Panjer's and related families of distributions in risk theory

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The present talk gives some extensions of Panjer's (1981) family in risk theory, which were developed by Sundt and Jewell (1981), Schröter (1990), Sundt (1992), and Kitano, Shimizu and Ong (2005). A family of distributions which include the generalized Charlier series distribution given by Kitano et al. is proposed and its probability mass function and descending factorial moments are studied.

References (in chronological order)

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3. Schröter, K. J. (1990). On a family of counting distributions and recursions for related compound distributions, Scandinavian Actuarial Journal, 161–175.

4. Sundt, B. (1992). On some extensions of Panjer's class of counting distributions, Astin Bulletin, 22, 61–80.

5. Kitano, M., Shimizu, K. and Ong, S. H. (2005). The generalized Charlier series distribution as a distribution with two-step recursion, Statistics & Probability Letters, 75, 280–290.

Applications to meteorological elements such as the total amount of precipitation and snow

References

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2. Le Cam, L. (1961). A stochastic description of precipitation, Proc. of the Fourth Berkeley Symp. on Mathematical Statistics and Probability, Vol. III, Univ. Calif. Press, Berkeley, 165–186.

3. Siegel, A. F. (1985). Modelling data containing exact zeroes using zero degrees of freedom, J. R. Statist. Soc. B 47, 267–271.

Panjer's (1981) family for probability mass function:

$$P_k = \left(a + \frac{b}{k}\right) P_{k-1}, \quad k \geq 1$$

for some constants a and b specifying the claim number distribution.

Panjer's family of the number of claims is characterized by the binomial, Poisson and negative binomial distributions and a degenerate distribution at zero.

Sundt and Jewell (1981) family:

$$P_k = \left(a + \frac{b}{k}\right) P_{k-1}, \quad k \geq 2,$$

which contains the distributions of Panjer's family and the log-series distribution

$$P_k = \frac{1}{-\log(1-p)} \frac{p^k}{k}, \quad k \geq 1; \quad 0 < p < 1$$

as well as their modified distributions at zero

$$p_0 = w + (1-w)P_0; \quad p_k = (1-w)P_k, \quad k \geq 1$$

for $-P_0/(1-P_0) \leq w < 1$**.**

Schröter (1990) family:

$$P_k = \left(a + \frac{b}{k}\right) P_{k-1} + \frac{c}{k} P_{k-2}, \qquad k \geq 2$$

for some constants a, b and c specifying the claim number distribution.

Example 1: Hermite distribution (Kemp and Kemp, 1966), which is a compound or generalized or stopped-sum distribution whose random variable expression is

$$X = \begin{cases} \sum_{i=1}^{K} Z_i, & K > 0, \\ 0, & K = 0 \end{cases}$$

for $K \sim$ Poisson(λ), $Z_i \overset{\text{iid}}{\sim}$ Binomial$(2, p)$, independent of K. X can be viewed as the total amount of claims.

Example 2: Charlier series distribution (Ong, 1988), which is the sum of two independent distributions:

$$\textbf{Binomial}(N, p) \textbf{ and } \textbf{Poisson}(\lambda p)$$

$$\begin{aligned} P_k &= e^{-\lambda p}(\lambda p)^k q^N C_k(N; -\lambda q)/k! \\ &= \binom{N}{k} p^k (1-p)^{N-k} e^{-\lambda} {}_1F_1(N+1; N-k+1; \lambda q), \quad k \geq 0 \end{aligned}$$

for a positive integer N, $0 < p = 1 - q < 1$ and $\lambda > 0$, where $C_k(x; a)$ denotes the Charlier polynomial

$$C_k(x; a) = (x - k + 1)_k (-a)^{-k} {}_1F_1(-k; x - k + 1; a)$$

and ${}_1F_1$ the confluent hypergeometric function

$${}_1F_1(a; c; x) = \sum_{j=0}^{\infty} \frac{(a)_j}{(c)_j} \frac{x^j}{j!}.$$

Sundt (1992) family:

$$P_k = \sum_{i=1}^{n} \left(a_i + \frac{b_i}{k} \right) P_{k-i}, \quad k \geq 1 \quad (P_k = 0 \text{ if } k < 0)$$

for some vectors $a = (a_1, \ldots, a_n)', b = (b_1, \ldots, b_n)'$ specifying the claim number distribution.

Example ($n = 2$): Non-central Negative Binomial Distribution (Ong and Lee, 1979), which is a mixture distribution of K, where

$$K|\theta \sim \textbf{Poisson}(\theta), \ \theta|N \sim \textbf{Gamma}(v + N, p/q), \ N \sim \textbf{Poisson}(\lambda)$$

for $0 < p = 1 - q < 1, \ v > 0, \ \lambda > 0$ (the mean of Gamma(α, β) is $\alpha\beta$ and the variance is $\alpha\beta^2$).

Kitano, Shimizu and Ong (2005) (KSO in short) family:

$$P_k = \left(a + \frac{b}{k}\right) P_{k-1} + \left(c + \frac{d}{k} + \frac{e}{k-1}\right) P_{k-2}, \quad k \geq 2$$

for some constants a, b, c, d and e specifying the claim number distribution.

Example: Generalized Charlier Series Distribution $(c = 0)$, which is the sum of two independent distributions: Binomial(N, p) with probability generating function (pgf) $(pt+q)^N$ and a distribution with pgf

$$\frac{{}_1F_1(s; N+1; \lambda(pt+q))}{{}_1F_1(s; N+1; \lambda)}$$

for a non-negative integer N, $0 < p = 1 - q < 1,\ s > 0$ and $\lambda > 0$.

(continued)

The last is the confluent hypergeometric distribution generalized by a generalizing Bernoulli distribution, i.e., it is a compound distribution whose random variable expression is

$$X = \begin{cases} \sum_{i=1}^{K} Z_i, & K > 0, \\ 0, & K = 0, \end{cases}$$

where K obeys the confluent hypergeometric distribution (Hall, 1956; Bhattacharya, 1966) with vector parameter (a, b, θ) and pgf

$$G(t) = \frac{{}_1F_1(a; b; t\theta)}{{}_1F_1(a; b; \theta)}, \quad a > 0,\ b > 0,\ \theta > 0$$

and Z_i are independently distributed as a Bernoulli(p), independent of K.

Recursions for GCSD and KSO families:

1 . The GCSD probability mass function has a closed form and satisfies a three-term recursion based on the recursion for the confluent hypergeometric function.

2 . The r-th descending factorial moment of the GCSD has a closed form and satisfies a three-term recursion based on the same reason as the above.

3 . The probability mass function of

$$X = \begin{cases} \sum_{i=1}^{K} Z_i, & K > 0, \\ 0, & K = 0 \end{cases}$$

satisfies a recursion if K belongs to the KSO family and Z_i are independent of K and independently distributed random variables which take non-negative integers.

An extension of the GCSD:

Hint

pgf of GCSD

$$E(t^X) = (pt+q)^N \frac{{}_1F_1(s; N+1; \lambda(pt+q))}{{}_1F_1(s; N+1; \lambda)}$$

for a non-negative integer N**,** $0 < p = 1-q < 1,\ s > 0$ **and** $\lambda > 0$**.**

pgf of extended NNBD (Ong and Shimizu, submitted)

$$E(t^X) = \left(\frac{p}{1-qt}\right)^k \frac{{}_1F_1(k; c; \lambda p/(1-qt))}{{}_1F_1(k; c; \lambda)}, \quad -1 < qt < 1$$

for $k > 0,\ 0 < p = 1-q < 1,\ c > 0$ **and** $\lambda > 0$**.**

(continued)

pgf of the sum of independent

and
$$X_1 \sim \textbf{Binomial}(n, p_2/(p_1+p_2))$$
$$X_2 \sim \textbf{NegativeBinomial}(n, p_3)$$

for a positive integer n, $p_1 > 0$, $p_2 > 0$, $p_3 > 0$ and $p_1+p_2+p_3 = 1$ ($\mathbf{GIT}_{3,1}$ of Aoyama, Shimizu and Ong, 2008, AISM: a particular member of Kemp's class of convolution of pseudo-binomial variables)

$$E(t^{X_1+X_2}) = \left(\frac{p_1+p_2t}{p_1+p_2}\right)^n \left(\frac{1-p_3}{1-p_3t}\right)^n = \left(\frac{p_1+p_2t}{1-p_3t}\right)^n,$$

which reduces to a shifted negative binomial if $p_1 \to 0$, to a negative binomial if $p_2 \to 0$, and to a binomial if $p_3 \to 0$.

Proposed distribution with pgf:

$$p(t) = \left(\frac{1-p_3}{1-p_3 t}\right)^{k_1} \left(\frac{p_1+p_2 t}{p_1+p_2}\right)^{k_2} \frac{{}_1F_1\left(k_1; k_2+1; \lambda\left(\frac{p_1+p_2 t}{1-p_3 t}\right)\right)}{{}_1F_1(k_1; k_2+1; \lambda)}$$

where $k_1 \geq 0$, k_2 **is a non-negative integer (for simplicity; possible for** $k_2 + 1 \leq 0$**),** $p_1, p_2, p_3 > 0$, $p_1 + p_2 + p_3 = 1$, $\lambda > 0$.

The proposed distribution reduces to a GCSD with pmf

$$p(t) = (p_1 + p_2 t)^{k_2} \frac{{}_1F_1\left(k_1; k_2+1; \lambda(p_1+p_2 t)\right)}{{}_1F_1(k_1; k_2+1; \lambda)}$$

if $p_3 \to 0$ **and to an extended NNBD (restricted form because** k_2 **is a non-negative integer) with pmf**

$$p(t) = \left(\frac{p_1}{1-p_3 t}\right)^{k_1} \frac{{}_1F_1\left(k_1; k_2+1; \lambda\left(\frac{p_1}{1-p_3 t}\right)\right)}{{}_1F_1(k_1; k_2+1; \lambda)}$$

if $p_2 \to 0$**.**

Formulation:

$N \sim$ **Confluent Hypergeometric Distribution** (k_1, k_2+1, λ) **with pgf**

$$E(t^N) = \frac{{}_1F_1(k_1; k_2+1; t\lambda)}{{}_1F_1(k_1; k_2+1; \lambda)}$$

$X_j \overset{\text{iid}}{\sim}$ $\text{GIT}_{3,1}(1; p_1, p_2, p_3)$ **with pgf** $E(t^{X_j}) = \frac{p_1+p_2t}{1-p_3t}$, $X_j \perp N$

$Y \sim$ NegativeBinomial (k_1, p_3), $Y \perp X_j, N$

$Z \sim$ Binomial $(k_2, \frac{p_2}{p_1+p_2})$, $Z \perp Y, X_j, N$

Then $U = \sum_{j=1}^{N} X_j + Y + Z$ $(\sum_{j=1}^{N} X_j = 0$ **if** $N = 0)$ **has pgf**

$$p(t) = \left(\frac{1-p_3}{1-p_3t}\right)^{k_1} \left(\frac{p_1+p_2t}{p_1+p_2}\right)^{k_2} \frac{{}_1F_1\left(k_1; k_2+1; \lambda\left(\frac{p_1+p_2t}{1-p_3t}\right)\right)}{{}_1F_1(k_1; k_2+1; \lambda)}.$$

pmf of the proposed distribution:

(1) $u \le k_2$,

$$P_u = \sum_{s=0}^{u} \frac{(k_1)_{u-s}}{(u-s)!}(1-p_3)^{k_1} {p_3}^{u-s} \binom{k_2}{s} \times \left(\frac{p_1}{p_1+p_2}\right)^{k_2-s} \left(\frac{p_2}{p_1+p_2}\right)^{s} \frac{{}_1F_1(k_1+u-s; k_2+1-s; \lambda p_1)}{{}_1F_1(k_1; k_2+1; \lambda)}.$$

(continued)

(2) $u > k_2$,

$$
\begin{aligned}
P_u &= \sum_{s=0}^{k_2} \frac{(k_1)_{u-s}}{(u-s)!}(1-p_3)^{k_1} {p_3}^{u-s} \binom{k_2}{s} \\
&\quad \times \left(\frac{p_1}{p_1+p_2}\right)^{k_2-s} \left(\frac{p_2}{p_1+p_2}\right)^{s} \frac{{}_1F_1(k_1+u-s; k_2+1-s; \lambda p_1)}{{}_1F_1(k_1; k_2+1; \lambda)} \\
&\quad + \sum_{s=k_2+1}^{u} \frac{(k_1)_{u-k_2}}{(u-s)!}(1-p_3)^{k_1} {p_3}^{u-s} \\
&\quad \times \frac{k_2!}{s!(s-k_2)!} \lambda^{s-k_2} \frac{p_2^s}{(p_1+p_2)^{k_2}} \frac{{}_1F_1(k_1+u-k_2; s-k_2+1; \lambda p_1)}{{}_1F_1(k_1; k_2+1; \lambda)}.
\end{aligned}
$$

In (1) $u \le k_2$**,**

$$P_u = \sum_{s=0}^{u} \frac{(k_1)_{u-s}}{(u-s)!}(1-p_3)^{k_1} {p_3}^{u-s} \binom{k_2}{s} \times \left(\frac{p_1}{p_1+p_2}\right)^{k_2-s} \left(\frac{p_2}{p_1+p_2}\right)^{s} \frac{{}_1F_1(k_1+u-s; k_2+1-s; \lambda p_1)}{{}_1F_1(k_1; k_2+1; \lambda)},$$

if $p_3 \to 0$ **(GCSD), then each term of the rhs has zero for** $s = 0, \ldots, u-1$ **and**

$$P_u = \binom{k_2}{u} \left(\frac{p_1}{p_1+p_2}\right)^{k_2-u} \left(\frac{p_2}{p_1+p_2}\right)^{u} \frac{{}_1F_1(k_1; k_2+1-u; \lambda p_1)}{{}_1F_1(k_1; k_2+1; \lambda)},$$

and if $p_2 \to 0$ **(extended NNBD: restricted form), then it has zero for** $s = 1, \ldots, u$ **and**

$$P_u = \frac{(k_1)_u}{u!}(1-p_3)^{k_1} {p_3}^{u} \left(\frac{p_1}{p_1+p_2}\right)^{k_2} \frac{{}_1F_1(k_1+u; k_2+1; \lambda p_1)}{{}_1F_1(k_1; k_2+1; \lambda)}.$$

In (2) $u > k_2$**, we have similar calculations.**

Recursion formulae for pmf:

$$P_u = \sum_{s=0}^{u} P_{u,s}$$

When the horizontal axis is $u - s$ and the vertical axis s,

(1) Horizontal $P_{u,s} = h_1 P_{u-1,s} + h_2 P_{u-2,s}$

(2) Vertical $P_{u,s} = v_1 P_{u-1,s-1} + v_2 P_{u-2,s-2}$

(3) Triangular $P_{u,s} = t_1 P_{u,s-1} + t_2 P_{u-1,s-1}$

(4) Diagonal $P_{u,s} = d_1 P_{u,s-1} + d_2 P_{u,s-2}$

where h_i, v_i, t_i **and** d_i $(i = 1, 2)$ **are independent of** u **and** s**.**

r-th descending factorial moment:

Using the Vandermonde formula

$$u_{[r]} = \sum_{k=0}^{r} \binom{r}{k} (u-s)_{[k]} (s)_{[r-k]}, \quad u \in \mathbf{R}, \ s \in \mathbf{R},$$

where $u_{[r]} = u(u-1)\cdots(u-r+1)$ **if** $r \geq 1$, $= 1$ **if** $r = 0$, **we have**

(1) $k_2 + 1 > 0, \ r - k \leq k_2$,

$$\mu_{[r]} = \sum_{k=0}^{r} \binom{r}{k} \frac{(k_1)_k {p_3}^k}{(1-p_3)^k} \frac{(k_2 - r + k + 1)_{r-k} {p_2}^{r-k}}{(p_1+p_2)^{r-k}} \times \frac{{}_1F_1(k_1+k; k_2-(r-k)+1; \lambda)}{{}_1F_1(k_1; k_2+1; \lambda)}.$$

(2) $k_2 + 1 > 0,\ r - k > k_2,$

$$\mu_{[r]} = \sum_{k=\max(0,r-k_2)}^{r} \binom{r}{k} \frac{(k_1)_k {p_3}^k}{(1-p_3)^k} \frac{(k_2 - r + k + 1)_{r-k} {p_2}^{r-k}}{(p_1 + p_2)^{r-k}}$$

$$\times \frac{{}_1F_1(k_1 + k; k_2 - (r-k) + 1; \lambda)}{{}_1F_1(k_1; k_2 + 1; \lambda)}$$

$$+ \sum_{k=0}^{\max(0,r-k_2)-1} \binom{r}{k} \frac{(k_1)_{r-k_2} {p_3}^k}{(1-p_3)^k} \frac{k_2! \lambda^{(r-k)-k_2} {p_2}^{r-k}}{((r-k) - k_2)!(p_1 + p_2)^{r-k}}$$

$$\times \frac{{}_1F_1(k_1 - k_2 + r; 1 + (r-k) - k_2; \lambda)}{{}_1F_1(k_1; k_2 + 1; \lambda)}.$$

Special cases:

(1) $p_2 \to 0$ (**GCSD**)
if $r \le k_2$**,**

$$\mu_{[r]} = (k_2)_{[r]}\, {p_2}^r \, \frac{{}_1F_1(k_1; k_2+1-r; \lambda)}{{}_1F_1(k_1; k_2+1; \lambda)}.$$

if $r > k_2$**,**

$$\mu_{[r]} = \frac{k_2!}{(r-k_2)!}\, \lambda^{r-k_2}\, p_2^r\, (k_1)_{r-k_2}\, \frac{{}_1F_1(k_1+r-k_2; r-k_2+1; \lambda)}{{}_1F_1(k_1; k_2+1; \lambda)}.$$

(2) $p_3 \to 0$ (**extended NNBD: restricted form**)

$$\mu_{[r]} = (k_1)_r \left(\frac{p_3}{1-p_3}\right)^r \frac{{}_1F_1(k_1+r; k_2+1; \lambda)}{{}_1F_1(k_1; k_2+1; \lambda)}.$$

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