f)}{4\pi\varepsilon}\int_{|u|\le\varepsilon}\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}du - \frac{i\sqrt{2\pi}a(f)}{4\pi\varepsilon}\int_{|u|\varepsilon}\overline{\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}}$$

$$+\frac{|i\sqrt{2\pi}a(f)|^2}{4\pi\varepsilon}\int_{|u|\le\varepsilon} du$$

$$= \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} |s(u+\varepsilon,f)-s(u-\varepsilon,f)|^2 du + ic_0\overline{a(f)} - i\overline{c}_0 a(f) + |a(f)|^2$$

$$= \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} |s(u+\varepsilon,f)-s(u-\varepsilon,f)|^2 du + |ic_0|^2 + |ic_0 + a(f)|^2$$

$$= \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} |s(u+\varepsilon,f)-s(u-\varepsilon,f)|^2 du - |c_0|^2 + |\tilde{c}_0|^2,$$

where $\tilde{c}_0 = ic_0 + a(f)$

Therefore we have proved the required formula and we can conclude that $\tilde{f}_1 \in S_0$.

(ii) On the case Theorem B_2. We have

Theorem B_2^* Let us suppose that $f \in S$ and the hypothesis (C_λ) and the condition (R_2) are satisfied. Then we have that $\tilde{f}_1 \in S$ and the following equality

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} e^{iux}\,|s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} e^{iux}\,|s(u+\varepsilon,f)-s(u-\varepsilon,f)|^2\,du - |c_0|^2 + |\tilde{c}_0|^2$$

Moreover ,we have by Theorem W_1 (c.f. II, ibid. pp.25~28)

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{f}_1(x+t)\overline{\tilde{f}_1(t)}dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x+t)\overline{f(t)}dt - |c_0|^2 + |\tilde{c}_0|^2.$$

Proof. We have

$$J = \frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} e^{iux} \,|\, s(u+\varepsilon,\tilde{f}_1) - s(u-\varepsilon,\tilde{f}_1)\,|^2\, du$$

$$= \frac{1}{4\pi\varepsilon}\int_{|u|\ge\varepsilon} e^{iux}\,|\,''\,|^2\, du + \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} e^{iux}\,|\,''\,|^2\, du = J_1 + J_2, \quad say.$$

We have by the Theorem A (c.f. *ibid. p.4)*

$$J_1 = \frac{1}{4\pi\varepsilon}\int_{|u|\ge\varepsilon} e^{iux}\,|\, s(u+\varepsilon,\tilde{f}_1) - s(u-\varepsilon,\tilde{f}_1)\,|^2\, du$$

$$= \frac{1}{4\pi\varepsilon}\int_{|u|\ge\varepsilon} e^{iux}\,|\,(-i signu)\{s(u+\varepsilon,f) - s(u-\varepsilon,f)\}\,|^2\, du$$

$$= \frac{1}{4\pi\varepsilon}\int_{|u|\ge\varepsilon} e^{iux}\,|\, s(u+\varepsilon,f) - s(u-\varepsilon,f)\,|^2\, du.$$

and also we have

$$J_2 = \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} e^{iux}\,|\, s(u+\varepsilon,\tilde{f}_1) - s(u-\varepsilon,\tilde{f}_1)\,|^2\, du$$

$$= \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} (e^{iux}-1)\,|\, s(u+\varepsilon,\tilde{f}_1) - s(u-\varepsilon,\tilde{f}_1)\,|^2\, du$$

$$+ \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} |\, s(u+\varepsilon,\tilde{f}_1) - s(u-\varepsilon,\tilde{f}_1)\,|^2\, du.$$

Since $\tilde{f}_1 \in S_0$ by Theorem B_1^* and $e^{iux} - 1 = O(\varepsilon)$ $(\varepsilon \to 0)$, we have

$$\frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} (e^{iux}-1)\,|\, s(u+\varepsilon,\tilde{f}_1) - s(u-\varepsilon,\tilde{f}_1)\,|^2\, du = O(\varepsilon)\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |\,\tilde{f}_1(x)\,|^2\, dx$$

$$= o(1) \quad (\varepsilon \to 0)$$

(c.f. I. ibid. pp.21~22).

Moreover applying the condition (R_2), we have

$$\frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} |\, s(u+\varepsilon,\tilde{f}_1) - s(u-\varepsilon,\tilde{f}_1)\,|^2\, du$$

$$= \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} |\, s(u+\varepsilon,f) - s(u-\varepsilon,f)\,|^2\, du - |\,c_0\,|^2 + |\,\tilde{c}_0\,|^2$$

Thus we have

$$J_2 = \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} e^{iux}\,|\, s(u+\varepsilon,f) - s(u-\varepsilon,f)\,|^2\, du - |\,c_0\,|^2 + |\,\tilde{c}_0\,|^2 + o(1) \quad (\varepsilon \to 0)$$

Therefore we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} e^{iux}\,|\, s(u+\varepsilon,\tilde{f}_1) - s(u-\varepsilon,\tilde{f}_1)\,|^2\, du$$

$$= \lim_{\varepsilon \to 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} e^{iux} \,|\, s(u+\varepsilon, f) - s(u-\varepsilon, f) \,|^2\, du - |\, c_0 \,|^2 + |\, \tilde{c}_0 \,|^2 .$$

Thus we can conclude that $\tilde{f}_1(x) \in S$ by the Theorem W_1 (c.f. II, ibid. pp.25~28) and we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \tilde{f}_1(x+t)\overline{\tilde{f}_1(t)}dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x+t)\overline{f(t)}dt - |\, c_0 \,|^2 + |\, \tilde{c}_0 \,|^2 .$$

(iii) On the case Theorem C. We have

Theorem C^*. Let us suppose that $f(x)$ is a B^2-almost periodic function and satisfies condition (R_2). Let us write its Fourier series as follows

$$f(x) \sim \sum_n c_n e^{i\lambda_n x} .$$

Then its G.H.T. $\tilde{f}_1(x)$ is also a function of B^2-almost periodic and has its Fourier series as follows

$$\tilde{f}_1(x) \sim \sum_n \tilde{c}_n e^{i\lambda_n x} ,$$

where

$$\tilde{c}_n = \begin{cases} (-i sign \lambda_n) c_n & (n = 1,2,3,\ldots) \\ ic_0 + a(f) & (n = 0). \end{cases}$$

Proof. Since $f(x)$ is a function of B^2-almost periodic and so it belongs to the class S and satisfies the condition (R_2), we have that its G.H.T. $\tilde{f}_1(x)$ belongs to the class S and satisfies the hypothesis $(\tilde{C}_\lambda)$ $(\forall real\ \lambda)$.

Let us denote the set $\Lambda = \{\lambda_n, n = 0,1,2,\ldots\}$. Then we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x) e^{-i\lambda x} dx = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \{s(u+\varepsilon, f) - s(u-\varepsilon, f)\} du$$

$$= \begin{cases} 0 & (\lambda \notin \Lambda) \\ c_n & (\lambda \in \Lambda,\ \ \lambda = \lambda_n,\ n = 0,1,2,\ldots). \end{cases}$$

Then we have by the Theorem A and condition (R_2)

(i) $\lambda \neq 0$

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\left\{s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)\right\}$$

$$=\lim_{\varepsilon\to 0}\frac{(-isign\lambda)}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}du$$

$$=(-isign\lambda)\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)e^{-i\lambda x}dx=(-isign\lambda)c_n$$

(ii) $\lambda=0$

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\left\{s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)\right\}$$

$$=\lim_{\varepsilon\to 0}\frac{i}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}du+a(f)=\ ic_0+a(f).$$

Therefore we have by the Theorem F_3 (c.f. ibid.VI,15.4) the hypothesis $(\tilde{C}_\lambda)\ (\forall real\ \lambda)$ is satisfied and we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{f}_1(x)e^{-i\lambda x}dx=\begin{cases}0 & (\lambda\notin\Lambda)\\ \tilde{c}_n & (\lambda\in\Lambda,\ \ \lambda=\lambda_n,\ n=0,1,2,\ldots).\end{cases}$$

Since $f(x)$ is to be B^2-almost periodic, we have $\sum_n|c_n|^2<\infty$ and $\sum_n|\tilde{c}_n|^2<\infty$ too.

Therefore we shall conclude that $\tilde{f}_1(x)$ is to be almost periodic and has its Fourier series as follows

$$\tilde{f}_1(x)\sim\sum_n\tilde{c}_n e^{i\lambda_n x}.$$

16.5 On the spectral analysis and synthesis of G.H.T. $\tilde{f}_1(x)$

Now we shall going to construct the theory of spectral analysis and synthesis of G.H.T. $\tilde{f}_1(x)$.

First of all we should remark the following results.
Let us suppose that $f(x)$ belongs to the class S and satisfies the hypothesis (C_λ) and

the condition (R_2). Then $\tilde{f}_1(x)$ belongs to the class S by the Theorem B_2^* (c.f. ibid. VI,p.149) and it satisfies the hypothesis $(\tilde{C}_\lambda)$ by the Theorem F_4 (c.f. ibid. VI, p147).

Let us denote that $\varphi(x)=\varphi(x,f)$ as the auto-correlation function of $f(x)$ and $\sigma(u)=\sigma(u,\varphi)$ as the G.F.T. of $\varphi(x)$. Then since $\sigma(u)$ is a function to be bounded and monotone increasing , there exist the set $\Lambda=\{\lambda_n, n=0,1,2,,...\}$ at most countable and the $\sigma(u)$ is discontinuous of first kind there and continuous elsewhere. Then we have

$$c_\lambda = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\lambda x}dx$$

$$=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}du$$

where $c_\lambda=0\ \ (\lambda\notin\Lambda)$ and $c_{\lambda_n}\neq 0\ \ (\lambda\in\Lambda)$. We shall denote c_n instead of c_{λ_n} and promise that $\lambda_0=0$ and we may permit $c_0=0$.

We have also

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\varepsilon,f)-s(u-\varepsilon,f)|^2du=\frac{\sigma(\lambda+0,\varphi)-\sigma(\lambda-0,\varphi)}{\sqrt{2\pi}}$$

by the Lemma E* (c.f. ibid. VI, p.135). Then we have by the Schwaltz inequality

$$|c_\lambda|^2\leq\frac{\sigma(\lambda+0,\varphi)-\sigma(\lambda-0,\varphi)}{\sqrt{2\pi}}\qquad(\forall real\ \lambda)$$

Now let us suppose that $f(x)$ belongs to the class S and satisfies the hypothesis (C_λ) and condition (R_2). We shall try the same problem as $f(x)$ to its G.H.T. $\tilde{f}_1(x)$. We shall state them step by step steadily for the sake of completeness.

Step(i) Let us define $\tilde{\varphi}(x)=\varphi(x,\tilde{f}_1)$ as the auto-correlation function of $\tilde{f}_1(x)$ and $\tilde{\sigma}(u)=\sigma(u,\tilde{\varphi})$ as the G.F.T. of $\tilde{\varphi}(x)$ respectively. Then we have if $\lambda\neq 0$

$$\tilde{c}_\lambda=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)\}du$$

$$= \lim_{\varepsilon \to 0} \frac{(-isign\lambda)}{2\varepsilon\sqrt{2\pi}} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \{s(u+\varepsilon, f) - s(u-\varepsilon, f)\} du$$

$$= (-isign\lambda) c_\lambda$$

and if $\lambda = 0$

$$\tilde{c}_0 = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{-\varepsilon}^{\varepsilon} \{s(u+\varepsilon, \tilde{f}_1) - s(u-\varepsilon, \tilde{f}_1)\} du$$

$$= \lim_{\varepsilon \to 0} \frac{i}{2\varepsilon\sqrt{2\pi}} \int_{-\varepsilon}^{\varepsilon} \{s(u+\varepsilon, f) - s(u-\varepsilon, f)\} du$$

$$+ \lim_{\varepsilon \to 0} \frac{2}{2\varepsilon\sqrt{2\pi}} \int_{-\varepsilon}^{\varepsilon} r_1(u+\varepsilon, f) du + \lim_{\varepsilon \to 0} \frac{2}{2\varepsilon\sqrt{2\pi}} \int_{-\varepsilon}^{\varepsilon} \left\{ r_2(u+\varepsilon, f) - \sqrt{\frac{\pi}{2}} a(f) \right\} du + a(f)$$

$$= ic_0 + a(f)$$

by the Theorem A (c.f. ibid. I, p.4) and hypotheses (C_λ), $(\tilde{C}_\lambda)$ and condition (R_2) (c.f. ibid. VI, Theorem F_4, 147 and p.143). Therefore we have

$$\tilde{c}_\lambda = \begin{cases} (-isign\lambda) c_\lambda & (\lambda \neq 0) \\ ic_0 + a(f) & (\lambda = 0). \end{cases}$$

Step (ii) Let us denote $\varphi(x) = \varphi(x, f)$ and $\tilde{\varphi}(x) = \varphi(x, \tilde{f}_1)$ the auto-correlation function of $f(x)$ and $\tilde{f}_1(x)$ respectively. Let us also denote $\sigma(u) = \sigma(u, \varphi)$ and $\tilde{\sigma}(u) = \sigma(u, \tilde{\varphi})$ the G.F.T. of $\varphi(x)$ and $\tilde{\varphi}(x)$ respectively. Then we have

$$\lim_{\varepsilon \to 0} \frac{1}{4\pi\varepsilon} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} | s(u+\varepsilon, f) - s(u-\varepsilon, f) |^2 du = \frac{\sigma(\lambda+0, \varphi) - \sigma(\lambda-0, \varphi)}{\sqrt{2\pi}} \qquad (\forall real\ \lambda)$$

by the Lemma E^* (c.f. ibid. VI, p.153). Since $\tilde{f}_1(x)$ satisfies the hypothesis $(\tilde{C}_\lambda)$ with the condition (R_2) as for $f(x)$, we have also

$$\lim_{\varepsilon \to 0} \frac{1}{4\pi\varepsilon} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} | s(u+\varepsilon, \tilde{f}_1) - s(u-\varepsilon, \tilde{f}_1) |^2 du = \frac{\sigma(\lambda+0, \tilde{\varphi}) - \sigma(\lambda-0, \tilde{\varphi})}{\sqrt{2\pi}} \qquad (\forall real\ \lambda)$$

Therefore we have by the Theorem A

if $\lambda \neq 0$ $$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)|^2du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|(-i\,signu)\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}|^2\ du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\varepsilon,f)-s(u-\varepsilon,f)|^2du$$

and if $\lambda = 0$ $$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)|^2du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|i\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}+2r_1(u+\varepsilon,f)+2r_2(u+\varepsilon,f)|^2\ du$$

where the integrand rewrite as follows

$$i\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}+2r_1(u+\varepsilon,f)+2r_2(u+\varepsilon,f)$$

$$=i\left\{s(u+\varepsilon,f)-s(u-\varepsilon,f)-i\sqrt{2\pi}a(f)\right\}+2r_1(u+\varepsilon,f)+2\left\{r_2(u+\varepsilon,f)-\sqrt{\frac{\pi}{2}}a(f)\right\}$$

and since we could apply the Minkowski inequality , we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)|^2du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|i\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}-i\sqrt{2\pi}a(f)|^2\ du$$

Furthermore we shall expand the integrand as follows

$$\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|i\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}-i\sqrt{2\pi}a(f)|^2\ du$$

$$=\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|s(u+\varepsilon,f)-s(u-\varepsilon,f)|^2du$$

$$-\frac{\overline{a(f)}}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}du-\frac{a(f)}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\overline{\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}du}+|a(f)|^2$$

and we shall also remark that

$$c_0=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)dx=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}du$$

by the one-sided Wiener formula (c.f. ibid. VI, p.135). Therefore we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)|^2du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|i\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}-i\sqrt{2\pi}a(f)|^2\ du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|s(u+\varepsilon,f)-s(u-\varepsilon,f)|^2du\quad+|ic_0+a(f)|^2-|c_0|^2$$

Thus we shall conclude that

$$\frac{\sigma(\lambda+0,\tilde{\varphi})-\sigma(\lambda-0,\tilde{\varphi})}{\sqrt{2\pi}}=\begin{cases}\dfrac{\sigma(\lambda+0,\varphi)-\sigma(\lambda-0,\varphi)}{\sqrt{2\pi}} & (\lambda\neq 0)\\[2ex] \dfrac{\sigma(0+,\varphi)-\sigma(0-,\varphi)}{\sqrt{2\pi}}+|ic_0+a(f)|^2-|c_0|^2 & (\lambda=0)\end{cases}$$

Since $\tilde{f}_1(x)$ belongs to the class S and satisfies hypothesis $(\tilde{C}_\lambda)$, we have that $\tilde{\sigma}(u)=\sigma(u,\tilde{\varphi})$ is a bounded and monotone increasing function. and so there exists $\tilde{\Lambda}$ the set of λ to be at most countable and $\tilde{\sigma}(u)$ is discontinuous of the first kind there and continuous elsewhere. By the arguments of Step(i) and Step(ii)above, we shall conclude that the set $\tilde{\Lambda}$ is just the same the set Λ and we have

$$\frac{\tilde{\sigma}(\lambda+0)-\tilde{\sigma}(\lambda-0)}{\sqrt{2\pi}}=\begin{cases}\dfrac{\sigma(\lambda+0)-\sigma(\lambda-0)}{\sqrt{2\pi}} & (\lambda\neq 0)\\[2ex] \dfrac{\sigma(0+)-\sigma(0-)}{\sqrt{2\pi}}+|ic_0+a(f)|^2-|c_0|^2 & (\lambda=0)\end{cases}$$

and furthermore we shall write the above formula as follows

$$\frac{\tilde{\sigma}(\lambda+0)-\tilde{\sigma}(\lambda-0)}{\sqrt{2\pi}}=\begin{cases}0, & (\lambda\notin\Lambda)\\ \dfrac{\sigma(\lambda_n+0)-\sigma(\lambda_n-0)}{\sqrt{2\pi}}, & (\lambda_n\in\Lambda, n=1,2,3,...)\\ \dfrac{\sigma(0+)-\sigma(0-)}{\sqrt{2\pi}}+|ic_0+a(f)|^2-|c_0|^2, & (\lambda=0)\end{cases}$$

Step(iii) In the first we shall remark that the following formula is satisfied

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{f}_1(x)e^{-i\lambda x}dx=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)\}du\qquad(\forall real\ \lambda).$$

in the sense that either of the limit exists, then the other limit exists and assume the same value (c.f. ibid.VI, Theorem F_3 ,pp.141~2; Hypothesis $(\tilde{C}_\lambda)$,p.146;Theore B_2^* ,p149).

Then we have by the Schwaltz inequality

$$|\tilde{c}_\lambda|^2=|\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)\}du|^2$$

$$\leq\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\varepsilon,\tilde{f}_1)-s(u-\varepsilon,\tilde{f}_1)|^2\,du$$

$$=\frac{\sigma(\lambda+0,\tilde{\varphi})-\sigma(\lambda-0,\tilde{\varphi})}{\sqrt{2\pi}}.$$

Therefore we have

$$\sum_n|\tilde{c}_n|^2\leq\frac{\sigma(\infty,\tilde{\varphi})-\sigma(-\infty,\tilde{\varphi})}{\sqrt{2\pi}}<\infty$$

and so there exists B^2 -almost periodic function $\tilde{g}(x)$ and its Fourier series

$$\tilde{g}(x)\sim\sum_n\tilde{c}_ne^{i\lambda_nx}$$

where we shall denote $\tilde{c}_n$ instead of $\tilde{c}_{\lambda_n}$.

Since we have already shown that $\tilde{f}_1(x)$ satisfies the hypothesis $(\tilde{C}_\lambda)$, we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{f}_1(x)e^{-i\lambda x}dx=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{g}(x)e^{-i\lambda x}dx\qquad(\forall real\ \lambda).$$

Then we shall set

$$\tilde{f}_1(x)-\tilde{g}(x)=\tilde{h}(x),\quad say$$

and we shall prove that $\tilde{h}(x)$ belongs to the N.Wiener class S. This can be done just the same arguments as the decomposition

$$f(x)-g(x)=h(x)$$

of the Theorem E^* (c.f. ibid. VI, 15.3, Step.(iii), p.137).

Thus we have proved

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{h}(x+t)\overline{\tilde{h}(t)}dt=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{f}_1(x+t)\overline{\tilde{f}_1(t)}dt-\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{g}(x+t)\overline{\tilde{g}(t)}dt$$

Step. (iv) We shall also consider the auto-correlation functions of $\tilde{f}_1$, $\tilde{g}$ and $\tilde{h}$

$$\tilde{\varphi}(x)=\varphi(x,\tilde{f}_1),\quad \tilde{\psi}(x)=\psi(x,\tilde{g})\quad\text{and}\quad \tilde{\chi}(x)=\chi(x,\tilde{h})$$

and their G.F.T. $\sigma(u,\tilde{\varphi}),\sigma(u,\tilde{\psi})$ and $\sigma(u,\tilde{\chi})$.

Then we have proved in the Step (iii)

$$\varphi(x,\tilde{f}_1)=\psi(x,\tilde{g})+\chi(x,\tilde{h})$$

and therefore we have also

$$\sigma(u,\tilde{\varphi})=\sigma(u,\tilde{\psi})+\sigma(u,\tilde{\chi}).$$

Thus we have proved that $\sigma(u,\tilde{\chi})$ is a bounded , monotone increasing function.
Therefore we could apply the N.Wiener theorem (c.f. N.Wiener[1], Theorem 24,pp.146-9)

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\tilde{h}(x)|^2dx=\sum_n\left\{\frac{\sigma(\lambda_n+0,\tilde{\varphi})-\sigma(\lambda_n-0,\tilde{\varphi})}{\sqrt{2\pi}}-|\tilde{c}_n|^2\right\}.$$

In particular $\sigma(u)=\sigma(u,\varphi)$ is continuous everywhere, then it is satisfied by the results of the Step (ii)

$$\frac{\sigma(\lambda_n+0,\tilde{\varphi})-\sigma(\lambda_n-0,\tilde{\varphi})}{\sqrt{2\pi}}=\frac{\sigma(\lambda_n+0,\varphi)-\sigma(\lambda_n-0,\varphi)}{\sqrt{2\pi}}=0,\ (\lambda_n\in\Lambda,\quad n=1,2,3,\ldots)$$

and

$$\frac{\sigma(0+,\tilde{\varphi})-\sigma(0-,\tilde{\varphi})}{\sqrt{2\pi}}=|a(f)|^2,\quad |\tilde{c}_0|^2=|a(f)|^2\quad(\lambda=0).$$

Therefore we have proved

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\tilde{h}(x)|^2dx=0.$$

We shall call it Theorem G_1.

17 The Spectral Analysis and Synthesis of the G.C.I. $C_1(z,f)$.

We shall define the G.C.I. of $f(x)$ as follows

$$f_1(z)=2C_1(z,f)=P.V.\frac{z+i}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}\frac{dt}{z-t} \qquad (z=x+iy,\ \ y>0)$$

and we shall going to construct the theory of spectral analysis and synthesis of G.C.I. $C_1(z,f)$. We shall state them step by step steadily for the sake of completeness.

Step (i) Let us suppose that $f(x)$ belongs to the class S and satisfies condition

(R_2) Let us put $F(x)=f(x)+i\tilde{f}_1(x)$ then $\tilde{f}_1(x)$ and $F(x)$ belong to the class W^2

and then we could apply Theorem A and Theorem D_3 to $f_1(z)=C_1(z,F)$ and therefore

we have

If $|u|>\varepsilon$, then we have

$$s(u+\varepsilon;f_1(z))-s(u-\varepsilon;f_1(z))=\frac{(1+signu)}{2}\left[\{s(u+\varepsilon;F)-s(u-\varepsilon;F)\}+r_0(u,y,\varepsilon;F)\right]$$

where

$$s(u+\varepsilon;F)-s(u-\varepsilon;F)=\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}+i\{s(u+\varepsilon;\tilde{f}_1)-s(u-\varepsilon;\tilde{f}_1)\}$$

$$=(1+signu)\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}$$

and then we have

$$s(u+\varepsilon;f_1(z))-s(u-\varepsilon;f_1(z))$$
$$=\frac{(1+signu)}{2}\left[2\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}+r_0(u,y,\varepsilon;F)\right]$$

where $$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|>\varepsilon}|r_0(u,y,\varepsilon;F)|^2\,du=0.$$

If $|u|<\varepsilon$, then we have

$$s(u+\varepsilon;f_1(z))-s(u-\varepsilon;f_1(z))=ir_1(u+\varepsilon;F)+ir_2(u+\varepsilon;F)+r_3(u+\varepsilon,y;F)$$

where

$$ir_1(u+\varepsilon;F)+ir_2(u+\varepsilon;F)=s(u+\varepsilon;F)-s(u-\varepsilon;F)$$
$$=\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}+i\{s(u+\varepsilon;\tilde{f}_1)-s(u-\varepsilon;\tilde{f}_1)\}$$
$$=2ir_1(u+\varepsilon;f)+2ir_2(u+\varepsilon;;f)$$

and then we have

$$s(u+\varepsilon;f_1(z))-s(u-\varepsilon;f_1(z))$$
$$=2ir_1(u+\varepsilon;f)+2ir_2(u+\varepsilon;;f)+r_3(u+\varepsilon,y;F)$$

where

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|<\varepsilon}|r_1(u,y,\varepsilon;f)|^2\,du=0 \quad \text{and} \quad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|(\ )\varepsilon}|r_3(u+\varepsilon,y;F)|^2\,du=0$$

and there exist constant $a(f)$ such as

$$(R_2) \qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|<\varepsilon}|r_2(u,y,\varepsilon;f)-\sqrt{\frac{\pi}{2}}a(f)|^2\,du\,.$$

Step (ii) Now we shall intend to prove that $f_1(z)$ belongs to the class S. This can be done by the application of Theorem W_1. We shall estimate it by the integration by parts and apply the Lemma E^*.

Let us estimate the following integral

$$\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}e^{iux}\,|s(u+\varepsilon;f_1(z))-s(u-\varepsilon;f_1(z))|^2\,du$$

$$=\frac{1}{4\pi\varepsilon}\int_{|u|\geq\varepsilon}e^{iux}\,|\,''\,|^2\,du+\frac{1}{4\pi\varepsilon}\int_{|u|\leq\varepsilon}e^{iux}\,|\,''\,|^2\,du=I_1+I_2,\quad say.$$

We have

$$I_1=\frac{1}{4\pi\varepsilon}\int_{|u|\geq\varepsilon}e^{iux}\,|\frac{(1+signu)}{2}\left[2\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}+r_0(u,y,\varepsilon;F)\right]|^2\,du$$

$$,\ =\frac{1}{2\pi\varepsilon}\int_{|\varepsilon}^{\infty}e^{iux}\,|e^{-yu}\{s(u+\varepsilon,f)-s(u-\varepsilon,f)\}|^2\,du+o(1)\quad(\varepsilon\to 0)$$

by the Minkowski inequality and we have

$$\frac{1}{\pi\varepsilon}\int_{\varepsilon}^{\infty}e^{iux}e^{-2yu}\,|s(u+\varepsilon,f)-s(u-\varepsilon,f)|^2\,du=\frac{e^{(ix-2y)u}}{\pi\varepsilon}\int_{\varepsilon}^{u}|s(v+\varepsilon;f)-s(v-\varepsilon;f)|^2\,dv\Big|_{u=\varepsilon}^{u=\infty}$$

$$-\frac{(ix-2y)}{\pi\varepsilon}\int_{\varepsilon}^{\infty}e^{(ix-2y)u}\left(\int_{\varepsilon}^{u}|s(v+\varepsilon;f)-s(v-\varepsilon;f)|^2\,dv\right)du$$

by the integration by parts.

Now we have by the Lemma E^*

$$\frac{1}{4\pi\varepsilon}\int_{\varepsilon}^{\infty}|s(v+\varepsilon;f)-s(v-\varepsilon;f)|^2dv\to\frac{\sigma(\infty)-\sigma(0+)}{\sqrt{2\pi}}\qquad(\varepsilon\to 0)$$

and

$$\frac{1}{4\pi\varepsilon}\int_{\varepsilon}^{u}|s(v+\varepsilon;f)-s(v-\varepsilon;f)|^2dv \to \frac{\sigma(u)-\sigma(0+)}{\sqrt{2\pi}} \quad a.e.\ u \quad (\varepsilon \to 0)$$

boundedly. Therefore we have

$$I_1 = -4(ix-2y)\int_{\varepsilon}^{\infty} e^{(ix-2y)u}\frac{\sigma(u)-\sigma(0+)}{\sqrt{2\pi}}du + o(1) \quad (\varepsilon \to 0)$$

Next we have

$$I_2 = \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} e^{iux}|s(u+\varepsilon;f_1(z))-s(u-\varepsilon;f_1(z))|^2\, du$$
$$= \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} e^{iux}|2ir_1(u+\varepsilon;f)+2ir_2(u+\varepsilon;f)+r_3(u+\varepsilon,y;F)|^2\, du$$

where let us remark the following properties

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|\le\varepsilon}|r_1(u+\varepsilon,f)|^2\, du = 0 \qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|\le\varepsilon}|r_3(u+\varepsilon,y;F)|^2\, du = 0$$

and the condition

$$(R_2) \quad \exists\ a(f) \ : \quad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|\le\varepsilon}|r_2(u+\varepsilon,f)-\sqrt{\frac{\pi}{2}}a(f)|^2\, du = 0\,.$$

Then we have by the Minkowski inequality

$$I_2 = \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon}|r_2(u+\varepsilon,f)-\sqrt{\frac{\pi}{2}}a(f)|^2\, du + |a(f)|^2 + o(1) = |a(f)|^2 + o(1)$$

$(\varepsilon \to 0)$

Therefore we have proved

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} e^{iux}|s(u+\varepsilon;f_1(z))-s(u-\varepsilon;f_1(z))|^2\, du$$
$$= -4(ix-2y)\int_{0}^{\infty} e^{(ix-2y)u}\frac{\sigma(u)-\sigma(0+)}{\sqrt{2\pi}}du + |a(f)|^2\,.$$

Thus we have proved $f_1(z) = 2C_1(z;f) = C_1(z;F)$ where $F(x) = f(x)+i\tilde{f}_1(x)$ belongs to the class S by the Theorem W_1. We shall present it as follows

$$\varphi(x;f_1(z))=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f_1(x+t,y)\overline{f_1(t,y)}dt$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}e^{iux}\,|\,s(u+\varepsilon;f_1(z))-s(u-\varepsilon;f_1(z))|^2\,du \quad (z=x+iy,\ \ y>0)$$

Step (iii) Let us suppose that $f(x)$ belongs to the class S and satisfies the hypothesis(C_λ) and the condition(R_2). Then we shall notice that G.H.T. $\tilde{f}_1(x)$ satisfies the hypothesis$(\tilde{C}_\lambda)$ too.

Now we shall prove

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f_1(z)e^{-i\lambda x}dx = \begin{cases}\dfrac{(1+sign\lambda)^2}{2}e^{-y\lambda}c_\lambda & (\lambda\neq 0)\\ \\ ia(f) & (\lambda=0)\end{cases}$$

where $z=x+iy,\ \ y>0$.

For this purpose we shall need the support of the Theorem F_3 (ibid. IV, p.141). There we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f_1(z)e^{-i\lambda x}dx=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon;f_1(z))-s(u-\varepsilon;f_1(z))\}du$$

where $z=x+iy,\ \ y>0$. Then we shall intend to estimate the formula in the right hand side with aide of the Theorems D_2, D_3 (c.f. ibid. III, pp.47~8) and Theorem A (c.f. ibid. I, p.4).

(i) The case $\lambda\neq 0$.

We have by the Theorem D_3

$$s(u+\varepsilon;f_1(z))-s(u-\varepsilon;f_1(z))=\frac{(1+signu)}{2}e^{-yu}\left(\{s(u+\varepsilon;F)-s(u-\varepsilon;F)\}+r_0(u,y,\varepsilon;F)\right)$$

where $F(x)=f(x)+i\tilde{f}_1(x)$ and we have by the Theorem A

$$s(u+\varepsilon;F)-s(u-\varepsilon;F)$$

$$=\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}+i\{s(u+\varepsilon;\tilde{f}_1)-s(u-\varepsilon;\tilde{f}_1)\}$$

$$= \{s(u+\varepsilon; f) - s(u-\varepsilon; f)\} + i(-isignu)\{s(u+\varepsilon; f) - s(u-\varepsilon; f)\}$$

$$= (1+signu)\{s(u+\varepsilon; f) - s(u-\varepsilon; f)\}$$

Therefore we have

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon; f_1(z)) - s(u-\varepsilon; f_1(z))\}du$$

$$= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\frac{(1+signu)^2}{2}e^{-yu}\{s(u+\varepsilon; f) - s(u-\varepsilon; f)\}du$$

$$= \frac{(1+sign\lambda)^2}{2}e^{-y\lambda}\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon; f) - s(u-\varepsilon; f)\}du$$

where we shall use the condition

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|\ge\varepsilon}|r_0(u,y,\varepsilon; F)|^2\,du = 0$$

for the estimation of the remainder term.

Since we have by the Theorem F_3

$$c_\lambda = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)e^{-i\lambda x}dx = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon; f) - s(u-\varepsilon; f)\}du$$

we have proved

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f_1(z)e^{-i\lambda x}dx = \frac{(1+sign\lambda)^2}{2}e^{-y\lambda}c_\lambda .$$

(ii) The case $\lambda = 0$. We have by the Theorem D_3

$$s(u+\varepsilon; f_1(z)) - s(u-\varepsilon; f_1(z))$$
$$= ir_1(u+\varepsilon; F) + ir_2(u+\varepsilon; F) + r_3(u+\varepsilon, y; F)$$

where $\quad ir_1(u+\varepsilon; F) + ir_2(u+\varepsilon; F) = s(u+\varepsilon; F) - s(u-\varepsilon; F).$

and we have by the Theorem A

$$s(u+\varepsilon; F) - s(u-\varepsilon; F) = \{s(u+\varepsilon; f) - s(u-\varepsilon; f)\} + i\{s(u+\varepsilon; \tilde{f}_1) - s(u-\varepsilon; \tilde{f}_1)\}$$

$$= 2ir_1(u+\varepsilon; f) + 2ir_2(u+\varepsilon; f)$$

Therefore we have

$$s(u+\varepsilon; f_1(z)) - s(u-\varepsilon; f_1(z)) = 2ir_1(u+\varepsilon; f) + 2ir_2(u+\varepsilon; f) + r_3(u+\varepsilon, y; F)$$

where we shall notice the following properties

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|\le\varepsilon} |r_1(u,y,\varepsilon;f)|^2\,du = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|\le\varepsilon} |r_3(u,y,\varepsilon;F)|^2\,du = 0$$

and the condition

$$(R_2) \qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|\le\varepsilon} \left|r_2(u,y,\varepsilon;f)-\sqrt{\frac{\pi}{2}}a(f)\right|^2 du = 0\,.$$

Therefore we have by the condition (R_2)

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{|u|\le\varepsilon}\{s(u+\varepsilon;f_1(z))-s(u-\varepsilon;f_1(z)\}du =$$

$$\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{|u|\le\varepsilon} 2ir_1(u+\varepsilon;f)du+\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{|u|\le\varepsilon} 2ir_2(u+\varepsilon;f)du+\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{|u|\le\varepsilon} r_3(u+\varepsilon,y;F)du$$

$$= \frac{2i}{2\varepsilon\sqrt{2\pi}}\int_{|u|\le\varepsilon}\left\{r_2(u+\varepsilon;f)-\sqrt{\frac{\pi}{2}}a(f)\right\}du+ia(f)+o(1)= ia(f)+o(1), \quad (\varepsilon\to 0).$$

Thus we have proved

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_1(z)dx = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{|u|\le\varepsilon}\{s(u+\varepsilon;f_1(z)-s(u-\varepsilon;f_1(z)\}du$$

$$= ia(f).$$

Remark. We shall consider as for $f_1(z)=2C_1(z,f)$ instead of $f_1(z)=C_1(z,F)$, $F(x)=f(x)+i\tilde{f}_1(x)$ and apply the first half part of the Theorem D_3. Then we have the followings.

(i) $|u|\ge\varepsilon$

$$s(u+\varepsilon;f_1(z))-s(u-\varepsilon;f_1(z))=(1+signu)e^{-yu}\left(\{s(u+\varepsilon;f)-s(u-\varepsilon;;f)\}+r_0(u,y,\varepsilon;f)\right)$$

where

$$\frac{1}{2\varepsilon}\int_{|u|\ge\varepsilon}|r_0(u,y,\varepsilon;f)|^2\,du = o(1), \quad (\varepsilon\to 0).$$

(ii) $|u|\le\varepsilon$

$$s(u+\varepsilon;f_1(z))-s(u-\varepsilon;f_1(z))=2ir_1(u+\varepsilon;f)+2ir_2(u+\varepsilon;f)+2r_3(u+\varepsilon;y;F)$$

where $F(x)=f(x)+i\tilde{f}_1(x)$ and

$$\frac{1}{2\varepsilon}\int_{|u|\le\varepsilon}|r_1(u+\varepsilon;f)|^2\,du=o(1), \quad \frac{1}{2\varepsilon}\int_{|u|<\varepsilon}|r_3(u+\varepsilon,y;F)|^2\,du=o(1), \quad (\varepsilon\to 0)$$

and

$$(R_2) \qquad \frac{1}{2\varepsilon}\int_{|u|\le\varepsilon} |r_1(u+\varepsilon;f)-\sqrt{\frac{\pi}{2}}a(f)|^2\,du = o(1) \qquad (\varepsilon\to 0).$$

Then we have

(i) $\lambda \neq 0$

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon;f_1(z))-s(u-\varepsilon;f_1(z))\}du$$

$$=(1+signu)e^{-y\lambda}\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}du\,.$$

(ii) $\lambda = 0$

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{|u|\le\varepsilon|}\{s(u+\varepsilon;f_1(z))-s(u-\varepsilon;;f_1(z))\}du$$

$$=\lim_{\varepsilon\to 0}\frac{2i}{2\varepsilon\sqrt{2\pi}}\int_{|u|\le\varepsilon}\left\{r_1(u+\varepsilon;f)-\sqrt{\frac{\pi}{2}}a(f)\right\}du+ia(f) \ =ia(f).$$

Therefore if we apply the Theorem F_3, then we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_1(z)e^{-i\lambda x}dx = \begin{cases}(1+sign\lambda)e^{-y\lambda}c_\lambda & (\lambda\neq 0)\\ \\ ia(f) & (\lambda=0)\end{cases}$$

where $z=x+iy,\ \ y>0$ and

$$c_\lambda=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x))e^{-i\lambda x}dx \qquad (\forall \ \ real\lambda)$$

Thus we have obtained the same results as above. The second proof, it may be more simple than that of the first.

Step (iv) There exists B_2-almost periodic function $G(z)$ $(z=x+iy, y>0)$ as for variable x and any $y>0$ of which Fourier series are as follows

$$G(z)\sim\sum_{n\neq 0}(1+sign\lambda_n)c_n e^{i\lambda_n z} \qquad (z=x+iy, y>0)$$

where

$$c_n = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x) e^{-i\lambda_n x} dx \qquad (n = 0,1,2,3,...)$$

Because we shall remark that

$$G(x,y) \sim ia(f) + \sum_{\lambda_n > 0} (1 + sign\lambda_n) c_n e^{-\lambda_n y} e^{i\lambda_n x}$$

where

$$c_n e^{-\lambda_n y} e^{i\lambda_n x} = c_n e^{i\lambda_n (x+iy)} = c_n e^{i\lambda_n z},$$

therefore we shall write $G(z)$ as for $G(x,y)$.

Then we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1(z) e^{-i\lambda x} dx = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G(z) e^{-i\lambda x} dx \qquad (\forall real \ \ \lambda).$$

Because if $\lambda \in \Lambda$, then $\lambda = \lambda_n$ for some n and so we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1(z) e^{-i\lambda_n x} dx = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G(z) e^{-i\lambda_n x} dx = \begin{cases} (1 + sign\lambda_n) e^{-\lambda_n y} c_n, & (n \neq 0) \\ ia(f), & (n = 0). \end{cases}$$

and if $\lambda \notin \Lambda$, then we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1(z) e^{-i\lambda x} dx = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G(z) e^{-i\lambda x} dx = 0.$$

Therefore if we set $f_1(z) - G(z) = H(z)$, then $H(z)$ belongs to the class S and we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} H(x+t+iy)\overline{H(t+iy)}dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1(x+t+iy)\overline{f_1(t+iy)}dt - \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G(x+t+iy)\overline{G(t+iy)}dt.$$

Now let us denote $\varphi(x,y; f_1(z))$, $\psi(x,y; G(z))$, and $\chi(x,y; H(z))$ as for their auto -correlation function of $f_1(z)$, $G(z)$, and $H(z)$ respectively. Let us denote also $\sigma(u,y;\varphi)$, $\sigma(u,y;\psi)$, and $\sigma(u,y;\chi)$ as for their G.F.T. respectively.

Then we have

$$\varphi(x,y; f_1(z)) = \psi(x,y; G(z)) + \chi(x,y; H(z))$$

and

$$\sigma(u,y;\varphi) = \sigma(u,y;\psi) + \sigma(u,y;\chi).$$

Since the $\sigma(u,y;\chi)$ is bounded and monotone increasing function, we have by the N.Wiener Theorem (c.f. N.Wiener[1] Theorem 24, pp.146~9)

$$\text{azws} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |H(x+iy)|^2 \ dt = \frac{\sigma(0+,y;\varphi) - \sigma(0-,y;\varphi)}{\sqrt{2\pi}} - |ia(f)|^2$$

$$+\sum_{\lambda_n>0}\left\{\frac{\sigma(\lambda_n+0,y;\varphi)-\sigma(\lambda_n-0,y;\varphi)}{\sqrt{2\pi}}-|2c_ne^{-\lambda_n y}|^2\right\}.$$

Step (v) We have by the Lemma E^*

$$\frac{\sigma(0+,y;\varphi)-\sigma(0-,y;\varphi)}{\sqrt{2\pi}}=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\varepsilon,y;f_1(z))-s(u-\varepsilon,y;f_1(z))|^2\,du$$

We have also by the Theorem A and Theorem D_3

(i) $\lambda>0$

$$\frac{\sigma(\lambda+0,y;\varphi)-\sigma(\lambda-0,y;\varphi)}{\sqrt{2\pi}}$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\varepsilon,y;f_1(z))-s(u-\varepsilon,y;f_1(z))|^2\,du$$

$$=4\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2\,du=4\frac{\sigma(\lambda+0,\varphi)-\sigma(\lambda-0;\varphi)}{\sqrt{2\pi}}$$

(ii) $\lambda=0$ If $|u|<\varepsilon$, we have

$$s(u+\varepsilon,y;f_1(z))-s(u-\varepsilon,y;f_1(z))=2ir_1(u+\varepsilon;f)+2ir_2(u+\varepsilon;f)+2r_3(u+\varepsilon;y;F)$$

where $F(x)=f(x)+i\tilde{f}_1(x)$.

We have also by the hypothesis (R_2) and he Minkowski inequality

$$\frac{\sigma(0+,y;\varphi)-\sigma(0-,y;\varphi)}{\sqrt{2\pi}}$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{|u|<\varepsilon}|s(u+\varepsilon,y;f_1(z))-s(u-\varepsilon,y;f_1(z))|^2\,du=|ia(f)|^2.$$

Therefore we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|H(x+iy)|^2dx=4\sum_{\lambda_n>0}\left\{\frac{\sigma(\lambda_n+0;\varphi)-\sigma(\lambda_n-0;\varphi)}{\sqrt{2\pi}}-|c_ne^{-\lambda_n y}|^2\right\}.$$

In particular if the $\sigma(u;\varphi)$ is continuous on $u>0$, then we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|H(x+iy)|^2dx=0\qquad(\forall\ y>0).$$

We shall call it as Theorem G_2.

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