On the Jones–Pewsey Distribution

Department of Mathematics
Keio University

Kunio Shimizu

The Jones–Pewsey (2005) distribution is a very flexible symmetric model on the sphere. It includes the von Mises as a special case as well as wrapped Cauchy, cardioid and Cartwright's power-of-cosine or equivalently Minh–Farnum (2003) distributions. The distribution is closely connected with t-distributions on the sphere (Shimizu and Iida, 2002), which are obtained by the conditioning method.

von Mises Distribution $\mathbf{VM}(\mu, \kappa)$ $\quad 0 \leq \mu < 2\pi,\ \kappa \geq 0$

$$f(\theta) = \frac{1}{2\pi I_0(\kappa)} \exp\left[\kappa \cos(\theta - \mu)\right], \quad 0 \leq \theta < 2\pi$$

I_p : **modified Bessel function of the first kind of order** p

$$I_p(\kappa) = \frac{1}{2\pi} \int_0^{2\pi} \cos(p\theta)\, e^{\kappa \cos\theta} d\theta = \sum_{r=0}^{\infty} \frac{1}{\Gamma(p+r+1) r!} \left(\frac{\kappa}{2}\right)^{2r+p}$$

μ : **mean direction,** $\quad \kappa$: **concentration**

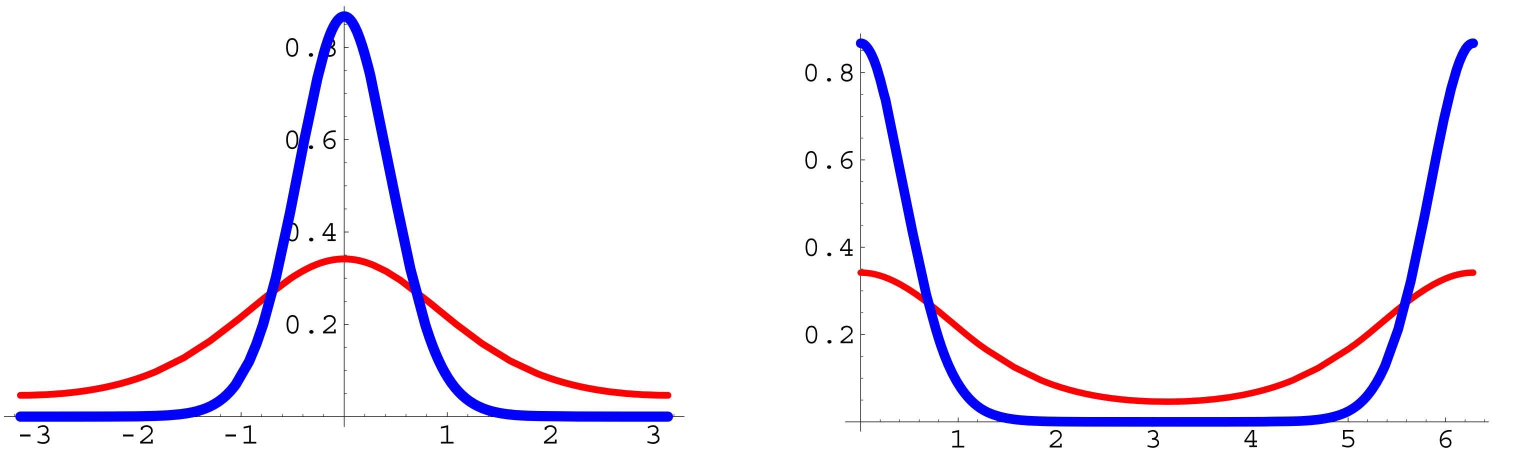


Figure 1: von Mises Distributions ($\mu = 0,\ \kappa = 1$ (red), $\mu = 0,\ \kappa = 5$ (blue), left: $-\pi \leq \theta < \pi$, right: $0 \leq \theta < 2\pi$)

Circular Data: wind directions, vanishing angles of birds, gene locations of bacterial genomes, etc.

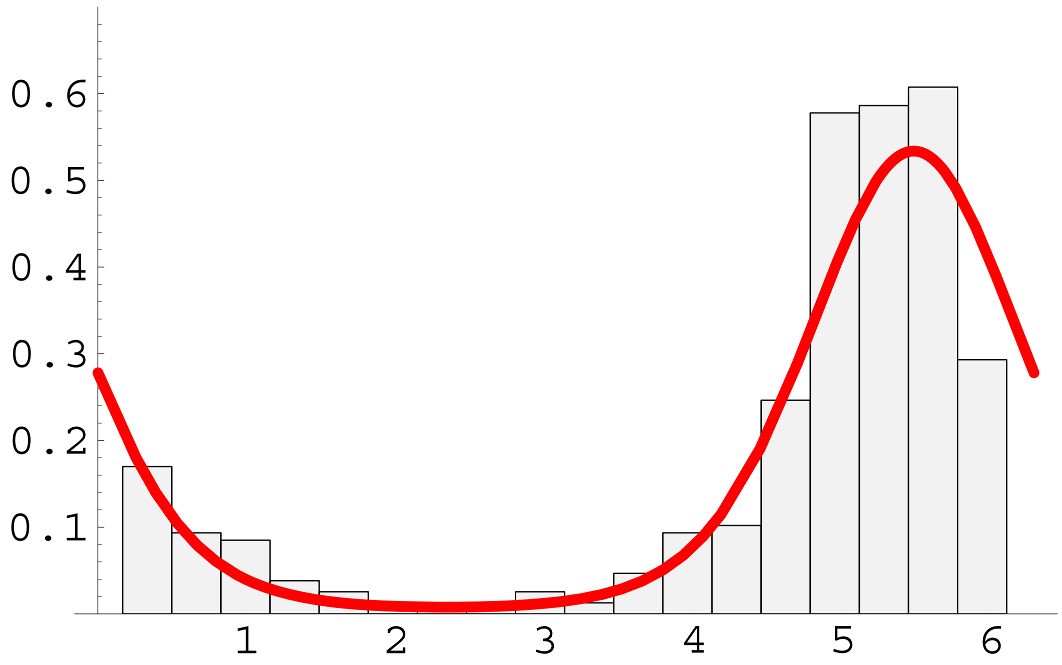


Figure 2: Histogram of mallard data with ML fit (von Mises)

Jones–Pewsey (2005) Distribution $0 \le \mu < 2\pi,\ \kappa \ge 0,\ \psi \ne 0$

$$f(\theta) = \frac{(\cosh(\kappa\psi) + \sinh(\kappa\psi)\cos(\theta - \mu))^{1/\psi}}{2\pi P_{1/\psi}(\cosh(\kappa\psi))}, \quad 0 \le \theta < 2\pi$$

$P_{1/\psi}$**: associated Legendre function of the first kind of degree** $1/\psi$

$$\begin{aligned} \int_0^\pi (z + \sqrt{z^2-1}\cos x)^{1/\psi} dx &= \pi P_{1/\psi} \quad (\psi > 0), \\ \int_0^\pi \frac{1}{(z + \sqrt{z^2-1}\cos x)^{-1/\psi}} dx &= \pi P_{-1/\psi-1} = \pi P_{1/\psi} \quad (\psi < 0) \end{aligned}$$

Special Cases:

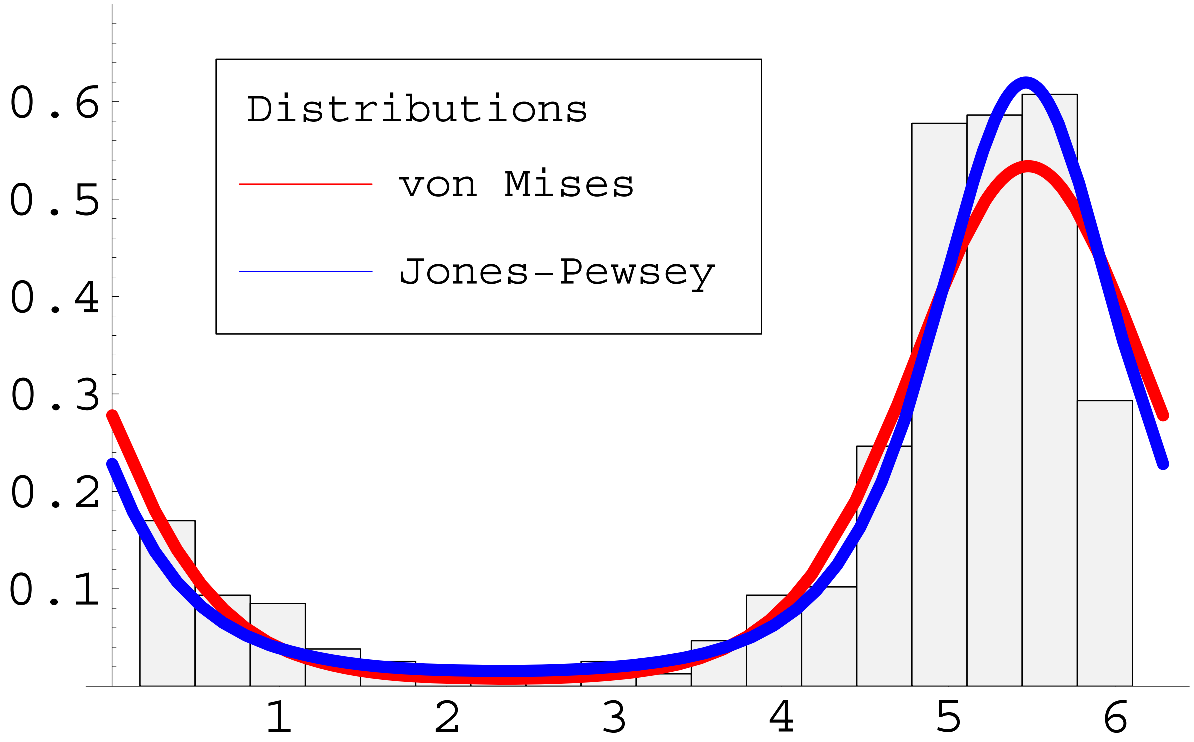
$\psi \to 0$, **von Mises distribution**
$\psi = 1$, **cardioid distribution**
$\psi = -1$, **wrapped Cauchy distribution**
$\psi > 0$, $\kappa \to \infty$, **Cartwright's power-of-cosine or Minh–Farnum (2003) distribution**

$$f(\theta) = \frac{2^{1/\psi - 1}\Gamma^2(1/\psi + 1)}{\pi\Gamma(2/\psi + 1)}(1 + \cos\theta)^{1/\psi}$$

Jones–Pewsey

$$f(\theta) = C(1 + \tanh(\kappa\psi)\cos\theta)^{1/\psi}$$

Figure 3: Histogram of the mallard data with ML fits (von Mises and Jones–Pewsey)



Methods of Construction:

ad-hoc, maximum entropy, characterization, wrapping, conditioning, offset, stereographic projection, Brownian motion, scale mixtures, etc.

von Mises distribution: conditioning method (Downs, 1966)

$(X, Y)' \sim N_2(\eta, I_2)$

polar transformation

$$\begin{cases} X = R\cos\Theta \\ Y = R\sin\Theta \end{cases} \qquad \eta = \rho(\cos\tau, \sin\tau)'$$

conditional distribution of Θ **given** $R = r$

$$(\Theta | R = r) \sim \mathbf{VM}(\tau, \rho r)$$

t-distribution on the circle: conditioning method (Shimizu and Iida, 2002)

$(X,Y)'|\sigma \sim N_2(\eta, \sigma^2 I_2)$, $\frac{dG(\sigma)}{d\sigma} = \frac{2^{1-n/2} n^{n/2}}{\Gamma(n/2)} \sigma^{-1-n} \exp\left(-\frac{n}{2\sigma^2}\right)$

polar transformation

$$\begin{cases} X = R\cos\Theta \\ Y = R\sin\Theta \end{cases} \qquad \eta = \rho(\cos\tau, \sin\tau)'$$

conditional distribution of Θ given $R = r$

$$f_t(\theta|r) = \frac{\{1-(2/n)\kappa_n(\rho|r)\cos(\theta-\tau)\}^{-n/2-1}}{2\pi\ {}_2F_1\left(n/4+1/2, n/4+1; 1; (2\kappa_n(\rho|r)/n)^2\right)},$$

$$\kappa_n(\rho|r) = \frac{\rho r}{1+(\rho^2+r^2)/n}$$

Gauss hypergeometric function

$$\begin{aligned} {}_2F_1(a,b;c;z) &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}, \quad |z|<1, \\ (a)_n &= a(a+1)\times\cdots\times(a+n-1) \text{ if } n\geq 1; \ =1 \text{ if } n=0 \end{aligned}$$

Relation between t- and Jones–Pewsey distributions

$$\frac{1}{\psi} = -\frac{n}{2} - 1, \ \tanh(\kappa\psi) = \frac{2}{n}\kappa_n(\rho|r)$$

Relation between associated Legendre and Gauss hypergeometric functions

$$P_{1/\psi}(z) = z^{1/\psi}\,{}_2F_1\left(-\frac{1}{2\psi}, -\frac{1}{2\psi}+\frac{1}{2}; 1; \frac{z^2-1}{z^2}\right)$$

The Jones–Pewsey distribution is a reparametrization of the t-distribution on the circle:

$$f(\theta) = \frac{\cosh^{1/\psi}(\kappa\psi)\,(1+\tanh(\kappa\psi)\cos(\theta-\mu))^{1/\psi}}{2\pi P_{1/\psi}(\cosh(\kappa\psi))}, \quad 0 \le \theta < 2\pi$$

$$f_t(\theta|r) = \frac{\{1-(2/n)\kappa_n(\rho|r)\cos(\theta-\tau)\}^{-n/2-1}}{2\pi\ {}_2F_1\left(n/4+1/2, n/4+1; 1; (2\kappa_n(\rho|r)/n)^2\right)}, \quad -\pi \le \theta < \pi$$

However, the Jones–Pewsey distribution is more flexible than the t-distribution on the circle because the range of parameters is extended.

Spherical case

Jones–Pewsey ditribution on the sphere

$$f(\mathbf{x}) = \frac{|\sinh(\kappa\psi)|^{p/2-1}}{2^{p/2-1}\Gamma(p/2)} \frac{(\cosh(\kappa\psi) + \sinh(\kappa\psi)\mathbf{x}'\boldsymbol{\mu})^{1/\psi}}{P^{1-p/2}_{1/\psi+p/2-1}(\cosh(\kappa\psi))}, \quad \mathbf{x} \in S^{p-1}$$

$$\kappa \geq 0, \ \psi \neq 0, \ \boldsymbol{\mu} \in S^{p-1} = \{\mathbf{y} \in \mathbb{R}^p | \parallel \mathbf{y} \parallel = 1\}$$

P^{η}_{ν}: **associated Legendre function of the first kind of degree ν and order η**

an alternative form

$$f(\mathbf{x}) = \frac{(1 + \tanh(\kappa\psi)\mathbf{x}'\boldsymbol{\mu})^{1/\psi}}{{}_2F_1\left(-1/(2\psi), -1/(2\psi) + 1/2; p/2; \tanh^2(\kappa\psi)\right)}, \quad \mathbf{x} \in S^{p-1}$$

Relation between the associated Legendre and Gauss hypergeometric functions

$$P^{1-p/2}_{1/\psi+p/2-1}\left(\cosh(\kappa\psi)\right) = P^{1-p/2}_{-1/\psi-p/2}\left(\cosh(\kappa\psi)\right)$$

$$= \quad \frac{2^{1-p/2}}{\Gamma(p/2)} \frac{\left(\cosh(\kappa\psi)\right)^{1/\psi}}{\left(\sinh^2(\kappa\psi)\right)^{(2-p)/4}} \, {}_2F_1\left(-\frac{1}{2\psi}, -\frac{1}{2\psi} + \frac{1}{2}; \frac{p}{2}; \tanh^2(\kappa\psi)\right)$$

If $\psi < 0$**,** ${}_2F_1$ **converges and takes a positive value since** $|\tanh(\kappa\psi)| < 1$**.**

If $\psi > 0$**, from the transformation formula**

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z), $$

we have

$$ \begin{aligned} & {}_2F_1\left(-\frac{1}{2\psi}, -\frac{1}{2\psi}+\frac{1}{2}; \frac{p}{2}; \tanh^2(\kappa\psi)\right) \\ = \quad & \frac{1}{(\cosh(\kappa\psi))^{p-1+2/\psi}}\,{}_2F_1\left(\frac{p}{2}+\frac{1}{2\psi}, \frac{p}{2}+\frac{1}{2\psi}-\frac{1}{2}; \frac{p}{2}; \tanh^2(\kappa\psi)\right), \end{aligned} $$

whose value is positive since $p \geq 2$.

Thus, the situation is almost similar to the circular case.

Discussions

(a) The J–P distribution is a very flexible model for symmetric data sets on the circle/sphere, but the wrapped Cauchy, a special case of the J–P distribution, plays the central role of circular-circular regression (Kato, Shimizu and Shieh, to appear).

(b) The t-distributions on the sphere are derived from a scale mixture of normals. Generalized Laplace distributions on the sphere are obtainable by using a different weight from that for t-distributions (Siew, poster session of this workshop).

(c) The Minh–Farnum distribution, a special case of the J–P, is induced by a stereographic projection (Möbius transformation) of the t-distribution with n degrees of freedom (df) on the real line:

$$x = u + v\,\frac{\sin\theta}{1+\cos\theta} = u + v\tan\frac{\theta}{2}, \qquad -\pi \le \theta < \pi$$

with $u = 0$, $v = \sqrt{m}$, $m = 2n+1$. The t-distribution on the line goes to the standard normal as n tends to infinity, but the Minh–Farnum distribution goes to a one-point distribution as n tends to infinity.

This problem is solved by introducing the stereographic projection with $u = 0$ and $v > 0$ independent of n.

The resulting distribution (not included in the J–P family) induced by the t-distribution with df $m = 2n + 1$ on the line has the density function

$$f_n(\theta) = \frac{1}{2\sqrt{m}B(m/2, 1/2)} v \left(1 + \tan^2 \frac{\theta}{2}\right) \left(1 + \frac{v^2}{m} \tan^2 \frac{\theta}{2}\right)^{-(m+1)/2},$$

which converges to a circular distribution with density

$$f(\theta) = \frac{v}{2\sqrt{2\pi}} \left(1 + \tan^2 \frac{\theta}{2}\right) \exp\left(-\frac{v^2}{2} \tan^2 \frac{\theta}{2}\right)$$

as n tends to infinity, which is just the circular distribution induced by the stereographic projection of the standard normal on the line. A more extended form is proposed by Abe (poster session of this workshop).

References

[1] Jones, M. C. and Pewsey, A. (2005). A family of symmetric distributions on the circle, *Journal of the American Statistical Association*, 100(472), 1422–1428.

[2] Kato, S., Shimizu, K. and Shieh, G. S. (to appear). A circular-circular regression model, Statistica Sinica.

[3] Minh, D-L. and Farnum, N. R. (2003). Using bilinear transformations to induce probability distributions, Communications in Statistics–Theory and Methods, 32(1), 1–9.

[4] Shimizu, K. and Iida, K. (2002). Pearson type VII distributions on spheres, *Communications in Statistics–Theory and Methods*, 31(4), 513–526.