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On the theory of generalized Hilbert transforms
Chapter V
The spectre analysis and synthesis on the N.Wiener class S

by

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ABSTRACT

We shall continue the problem of spectrum of function of the N.Wiener class S after the preceding section 13 in this research report IV and we shall present the more fine and advanced results.

14. The Spectral Analysis on the N.Wiener class S.

We shall intend to construct the theory of spectral analysis on the N.Wiener class S nuder the hypothesis of which relaxed as in section 13 of IV in this series (c.f. pp.105~114).

We shall denote as before that the function $f(x)$ of the N.Wiener class S and $\varphi(x)$ of its correlation function and also that $s(u)$ and $\sigma(u)$ the G.F.T. of f and φ respectively.

We shall set the presupposed conditions as follows. There exists

$$(C_\lambda) \qquad c_\lambda = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x) e^{-i\lambda x}\, dx \quad (\forall \, real \ \lambda)$$

and

$$(D_\lambda) \qquad d_\lambda = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varphi(x) e^{-i\lambda x}\, dx \quad (\forall real \ \lambda)$$

respectively.

Let us denote

$$\varphi_\varepsilon(x) = \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} e^{i\lambda x} \,|\, s(u+\varepsilon; f) - s(u-\varepsilon; f) \,|^2 \, du$$

and

$$\sigma_\varepsilon(u)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\varphi_\varepsilon(x)\frac{e^{-iux}-1}{-ix}dx+\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\left[\int_{-A}^{-1}+\int_{1}^{A}\right]\varphi_\varepsilon(x)\frac{e^{-iux}}{-ix}dx.$$

Then we have

$$P.V.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi_\varepsilon(x)\frac{e^{-iux}-1}{-ix}dx=\frac{1}{2\varepsilon\sqrt{2\pi}}\int_0^u|s(v+\varepsilon;f)-s(v-\varepsilon;f)|^2dv$$

for any finite range of u and any positive number ε and we have

$$\frac{1}{2\varepsilon\sqrt{2\pi}}\int_0^u|s(v+\varepsilon;f)-s(v-\varepsilon;f)|^2dv=\sigma_\varepsilon(u)+C_\varepsilon \qquad \text{a.e. u.}$$

where

$$C_\varepsilon=P.V.\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-1}+\int_{1}^{\infty}\right]\frac{\varphi_\varepsilon(x)}{ix}dx.$$

We shall refer the above formula to the Research Report III,pp.55~57.

Let us notice that $\sigma_\varepsilon(u)$ is defined on the space L^2 and so it is indeterminable on the set of measure 0. Therefore we shall intend to define $\sigma_\varepsilon(u)$ on the indeterminable set by the above formula for any point of u. Then $\sigma_\varepsilon(u)$ is defined everywhere and it is bounded continuous and monotone increasing function.

Now we shall consider the sequence $\{\sigma_\varepsilon(u)\}$ of which just defined

Since we have

$$\underset{\varepsilon\to 0}{l.i.m.}\sigma_\varepsilon(u)=\sigma(u) \qquad (L^2)$$

and the $\sigma_\varepsilon(u)$ is a bounded continuous and monotone increasing function of u for each fixed positive number ε. Then applying the Paley-Wiener Lemma [2](c.f. p.135),we could conclude that

$$\lim_{\varepsilon\to 0}\sigma_\varepsilon(u)=\sigma(u) \qquad a.e.\ u$$

and $\sigma(u)$ is also a bounded and monotone increasing function on the set of u where $\sigma(u)$ is defined.

14.1 The refinement of properties of $\sigma(u)$

First of all, we shall intend to clear the properties of $\sigma(u)$. The reader should refer

Lemma D_4 of III in this series of research report [3](c.f. pp.52~60).

Now we shall intend to refinement of the properties of the $\sigma(u)$ as follows.

Let us denote the set D of u where the sequence $\{\sigma_\varepsilon(u)\}$ is convergent and the set E of u where it is not convergent. Then we have $D \cup E = (-\infty, +\infty)$, $m(E) = 0$.

We shall define

$$\underline{\sigma}(u) = \sup_{\substack{v<u \\ v\in D}} \sigma(v), \quad \overline{\sigma}(u) = \inf_{\substack{u<v \\ v\in D}} \sigma(v) \quad \text{and} \quad \sigma^*(u) = \frac{\underline{\sigma}(u) + \overline{\sigma}(u)}{2}$$

respectively.

We shall intend to investigate properties of the $\sigma^*(u)$

(i) The $\sigma^*(u)$ is defined at every point of u and bounded, monotone increasing function .

Proof. Since and $\overline{\sigma}(u)$ are $\underline{\sigma}(u)$ both to be bounded and so $\sigma^*(u)$ does too.

Since $\sigma(u)$ is bounded and monotone increasing on D, we have for any pair of (u', u'') such as $u' < u''$

$$\sup_{\substack{v'<u' \\ v'\in D}} \sigma(v') \le \sup_{\substack{v''<u'' \\ v''\in D}} \sigma(v'')$$

and so we have

$$\underline{\sigma}(u') \le \underline{\sigma}(u'').$$

Then $\underline{\sigma}(u)$ is defined at every point of u and a bounded ,monotone increasing function.

Similarly we have the same property as $\overline{\sigma}(u)$ too.

Therefore we have

$$\sigma^*(u') = \frac{\underline{\sigma}(u') + \overline{\sigma}(u')}{2} \le \frac{\underline{\sigma}(u'') + \overline{\sigma}(u'')}{2} = \sigma^*(u'').$$

(ii) The $\sigma^*(u)$ satisfies at every point of u the following properties

$$\sigma^*(u-0) \le \underline{\sigma}(u) \quad \text{and} \quad \overline{\sigma}(u) \le \sigma^*(u+0).$$

Proof. Since $\sigma(u)$ is bounded, monotone increasing function, we have for any pair of (u', u'') such as $u' < u''$

$$\overline{\sigma}(u') = \inf_{\substack{u'<v' \\ v'\in D}} \sigma(v') = \inf_{\substack{u'<v'<u'' \\ v'\in D}} \sigma(v') \le \sup_{\substack{u'<v''<u'' \\ v''\in D}} \sigma(v'') = \sup_{\substack{v''<u'' \\ v''\in D}} \sigma(v'') = \underline{\sigma}(u'').$$

Therefore we have proved for any pair of (v,u) such as $v<u$,

$$\underline{\sigma}(v) \le \overline{\sigma}(v) \le \underline{\sigma}(u)$$

and so

$$\sigma^*(v) = \frac{\underline{\sigma}(v) + \overline{\sigma}(v)}{2} \le \underline{\sigma}(u).$$

Then if we tend $v \uparrow u$, we have by the property(i) of $\sigma^*(u)$

$$\sigma^*(u-0) \le \underline{\sigma}(u).$$

Similarly we have

$$\overline{\sigma}(u) \le \sigma^*(u+0).$$

(iii) The $\sigma^*(u) = \sigma(u)$ on the set D and continuous at any point u of the set D.

Proof. First of all we shall remark that

$$\underline{\sigma}(u) \le \sigma(u) \le \overline{\sigma}(u) \qquad (\forall u \in D).$$

Let us suppose that $\overline{\sigma}(u) - \underline{\sigma}(u) > 0$ at a point u of the set D. In particular let us suppose that $\sigma(u) - \underline{\sigma}(u) = \eta > 0$.

Since $\lim_{\varepsilon \to 0} \sigma_\varepsilon(u) = \sigma(u)$, for any $\eta_1 > 0$ there exists $\varepsilon' > 0$ such that

(a) $$|\sigma_\varepsilon(u) - \sigma(u)| < \eta_1 \qquad (0 < \varepsilon < \varepsilon').$$

Since $\sigma_\varepsilon(v) \to \sigma_\varepsilon(u)$ as $v \in D$ and $v \uparrow u$, we have for any $\varepsilon\ (0 < \varepsilon < \varepsilon')$ to be fixed and for any $\eta_2 > 0$ there exists $\delta > 0$ such that

(b) $$|\sigma_\varepsilon(v) - \sigma(u)| < \eta_2 \qquad (u - \delta < v < u).$$

Since $\lim_{\varepsilon \to 0} \sigma_\varepsilon(v) = \sigma(v)$, for any $\eta_3 > 0$ there exists $\varepsilon'' > 0$ such that

(c) $$|\sigma_\varepsilon(v) - \sigma(v)| < \eta_3 \qquad (0 < \varepsilon < \varepsilon'')$$

Then we have for any $\varepsilon > 0$ to be fixed such as $0 < \varepsilon < min(\varepsilon', \varepsilon'')$

$$\sigma(v) > \sigma_\varepsilon(v) - \eta_3 > \sigma_\varepsilon(u) - \eta_2 - \eta_3 > \sigma(u) - \eta_1 - \eta_2 - \eta_3 = \sigma(u) - (\eta_1 + \eta_2 + \eta_3)$$

by combining the above estimations (a), (b) and (c). Then we could take $\eta_1 + \eta_2 + \eta_3 < \eta$ and we shall prove that there exists $v < u$ and $v \in D$ such that

$$\sigma(v) > \sigma(u) - \eta = \underline{\sigma}(u).$$

This lead to the contradiction that $\sigma(v) \le \underline{\sigma}(u) \qquad (\forall v < u, v \in D)$.

Thus we shall conclude that

$$\underline{\sigma}(u) = \sigma(u)$$

at any point u of the set D.

Similarly we shall conclude that

$$\overline{\sigma}(u) = \sigma(u) \qquad (\forall u \in D)$$

and

$$\sigma^*(u) = \frac{\underline{\sigma}(u) + \overline{\sigma}(u)}{2} = \sigma(u) \qquad (\forall u \in D).$$

Now we have

$$\underline{\sigma}(u) = \sup_{\substack{v<u \\ v\in D}} \sigma(v) = \sup_{\substack{v<u \\ v\in D}} \sigma^*(v).$$

Since $\sigma^*(u)$ is a bounded, monotone increasing function, we have for any given positive number $\varepsilon > 0$, there exists $v' < u$ such that

$$\sigma^*(u-0) - \varepsilon < \sigma^*(v'),$$

where since m(E)=0, we can pick up a point v' in the set D, then we have

$$\sigma^*(v') \le \sup_{\substack{v<u \\ v\in D}} \sigma^*(v) = \sup_{\substack{v<u \\ v\in D}} \sigma(v) = \underline{\sigma}(u).$$

Therefore we have

$$\sigma^*(u-0) - \varepsilon < \underline{\sigma}(u).$$

Thus we have proved

$$\sigma^*(u-0) \le \underline{\sigma}(u).$$

On the other hand for any point v in the set D such as $v < u$, we can pick up v' of the set D such as $v < v' < u$,then since

$$\sigma(v) \le \sigma(v') = \underline{\sigma}(v') = \overline{\sigma}(v'),$$

we have

$$\sigma(v) \le \sigma^*(v') = \frac{\underline{\sigma}(v') + \overline{\sigma}(v')}{2}.$$

By tending $v' \uparrow u$, we shall conclude that

$$\sigma(v) \le \sigma^*(u-0),$$

for any point v in the set D such as $v < u$.

Therefore we have

$$\underline{\sigma}(u) = \sup_{\substack{v<u \\ v\in D}} \sigma(v) \le \sigma^*(u-0).$$

Combining the two inequalities to be obtained, we have

$$\sigma^*(u-0) = \underline{\sigma}(u).$$

Similarly we have

$$.\sigma^*(u+0)=\overline{\sigma}(u).$$

But since we have $\overline{\sigma}(u)=\underline{\sigma}(u)=\sigma(u)$ at any point of u of the set D and so $\sigma^*(u)=\sigma(u)$, therefore we have

$$\sigma^*(u-0)=\sigma^*(u+0)=\sigma^*(u)$$

at any point u of the set D.

Thus we shall prove that

$$\sigma^*(u)=\sigma(u)$$

at any point u of the set D and $\sigma^*(u)$ is continuous bounded, and monotone increasing function on the set D.

(iv) The $\sigma^*(u)$ is discontinuous of the first kind for every point u of the set E and the set E is at most countable.

Proof. In the first we shall intend to prove that

$$\underline{\sigma}(u)<\overline{\sigma}(u)$$

for any point u of the set E.

Let us suppose that $\underline{\sigma}(u)=\overline{\sigma}(u)$ at a point u of the set E. Then any pairs of $\varepsilon,\varepsilon'>0$, we have

$$\sigma_\varepsilon(u-\varepsilon')\le\sigma_\varepsilon(u)\le\sigma_\varepsilon(u+\varepsilon').$$

Since the measure of the set E is 0 and so , for any point of u of the set E, there exists a sequence of points $\{u\pm\varepsilon'\}$ such that $u\pm\varepsilon'\in D$ and $\{\varepsilon'\}\downarrow 0$. In the first we shall intend $\{\varepsilon\}\downarrow 0$, then we have

$$\sigma(u-\varepsilon')\le\underline{lim}_{\varepsilon\to 0}\sigma_\varepsilon(u)\le\overline{lim}_{\varepsilon\to 0}\sigma_\varepsilon(u)\le\sigma(u+\varepsilon')$$

Next we shall intend the $\{\varepsilon'\}\downarrow 0$. Since the $\sigma(u)$ is bounded monotone increasing function on the set D, we have

$$\lim_{\varepsilon\to 0}\sigma(u-\varepsilon')=\sup_{\substack{v<u\\ v\in D}}\sigma(v)=\underline{\sigma}(u)$$

and

$$\lim_{\varepsilon\to 0}\sigma(u+\varepsilon')=\inf_{\substack{u<v\\ v\in D}}\sigma(v)=\overline{\sigma}(u).$$

Therefore we have

$$\underline{\sigma}(u) \le \varliminf_{\varepsilon\to 0} \sigma_\varepsilon(u) \le \varlimsup_{\varepsilon\to 0} \sigma_\varepsilon(u) \le \overline{\sigma}(u).$$

Since we have assumed that $\underline{\sigma}(u) = \overline{\sigma}(u)$ at the point u, we have

$$\varliminf_{\varepsilon\to 0} \sigma_\varepsilon(u) = \varlimsup_{\varepsilon\to 0} \sigma_\varepsilon(u)$$

and so $\lim_{\varepsilon\to 0} \sigma_\varepsilon(u)$ exists. Thus it lead to the contradiction of the point u of set E

where $\lim_{\varepsilon\to 0} \sigma_\varepsilon(u)$ does not exists.

Next we shall intend to prove that the set E is at most countable.
For any pair of (u,v) such as $u < v$, we have

$$\overline{\sigma}(u) = \inf_{\substack{u<u' \\ u'\in D}} \sigma(u') \le \sup_{\substack{v'<v \\ v'\in D}} \sigma(v') = \underline{\sigma}(v)$$

and $\underline{\sigma}(v) \le \overline{\sigma}(v)$. Therefore we have

$$\overline{\sigma}(u) \le \sigma^*(v) = \frac{\underline{\sigma}(v) + \underline{\sigma}(v)}{2}$$

Now we shall tend $v \downarrow u$, then we have

$$\overline{\sigma}(u) \le \sigma^*(u+0),$$

for any point u of the set E.

Similarly we have

$$\sigma^*(u-0) \le \underline{\sigma}(u),$$

for any point u of the set E and therefore we have

$$\sigma^*(u+0) - \sigma^*(u-0) \ge \overline{\sigma}(u) - \underline{\sigma}(u) > 0,$$

for any point u of the set E.

Since the $\sigma^*(u)$ is bounded, monotone increasing function, we shall conclude that the $\sigma^*(u)$ is discontinuous of the first kind at each point u of the set E and the set E is at most countable.

Thus we have proved that the function $\sigma^*(u)$ could be defined everywhere and satisfies the following properties.

The function $\sigma^*(u)$ is bounded, monotone increasing function of u and furthermore

(a) On any point u of the set D, we have $\sigma^*(u) = \sigma(u)$ and moreover

$$\sigma^*(u-0) = \sigma^*(u+0) = \sigma^*(u)$$

and so it is continuous there.

(b) On any point u of the set E, we have

$$\sigma^*(u-0) \le \underline{\sigma}(u) \quad \text{and} \quad \overline{\sigma}(u) \le \sigma^*(u+0)$$

and so it is discontinuous of the first kind and has the magnitude of jump

$$\sigma^*(u+0) - \sigma^*(u-0) \ge \overline{\sigma}(u) - \underline{\sigma}(u) > 0$$

there.

It should be remarked that the set E is at most countable.

14.2 The spectral synthesis of function of the N.Wiener class S.

Let us begin to prove the general properties of function $f(x)$ in the N.Wiener class S under the hypothesis stated in the beginning of section 14.1 as follows.

There exists

$$(C_\lambda) \qquad c_\lambda = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x) e^{-i\lambda x}\, dx \quad (\forall real\ \lambda)$$

and

$$(D_\lambda) \qquad d_\lambda = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varphi(x) e^{-i\lambda x}\, dx \quad (\forall real\ \lambda)$$

respectively.

Then we shall prove

Theorem E. Let us suppose that the function $f(x)$ of which belongs to the N.Wiener class S and satisfies the conditions (C_λ) and (D_λ) for all real λ.

Then we have

(I) $$|c_\lambda|^2 \le d_\lambda \qquad (\forall real\ \lambda)$$

(II) There exist at most countable set of real number $\lambda = \{\lambda_n\}$ $(n = 0,1,2,\cdots\cdots)$

and it satisfies followings

(i) $$c_{\lambda_n} \ne 0 \qquad (n = 0,1,2,\cdots\cdots),$$

where we shall denote the c_n instead of c_{λ_n} $(n = 0,1,2,\cdots\cdots)$ and $\lambda_0 = 0$ with $c_0 = 0$

may be permitted.

and

(ii) $$c_\lambda = d_\lambda = 0 \qquad (\forall \lambda \notin \Lambda)$$

There exist B_2 almost periodic function $g(x)$ of which can be expressed as its Fourier series

(iii) $$g(x) \sim \sum_n c_n e^{i\lambda_n x}.$$

If we decompose $f(x)$ as follows

(iv) $$f(x) = g(x) + h(x)$$

then we have

(v) $$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |h(x)|^2 dx = \frac{1}{2\pi} \sum_n (d_n - |c_n|^2)$$

After Prof.N.Wiener we shall denote $f(x)$ and $s(u)$ as its G.F.T. and also $\varphi(x)$ and $\sigma(u)$ as its correlation function and G.F.T. respectively. N.Wiener[1](c.f. p.159) also introduced the following functions

(21.175) $$\varphi_\varepsilon(x) = \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} e^{i\lambda x} | s(u+\varepsilon; f) - s(u-\varepsilon; f) |^2 \, du$$

and

(21.22) $$\sigma_\varepsilon(u) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \varphi_\varepsilon(x) \frac{e^{-iux}-1}{-ix} dx + \underset{A\to\infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \left[\int_{-A}^{-1} + \int_{1}^{A} \right] \varphi_\varepsilon(x) \frac{e^{-iux}}{-ix} dx$$

Then R.E.A.C.Paley-N.Wiener[2](c.f. p.135) proved that

$$\lim_{\varepsilon\to 0} \sigma_\varepsilon(u) = \sigma(u) \qquad (a.e.\ \ u).$$

We have proved in III in this research report [3] (c.f. pp.52~60)

Lemma D_4. Let us suppose that $f(x)$ belongs to the class S, then the following limit

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{-\varepsilon}^{u} | s(v+\varepsilon; f) - s(v-\varepsilon; f) |^2 dv = \lim_{\varepsilon\to 0} (\sigma_\varepsilon(u) - \sigma_\varepsilon(-\varepsilon))$$

exists and equals to

$$\sigma(u) - \sigma(-0) \qquad (a.e.\ \ u)$$

over any finite range of u.

Now we shall begin to refine Lemma D_4 for the sake of completeness and prove

Lemma E. Let us suppose that $f(x)$ belongs to the class S, then the following

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{\pm\varepsilon}^{u\pm\varepsilon} | s(v+\varepsilon; f) - s(v-\varepsilon; f) |^2 dv = \lim_{\varepsilon\to 0} (\sigma_\varepsilon(u\pm\varepsilon) - \sigma_\varepsilon(\pm\varepsilon))$$

exists and equals to

$$\sigma(u \pm 0) - \sigma(\pm 0)$$

for any point u, respectively.

Proof of Lemma E.

(i) Let us suppose that $0 \in D$, then we have $\lim_{\varepsilon \to 0} \sigma_\varepsilon(0) = \sigma(0)$,where $\{\varepsilon\}$ is any sequence of positive number ε and tend to 0.

Since $\{\sigma_\varepsilon(u)\}$ is bonded, monotone increasing as function of u for each positive number ε, we have

$$\sigma_\varepsilon(-u) \le \sigma_\varepsilon(-\varepsilon) \le \sigma_\varepsilon(0) \qquad (0 < \varepsilon < u).$$

Then tending the $\varepsilon \to 0$, for any point u such as $-u \in D$ where $\sigma_\varepsilon(-u) \to \sigma(-u)$, we have

$$\sigma(-u) \le \underline{\lim}_{\varepsilon \to 0} \sigma_\varepsilon(-\varepsilon) \le \overline{\lim}_{\varepsilon \to 0} \sigma_\varepsilon(-\varepsilon) \le \sigma(0).$$

In the last tending $u \to 0$ such as $-u \in D$ where $\sigma(-u) \to \sigma(0) = \sigma(-0)$,thus we have

$$\lim_{\varepsilon \to 0} \sigma_\varepsilon(-\varepsilon) = \sigma(-0).$$

Similarly we have

$$\lim_{\varepsilon \to 0} \sigma_\varepsilon(+\varepsilon) = \sigma(+0).$$

(ii) Let us suppose that $0 \notin D$, that is $0 \in E$. Since the following limit $\lim_{\varepsilon \to 0} \sigma_\varepsilon(0)$ does not exist, we shall define

$$\sigma(0) = \frac{\sigma(-0) + \sigma(+0)}{2}$$

where

$$\sigma(-0) = \lim_{\substack{u \uparrow 0 \\ u \in D}} \sigma(u) \qquad \text{and} \qquad \sigma(+0) = \lim_{\substack{u \downarrow 0 \\ u \in D}} \sigma(u)$$

respectively.

Let us put $\sigma(+0) - \sigma(-0) = d > 0$ and define

$$\sigma^{\sim}(u) = \sigma(u) - dh(u)$$

where $h(u)$ is the Heaviside operator, that is as follows

$$h(u)=\begin{cases} 1 & (u>0) \\ \frac{1}{2} & (u=0) \\ 0 & (u<0). \end{cases}$$

Then $\sigma^{\sim}(u)$ is continuous at $u=0$. Because we have

$$\sigma^{\sim}(-0)=\sigma(-0)$$

$$\sigma^{\sim}(0)=\sigma(0)-\frac{\sigma(+0)-\sigma(-0)}{2}=\frac{\sigma(-0)+\sigma(+0)}{2}-\frac{\sigma(+0)-\sigma(-0)}{2}=\sigma(-0)$$

$$\sigma^{\sim}(+0)=\sigma(+0)-(\sigma(+0)-\sigma(-0))=\sigma(-0)$$

respectively. Now let us put

$$\sigma_{\varepsilon}^{\sim}(u)=\sigma_{\varepsilon}(u)-dh(u)$$

and consider the sequence $\{\sigma_{\varepsilon}^{\sim}(u)\}$ instead of $\{\sigma_{\varepsilon}(u)\}$, then it is continuous at any point u of the set D. Because we have

$$\lim_{\varepsilon\to 0}\sigma_{\varepsilon}^{\sim}(u)=\lim_{\varepsilon\to 0}\sigma_{\varepsilon}(u)-dh(u)=\sigma(u)-dh(u)=\sigma^{\sim}(u)$$

and $\sigma^{\sim}(u)$ is continuous at $u=0$. Then we have

$$\lim_{\varepsilon\to 0}\sigma^{\sim}(-\varepsilon)=\sigma^{\sim}(-0)$$

where

$$\sigma_{\varepsilon}^{\sim}(-\varepsilon)=\sigma_{\varepsilon}(-\varepsilon)-dh(-\varepsilon)=\sigma_{\varepsilon}(-\varepsilon) \quad \text{and} \quad \sigma^{\sim}(-0)=\sigma(-0)$$

Therefore we have

$$\lim_{\varepsilon\to 0}\sigma_{\varepsilon}(-\varepsilon)=\sigma(-0)$$

Similarly we have

$$\lim_{\varepsilon\to 0}\sigma^{\sim}(+\varepsilon)=\sigma^{\sim}(+0)$$

where

$$\sigma_{\varepsilon}^{\sim}(+\varepsilon)=\sigma_{\varepsilon}(+\varepsilon)-dh(+\varepsilon)=\sigma_{\varepsilon}(+\varepsilon)-(\sigma(+0)-\sigma(-0))$$

and

$$\sigma^{\sim}(+0)=\sigma(+0)-d=\sigma(+0)-(\sigma(+0)-\sigma(-0))=\sigma(-0)$$

Thus we have proved

$$\lim_{\varepsilon\to 0}\{\sigma_{\varepsilon}(+\varepsilon)-(\sigma(+0)-\sigma(-0))\}=\sigma(-0)$$

and thus we have proved

$$\lim_{\varepsilon\to 0}\sigma_{\varepsilon}(+\varepsilon)=\sigma(+0)$$

In general we shall prove by the same argument as above at a point $u=0$,

$$\lim_{\varepsilon\to 0}\sigma_{\varepsilon}(u\pm\varepsilon)=\sigma(u\pm 0)$$

at any point u.(Notice: In these cases, we should remark that we would use $\sigma(\pm 0)$ etc. instead as the ordinary notations $\sigma(0\pm)$ etc. respectively).

Proof of the Theorem E. First of all let us1 notice that we have

$$c_{\lambda}=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)e^{-i\lambda x}dx \qquad (\forall real \ \ \lambda).$$

Then we have by the hypotheses (C_{λ}) and the one sided Wiener formula,

$$|c_{\lambda}|^{2}=|\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)e^{-i\lambda x}dx|^{2}=|\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(v+\varepsilon;f)-s(v-\varepsilon;f)\}dv|^{2}$$

$$\leq\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(v+\varepsilon;f)-s(v-\varepsilon;f)|^{2}dv=\frac{\sigma(\lambda+0;\varphi)-\sigma(\lambda-0;\varphi)}{\sqrt{2\pi}}$$

and we have by the hypotheses (D_{λ}) and one-sided Wiener formula too

$$d_{\lambda}=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\varphi(x;f)e^{-i\lambda x}dx$$

$$=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{\sigma(u+\varepsilon;\varphi)-\sigma(u-\varepsilon;\varphi)\}du=\frac{\sigma(\lambda+0;\varphi)-\sigma(\lambda-0;\varphi)}{\sqrt{2\pi}}$$

for all real λ where we should apply the Lemma E in each of the last formula.
Then we have

$$|\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)e^{-i\lambda x}dx|^{2}\leq\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\varphi(x;f)e^{-i\lambda x}dx$$

for all real λ. Therefore we have

$$|c_{\lambda}|^{2}\leq d_{\lambda} \qquad (\forall real \ \lambda)$$

According to Lemma E, we shall expect as follows. Let us put the set $E=\{\lambda_{n}\}$ $(n=0,1,2,\cdots\cdots)$ as the set Λ and the set D as its complement. Let us put

$$d_\lambda = \begin{cases} 0, & \lambda \in D \\ \dfrac{\sigma(\lambda_n+0;\varphi)-\sigma(\lambda_n-0;\varphi)}{\sqrt{2\pi}}, & (n=0,1,2,\cdots\cdots) \end{cases}$$

and

$$c_\lambda = \begin{cases} 0, & \lambda \in D \\ c_n & (n=0,1,2,\cdots\cdots) \end{cases}$$

Then we shall acknowledge that these settings are true and coincide with the assertions of Theorem E.

We have

$$\sum_n |c_n|^2 \le \sum_n d_n = \sum_n \frac{\sigma(\lambda_n+0;\varphi)-\sigma(\lambda_n-0;\varphi)}{\sqrt{2\pi}} \le \frac{\sigma(+\infty;\varphi)-\sigma(-\infty;\varphi)}{\sqrt{2\pi}} < \infty.$$

Therefore we shall conclude that there exists a B_2-almost periodic function $g(x)$ of which Fourier series is as follows

$$g(x) \sim \sum_n c_n e^{i\lambda_n x}.$$

Then if we put

$$f(x)-g(x)=h(x)$$

say, and we shall consider correlation functions $\varphi(x;f)$, $\psi(x;g)$ and $\chi(x;h)$ of f,g,h ; their G.F.T. $\sigma(u;\varphi),\sigma(u;\psi)$ and $\sigma(u;\chi)$ of φ,ψ,χ respectively. Let us remark that

$$c_\lambda = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\lambda x}dx = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x)e^{-i\lambda x}dx \qquad (\forall real\ \lambda).$$

Then by repeating the same arguments as IV in this research report [4] (c.f. pp.105~108), we shall prove

$$\varphi(x;f)=\psi(x;g)+\chi(x;h)$$

and

$$\sigma(u;\varphi)=\sigma(u;\psi)+\sigma(u;\chi)$$

respectively.

Then by the Lemma E, $\sigma(u;\varphi)$ is continuous on the set D and discontinuous of the first kind with jump on the set E such as

$$d_n = \frac{\sigma(\lambda_n+0;\varphi)-\sigma(\lambda_n-0;\varphi)}{\sqrt{2\pi}}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{\lambda_n - \varepsilon}^{\lambda_n + \varepsilon} \{\sigma(u+\varepsilon;\varphi) - \sigma(u-\varepsilon;\varphi)\} du \qquad (n = 0,1,2,\cdots\cdots)$$

On the other hand , since $\sigma(u;\psi)$ is G.F.T. of $\psi(x;g)$ and $\psi(x;g)$ is the correlation function of B_2 -almost periodic function $g(x)$, we have

$$\sigma(u;\psi) = \begin{cases} \sqrt{2\pi} \sum_{\lambda_n < u} |c_n|^2 & (u \neq \lambda_m) \\ \sqrt{2\pi} \left(\sum_{\lambda_n < u} |c_n|^2 + \frac{1}{2} |c_m|^2 \right) & (u = \lambda_m) \end{cases}$$

Then $\sigma(u;\psi)$ is continuous on the set D and discontinuous of the first kind with jump on the set E such as

$$|c_n|^2 \leq \frac{\sigma(\lambda_n + 0;\psi) - \sigma(\lambda_n - 0;\psi)}{\sqrt{2\pi}} \qquad (n = 0,1,2,\cdots\cdots)$$

Thus we have

$$|c_n|^2 \leq d_n \qquad (n = 0,1,2,\cdots\cdots)$$

on the set E.
Furthermore we have

$$c_\lambda = d_\lambda = 0 ,$$

on the set D.
Therefore we shall prove that $\sigma(u;\chi)$ is bounded, monotone increasing function. Since $\sigma(u;\chi)$ is G.F.T. of $\chi(x;h)$ and $\chi(x;h)$ is the correlation function of $h(x)$, we have by the N.Wiener Theorem[1](Theorem 24,pp. 146~149)

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |h(x)|^2 \, dx = \frac{1}{2\pi} \sum_n (d_n - |c_n|^2).$$

In particular, if it is satisfied the condition

$$d_n - |c_n|^2 = 0 \qquad (n = 0,1,2,\cdots\cdots)$$

then we have

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |h(x)|^2 dx = 0 .$$

Thus we have proved the Theorem E completely.

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