Two Nested Families of Skew-symmetric Circular Distributions

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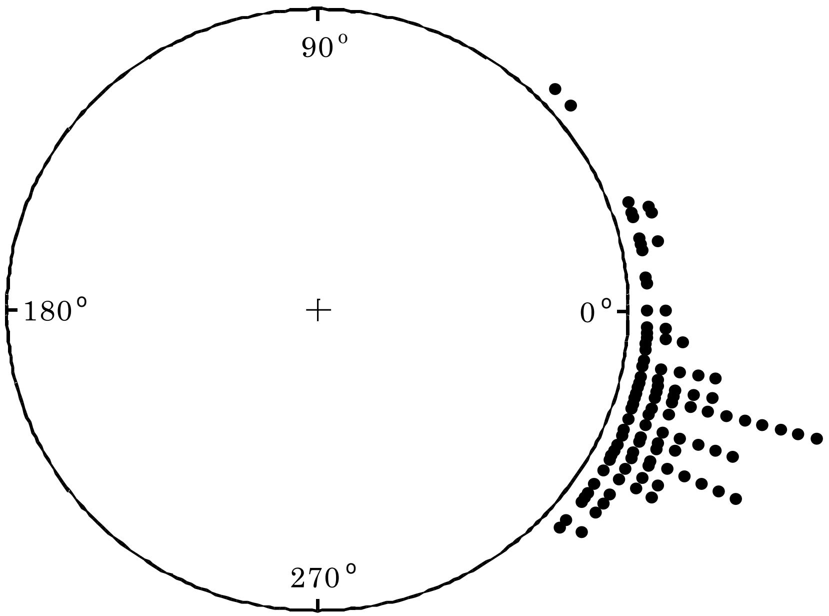
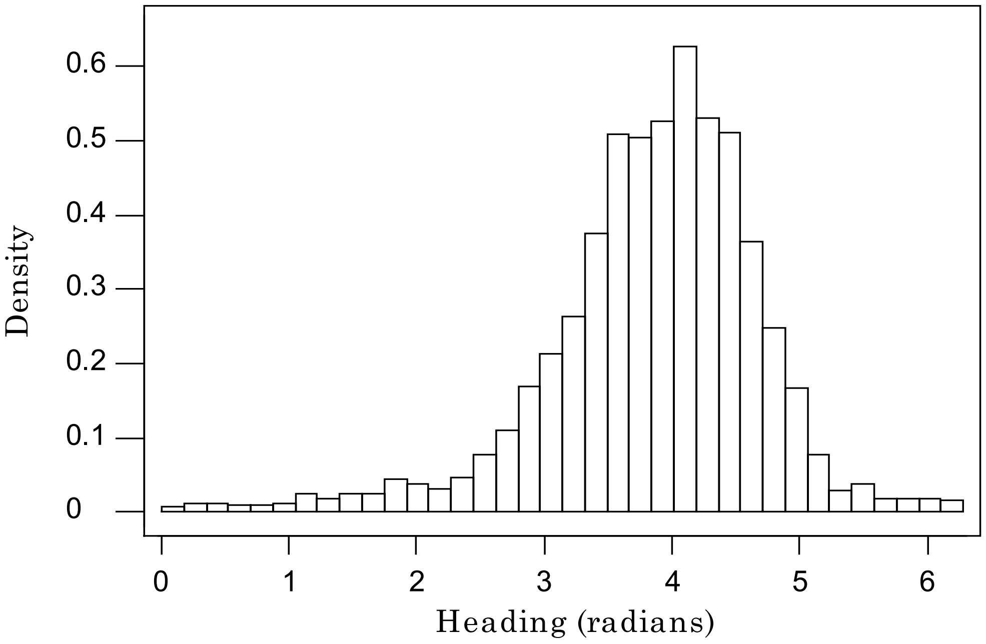
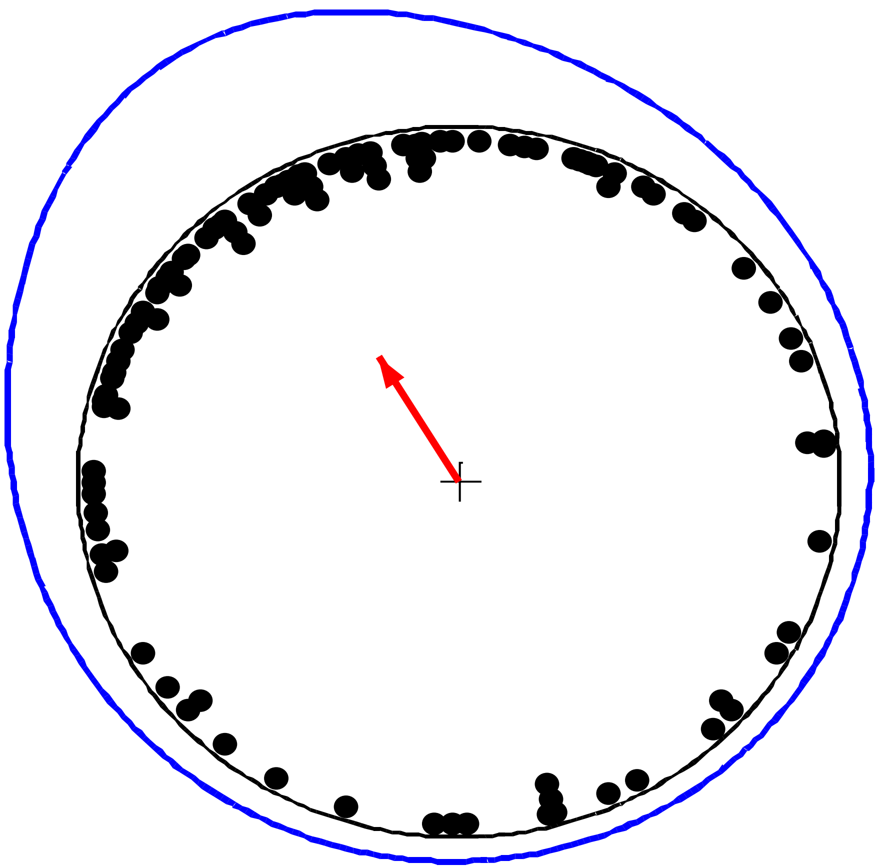
0. Circular Data

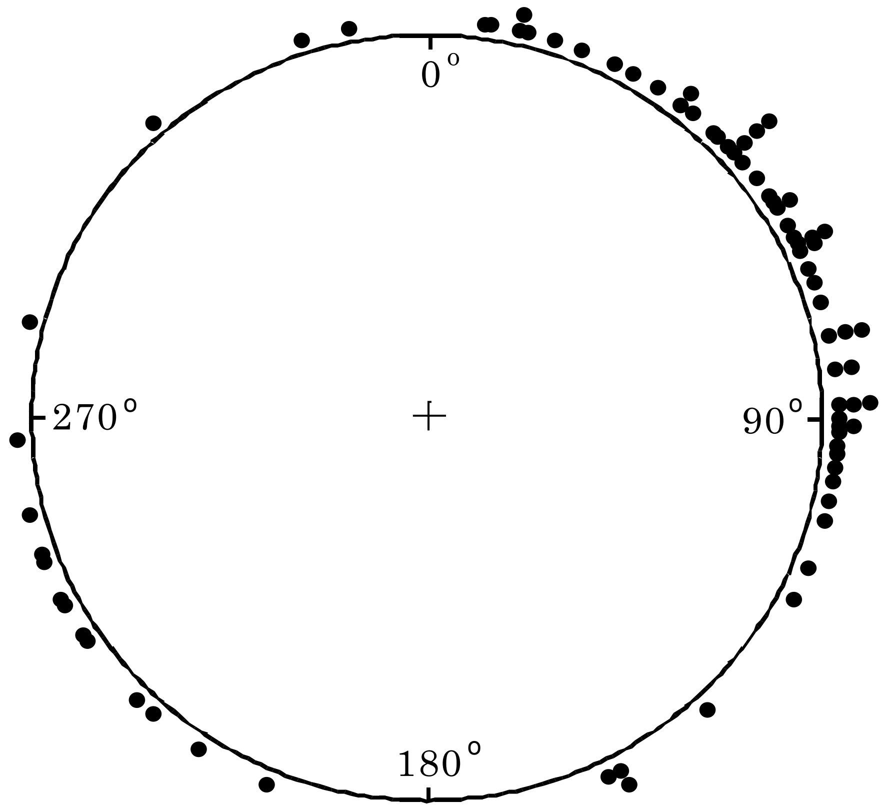
Asymmetric





Symmetric





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1. Introduction

Models for Circular Data

Best known circular models (von Mises, wrapped normal, wrapped Cauchy, cardioid,...) are all symmetric.

Circular data seldom symmetrically distributed (Mardia 1972, p. 10).

Skew-symmetric circular distributions include:

- Wrapped skew-normal: Pewsey (2000)
- Wrapped Laplace: Jammalamadaka & Kozubowski (2004)
- Wrapped stable: Pewsey (2008)

WGNL and WNL Unimodal Families

Obtained by wrapping the 5 parameter generalised normal-Laplace and 4 parameter normal-Laplace distributions, respectively onto the unit circle.

Both include the wrapped normal, wrapped Laplace and wrapped generalised Laplace distributions and can be derived from simple stochastic models involving Brownian motion on the circle.

2. GNL and NL Families

Generalised normal-Laplace (GNL) distribution on the real line does not, in general, have a closed form density. Its characteristic function is

$$\psi_{GNL}(s) = \left[\frac{\exp\left(i\eta s - \tau^2 s^2 / 2\right)}{(1 - ias)(1 + ibs)} \right]^{\zeta},$$

where $-\infty < \eta < \infty$ and a, b, ζ, $\tau^2 \geq 0$.

Reduces to the normal-Laplace (NL) distribution when $\zeta = 1$. Closed forms for the density and distribution function do exist for this special case.

A GNL rv can be represented as a convolution of independent Gaussian and generalised Laplace components and thus as

$$X = \eta\zeta + \tau\sqrt{\zeta}Z + aV_1 - bV_2 \text{ ,}$$

where Z, V_1 and V_2 are independent, $Z \sim \mathrm{N}(0,1)$ and V_1 and V_2 are identically distributed gamma rv's with shape parameter ζ and scale parameter 1. (Useful for simulation of GNL and WGNL rv's.)

Special cases: normal ($a = b = 0$); generalised Laplace ($\eta = \tau = 0$); (skew)-Laplace ($\eta = \tau = 0$ and $\zeta = 1$).

3. WGNL and WNL Families

If $X \sim$ GNL(η, τ^2, a, b, ζ) then the circular rv $\Theta = X \pmod{2\pi} \in [0, 2\pi)$ follows the wrapped generalised normal-Laplace (WGNL) distribution.

The characteristic function of Θ has complex Fourier coefficients

$$\psi_p = \psi_{GNL}(p) = \left[\frac{\exp\left(i\eta p - \tau^2 p^2 / 2\right)}{(1 - iap)(1 + ibp)} \right]^{\zeta}, \qquad (1)$$

for $p = 0, \pm 1, \pm 2, \ldots$.

Special cases: wrapped normal-Laplace (WNL) ($\zeta = 1$); wrapped normal ($a = b = 0$); wrapped generalised Laplace ($\eta = \tau = 0$); wrapped Laplace ($\eta = \tau = 0$ and $\zeta = 1$).

State of a Brownian Motion on the Circle

Consider a particle following a Brownian motion on the circle with infinitesimal mean drift $\mu \, \mathrm{d} t$, infinitesimal variance $\sigma^2 \, \mathrm{d} t$ and initial direction θ_0.

The direction of the particle at time t has a wrapped normal distribution, with characteristic function

$$\psi_p = \exp(i\theta_0 p)\exp\left(i\mu tp - \left(\sigma^2/2\right)tp^2\right).$$

Now suppose that the time, T, for which the Brownian motion has been evolving is such that

$$T = t_0 + \frac{1}{\lambda} G, \qquad (2)$$

where t_0 is a constant and G has a gamma distribution with unit scale parameter and shape parameter ζ.

Then the characteristic function of the direction of the particle after the random time T is given by

$$\psi_p = \exp\left[i(\theta_0 + \mu t_0)p - (\sigma^2/2)t_0 p^2\right]\left(\frac{\lambda}{\lambda - i\mu p + (\sigma^2/2)p^2}\right)^{\zeta},$$

which is of the form (1) with

$$\eta = \frac{\theta_0 + \mu t_0}{\zeta} \pmod{2\pi}, \qquad \tau^2 = \frac{\sigma^2 t_0}{\zeta},$$

$$a = \sqrt{\left(\frac{\mu}{2\lambda}\right)^2 + \frac{\sigma^2}{2\lambda}} + \frac{\mu}{2\lambda}, \qquad b = \sqrt{\left(\frac{\mu}{2\lambda}\right)^2 + \frac{\sigma^2}{2\lambda}} - \frac{\mu}{2\lambda}.$$

An alternative model to (2) is when a Brownian motion on the circle with initial direction following a wrapped normal distribution (with mean direction μ_0 and scale parameter σ_0^2, say), evolves for a gamma-distributed random time.

This results in the same WGNL distribution with μt_0 and $\sigma^2 t_0$ replaced by μ_0 and σ_0^2, respectively.

Special cases include:

- $\zeta = 1$, i.e. the random component of the evolution time is exponentially distributed. Then state after time T is WNL.
- $\mu = 0$, i.e. there is no mean drift in the circular Brownian motion. Then WGNL distribution is symmetric ($a = b = \sigma / \sqrt{2\lambda}$).
- $t_0 = 0$. Then T follows a gamma distribution and the state after the random time T follows the wrapped generalised Laplace distribution.
- $\zeta = 1$ and $t_0 = 0$. The state at time T is wrapped Laplace.

4. Fourier Series Representation of the WGNL Density

The density of a circular distribution can be represented in terms of its Fourier coefficients as

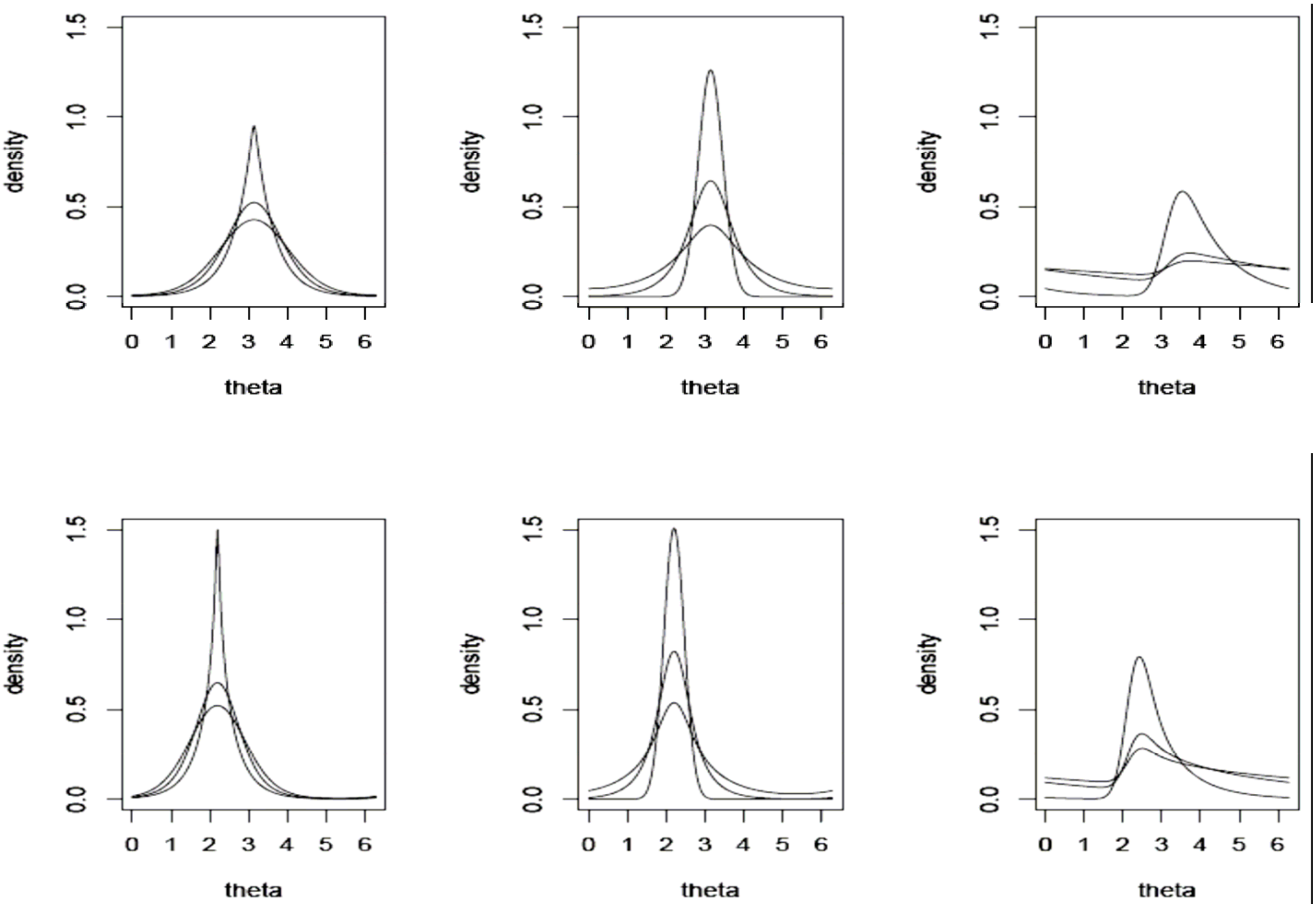
$$f(\theta) = \frac{1}{2\pi}\left\{1 + 2\sum_{p=1}^{\infty}\left[\alpha_p \cos(p\theta) + \beta_p \sin(p\theta)\right]\right\},$$

where $\psi_p = E(\cos p\Theta) + iE(\sin p\Theta) = \alpha_p + i\beta_p$, or as

$$f(\theta) = \frac{1}{2\pi}\left\{1 + 2\sum_{p=1}^{\infty}\rho_p \cos(p\theta - \mu_p)\right\},$$

where $\alpha_p = \rho_p \cos\mu_p$ and $\beta_p = \rho_p \sin\mu_p$. For the WGNL distribution

$$\rho_p = \left[\frac{\exp(-\tau^2 p^2)}{(1 + a^2 p^2)(1 + b^2 p^2)}\right]^{\zeta/2}.$$


density
theta
0.0
0.5
1.0
1.5
0 1 2 3 4 5 6

5. Maximum Likelihood Estimation

Involves the constrained numerical maximisation of

$$\ell\left(\eta,\tau^2,a,b,\zeta\right)=\sum_{i=1}^{n}\log f\left(\theta_i\right),$$

using a finite sum approximation to the density $f(\theta)$.

Proves beneficial to re-parameterise in terms of the mean direction, μ, and the mean resultant length, ρ, instead of η and τ^2.

Easily implemented using the optim routine of R. The same routine can be used to obtain the observed information matrix.

6. Example

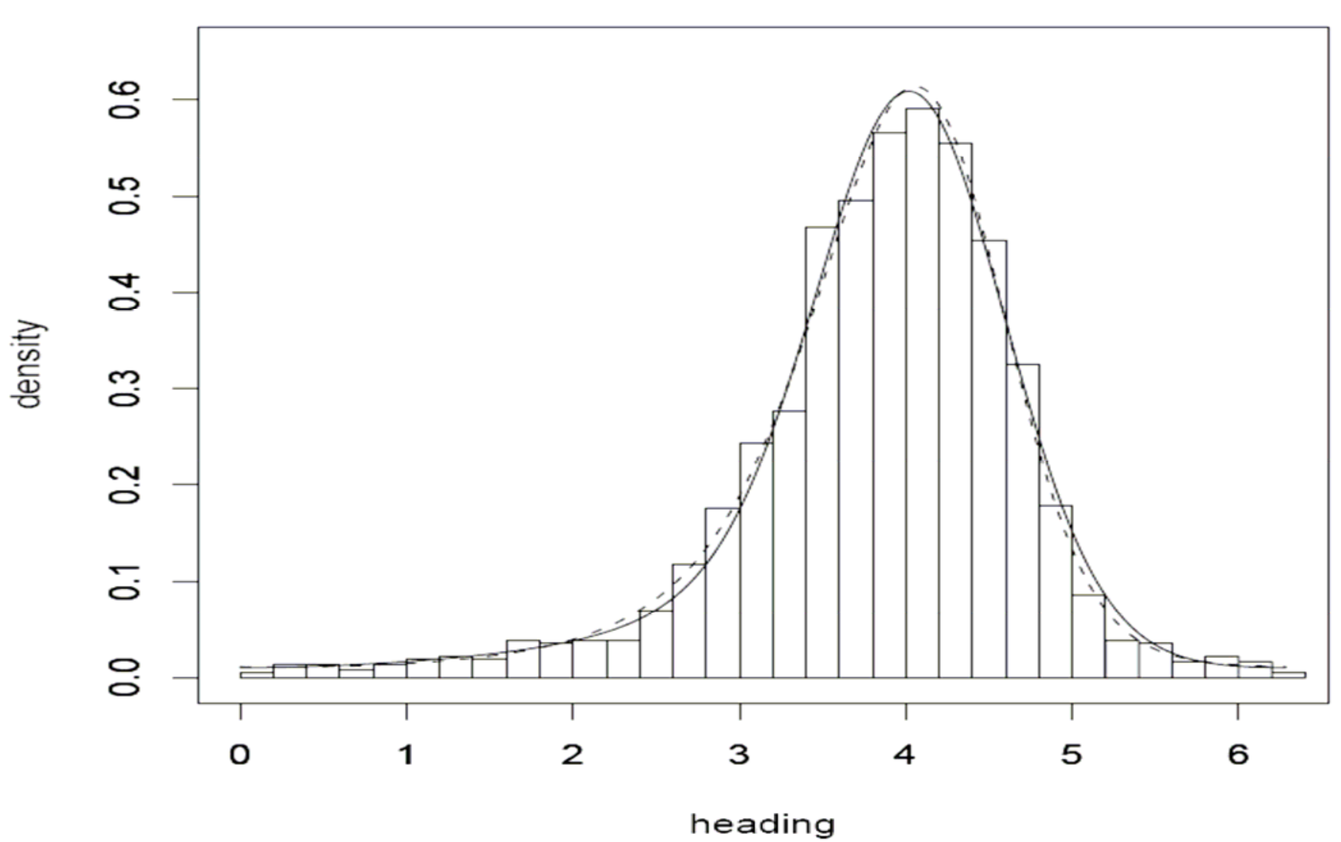
Headings of Migrating Birds

Data set consisting of $n = 1827$ ‘headings’ of birds - Bruderer & Jenni (1990).

Recorded near Stuttgart in Germany during the autumnal migration period of 1987.

A ‘heading’ is the direction, measured clockwise from north, of a bird's body during flight.

Circular reflective symmetry emphatically rejected by large-sample test of Pewsey (2002) ($p = 0.000$).



Distribution	ℓ	*# Par.*
Wrapped normal-Laplace	-2137.08	4
Two component von Mises mixture	-2131.10	5
Wrapped generalised normal-Laplace	-2129.83	5
Wrapped skew-normal + uniform mixture	-2128.03	4
Wrapped stable	-2127.73	4

References

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