Structured Hidden Markov Models

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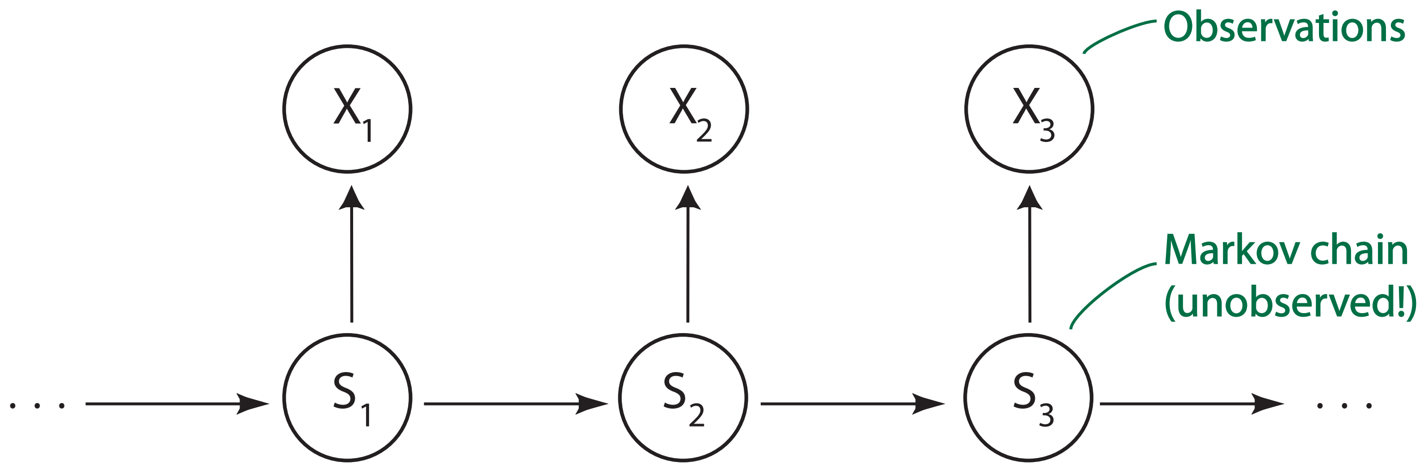
Outline

1. Hidden Markov Models
2. Structured Hidden Markov Models
3. Application - Daily Return Series
4. Application - Asset Allocation

1. Hidden Markov Models



Hidden Markov Models (HMM)/Markov-switching models are applied in various contexts, e.g.,

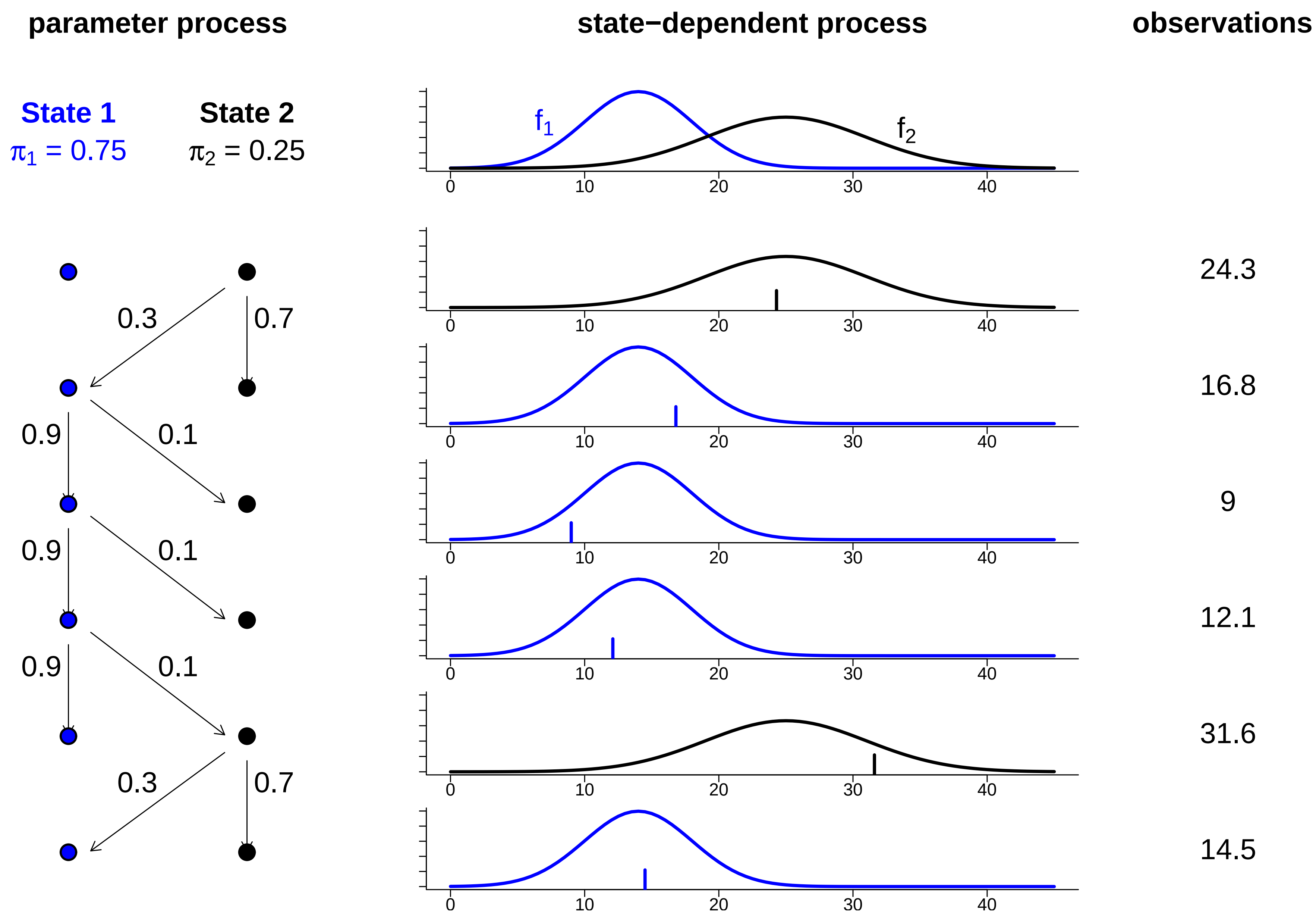
1. Daily return series (Rydén et al. 1998),

$$r_t \sim N(0, \sigma^2_{s_t}).$$

2. GDP, unemployment rate (Hamilton 1989, 2005)

$$y_t = c_{s_t} + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2)$$

1. Hidden Markov Models



1. Hidden Markov Models - Likelihood

The likelihood of a HMM can be written as

$$L = \boldsymbol{\pi} \boldsymbol{P}(x_1) \boldsymbol{T} \boldsymbol{P}(x_2) \boldsymbol{T} \cdots \boldsymbol{T} \boldsymbol{P}(x_T) \mathbf{1}^t \,,$$

with

$$\boldsymbol{P}(x_t) := \begin{pmatrix} p_1(x_t) & & & 0 \\ & p_2(x_t) & & \\ & & \ddots & \\ 0 & & & p_m(x_t) \end{pmatrix},$$

$$p_i(x_t) := P(X_t = x_t \mid S_t = i),\ i = 1, \ldots, m,$$

transition probability matrix $\boldsymbol{T}$, initial distribution $\boldsymbol{\pi}$, and $\mathbf{1} = (1, \ldots, 1)$ (Zucchini, 1997)

1. Hidden Markov Models - Parameter Estimation

Method: Usually, maximum-likelihood parameter estimation is used

Two 'competing' techniques

1. EM-algorithm (Baum et al. 1970)

   Pros: Stability/large circle of convergence

   Cons: Slow rate of convergence in the neighborhood of a maximum

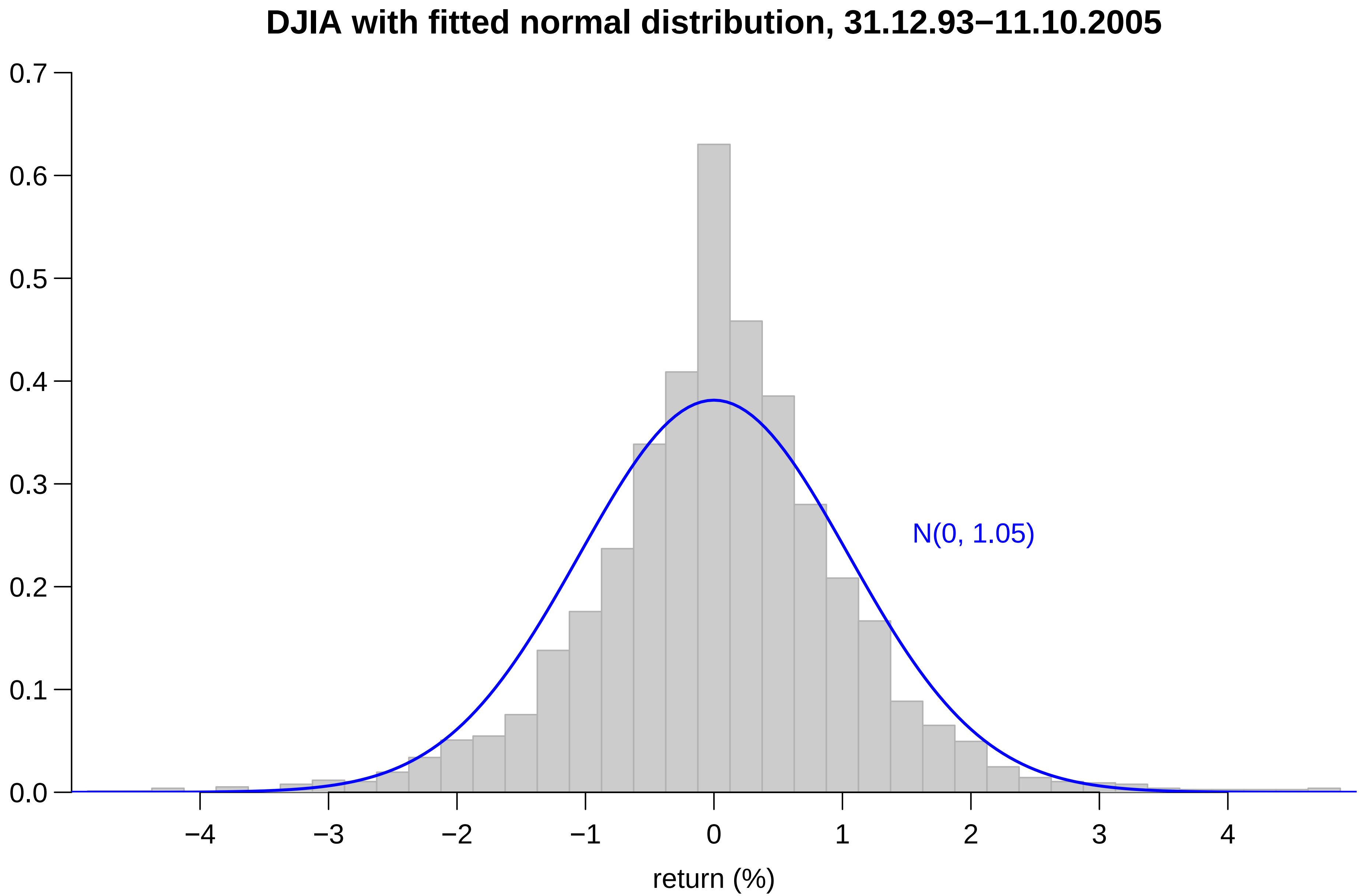
   M-Step may itself involve a numerical maximization in complex models (Wang, Puterman 1999)

2. Direct numerical maximization

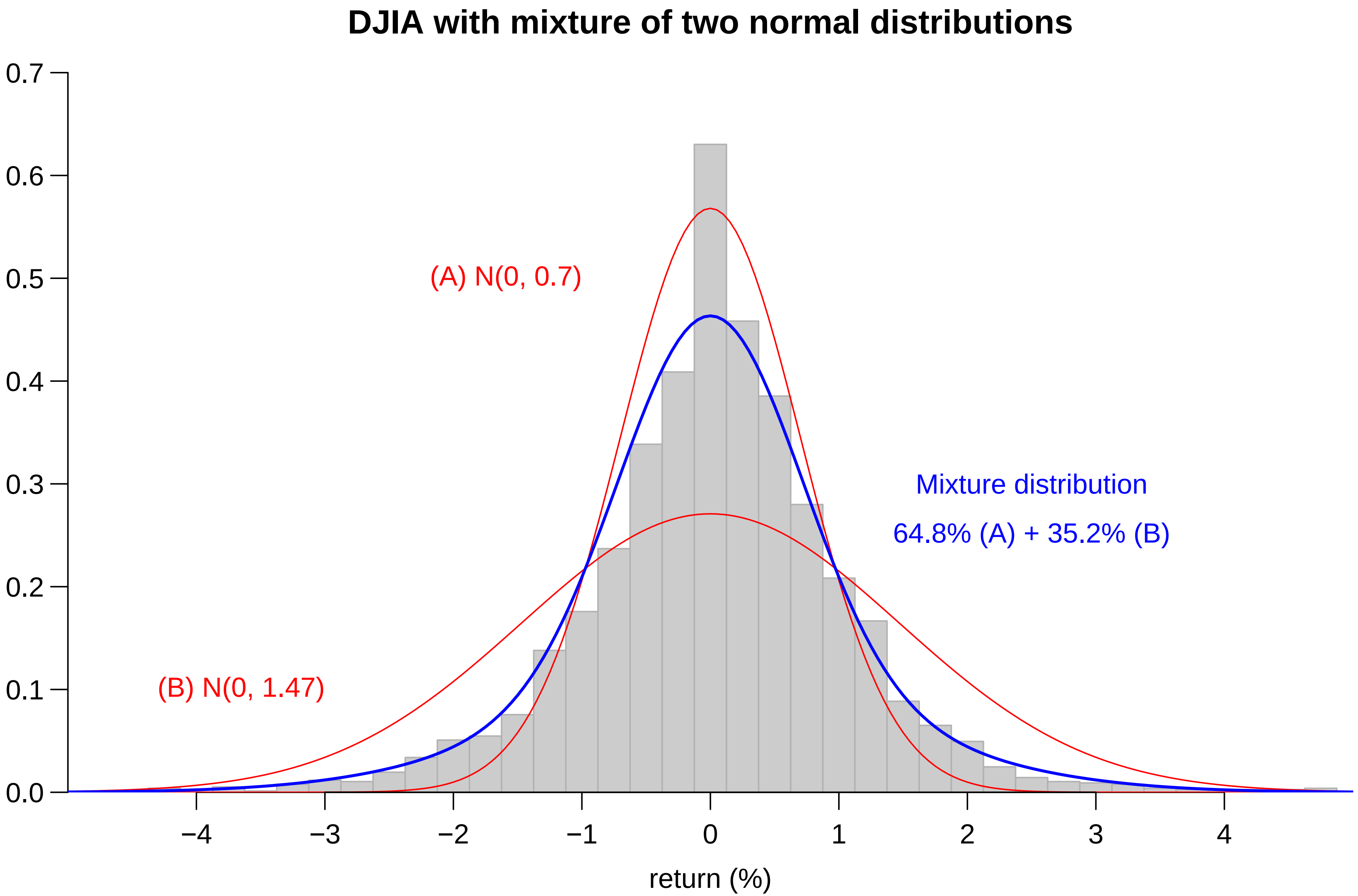
   Pros: Treatment of missing observations, easy to fit complex models, strong (superlinear) convergence in the neighborhood of a maximum

   Cons: Weak 'global convergence' properties

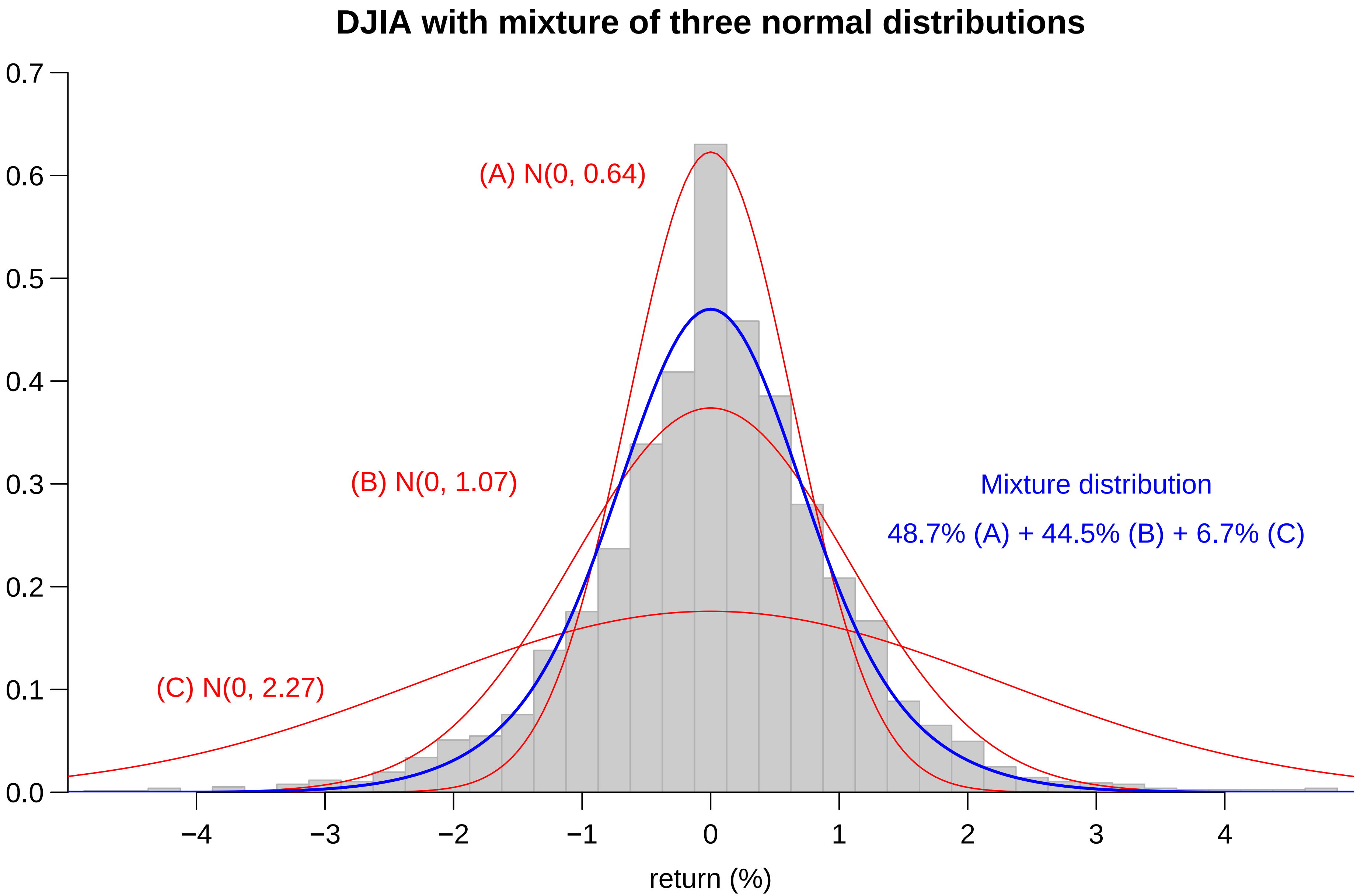
1. Hidden Markov Models - Daily Return Series Example



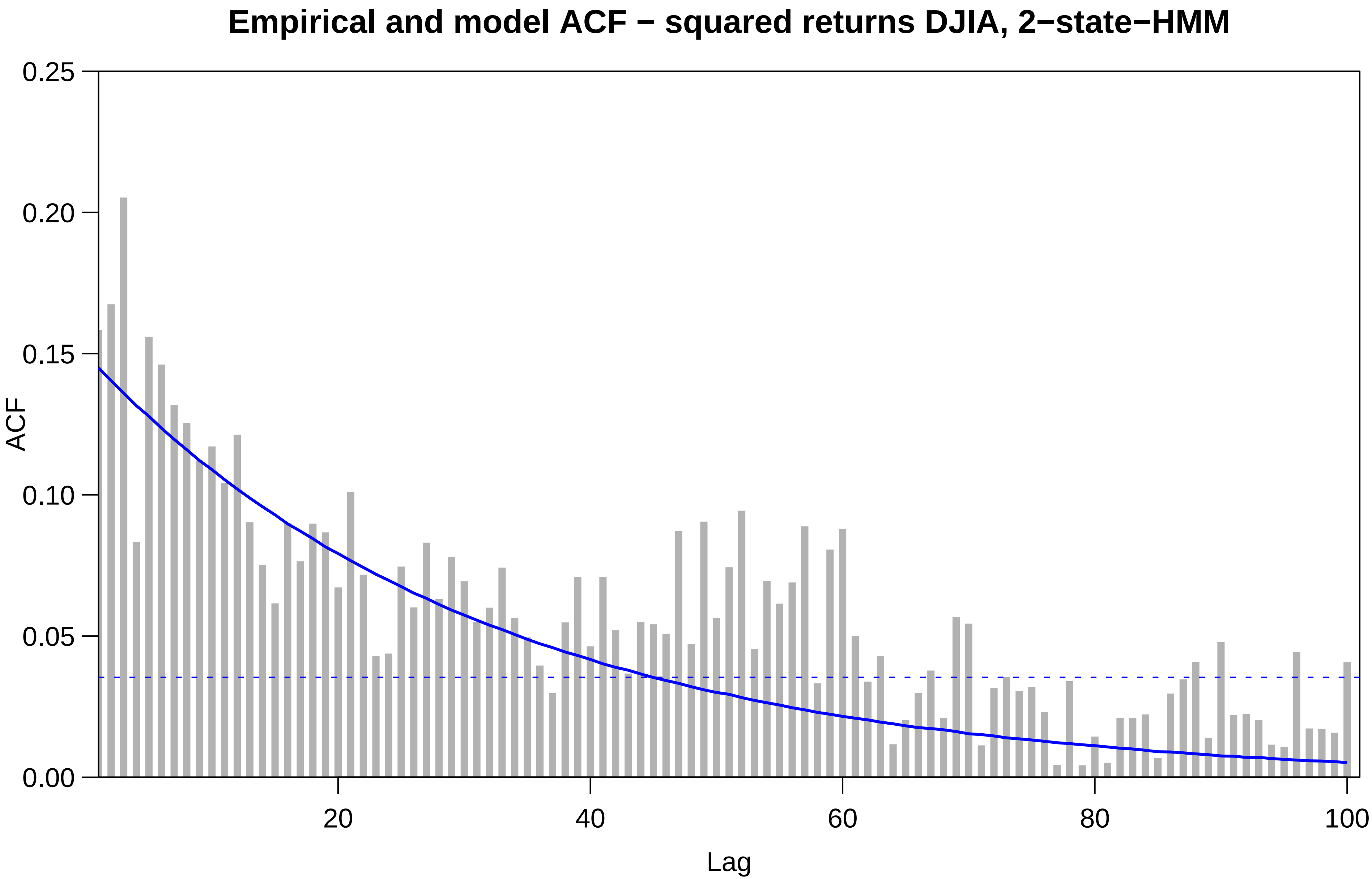
1. Hidden Markov Models - Daily Return Series Example



1. Hidden Markov Models - Daily Return Series Example



1. Hidden Markov Models - Daily Return Series Example



1. Hidden Markov Models - Limitations

Models in finance: 2 - 3 states

Reason: Number of parameters - m-state HMM (normal distributions)

$$\mathbf{m(m-1)+2m}$$

'High' number of states $\Rightarrow$

- Low-persistent regimes occur, depending on outliers (Rydén et al. 1998)
- Results highly dependent on initial values

Example: Daily returns from the DJIA, 31.12.93-11.10.2005 (3072 obs.),
HMM with $N(0, \sigma^2_{s_t})$ conditional distributions

$$T_2 = \begin{pmatrix} 0.989 & 0.011 \\ 0.020 & 0.980 \end{pmatrix}$$

1. Hidden Markov Models - Limitations

$$T_4 = \begin{pmatrix} \color{blue}{0.973} & 0.027 & 0.000 & 0.000 \\ 0.045 & \color{blue}{0.950} & 0.005 & 0.000 \\ 0.000 & 0.003 & \color{blue}{0.989} & 0.008 \\ 0.000 & 0.000 & 0.051 & \color{blue}{0.949} \end{pmatrix}$$

$$T_6 = \begin{pmatrix} \color{blue}{0.966} & 0.034 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.091 & \color{blue}{0.907} & 0.002 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.002 & \color{blue}{0.971} & 0.026 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.018 & \color{blue}{0.972} & 0.000 & 0.010 \\ 0.000 & 0.000 & 0.000 & 0.051 & \color{blue}{0.949} & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & \color{red}{0.414} & \color{red}{0.586} \end{pmatrix}$$

1. Hidden Markov Models - Limitations

! For less observations the TPM becomes instable much faster
($\sim$ 1500 daily returns $\Rightarrow$ max. 3 states)

Is the 'real world' modeled well with only 2-3 states?

Each states represents a level of risk (Volatility): For $\sigma_1 < \sigma_2 < \sigma_3$

- σ_1: Low risk state
- σ_2: Medium risk state
- σ_3: High risk state

Alternative: Models with many states

Problem: #parameters in the TPM

2. Structured Hidden Markov Models

Data: Series of daily returns

Model: HMM with conditional distributions,

$$P(R_t = r_t \mid S_t = i) \sim N(0, \sigma_i^2),\ i = 1, \ldots, m$$

Idea: Impose reasonable conditions on the

- Variances $\sigma_1^2, \ldots, \sigma_m^2$
- Design of the TPM

$\Rightarrow$ Reduce #parameters

2. Structured Hidden Markov Models

Structure - Variances

- Equidistant: $\sigma_i = \sigma_1 + \frac{i-1}{m-1}(\sigma_m - \sigma_1)$, i=1,...,m (2 parameters)

Structure - TPM

- Type I

$$\boldsymbol{T} = \begin{pmatrix} p & \frac{1-p}{m-1} & \cdots & \cdots & \frac{1-p}{m-1} \\ \frac{1-p}{m-1} & p & \frac{1-p}{m-1} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{1-p}{m-1} \\ \frac{1-p}{m-1} & \cdots & \cdots & \frac{1-p}{m-1} & p \end{pmatrix}$$

2. Structured Hidden Markov Models

- Type II

$$
\boldsymbol{T} = \begin{pmatrix}
p & 1-p & & & & & 0 \\
\frac{1-p}{2} & p & \frac{1-p}{2} & & & & \\
 & \frac{1-p}{2} & p & \frac{1-p}{2} & & & \\
 & & \ddots & \ddots & \ddots & & \\
 & & & \ddots & \ddots & \ddots & \\
 & & & & \frac{1-p}{2} & p & \frac{1-p}{2} \\
0 & & & & & 1-p & p
\end{pmatrix}
$$

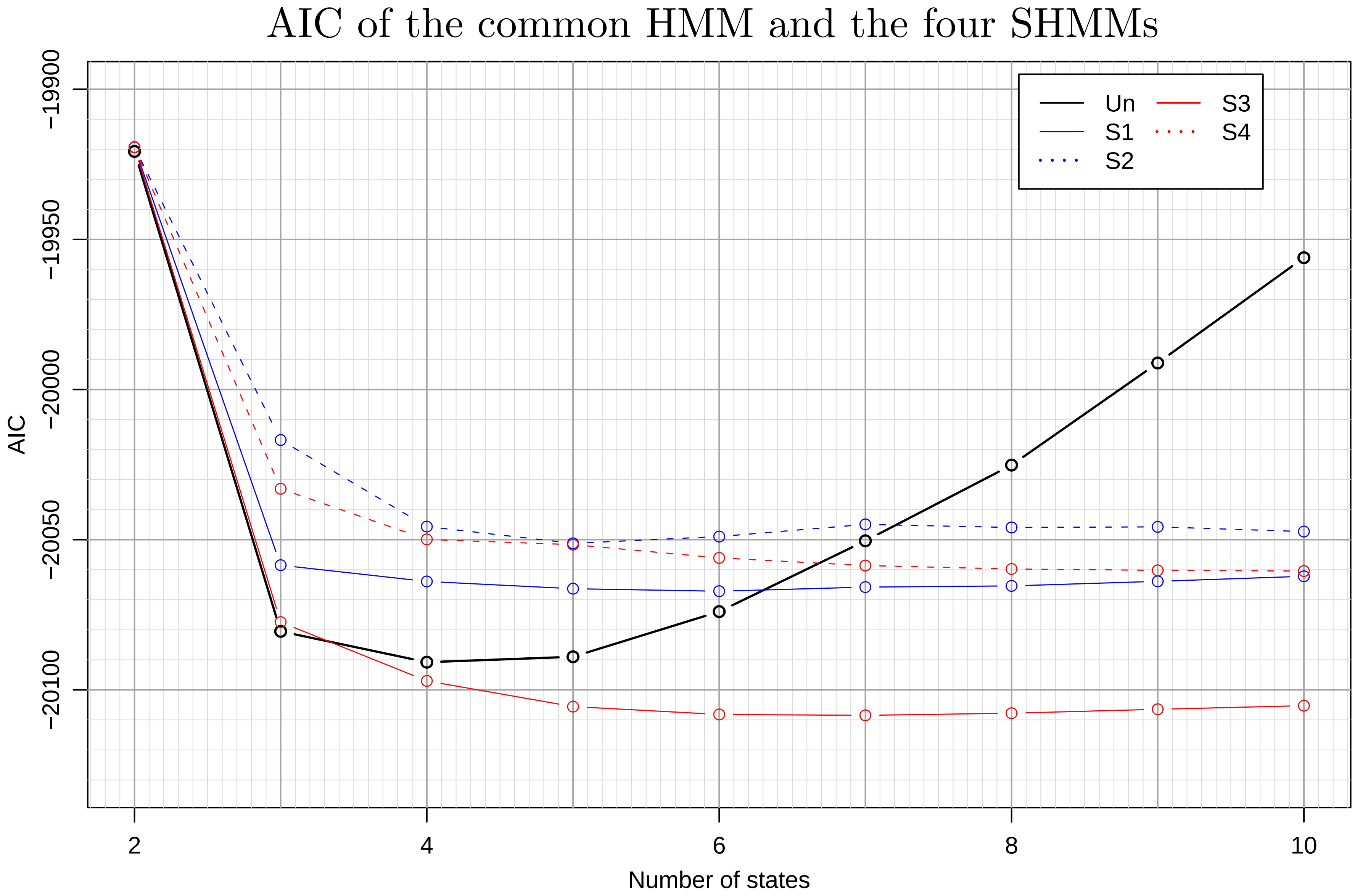
Both structured TPMs have one single parameter p, not $m \cdot (m-1)$

3. Application - Daily Return Series

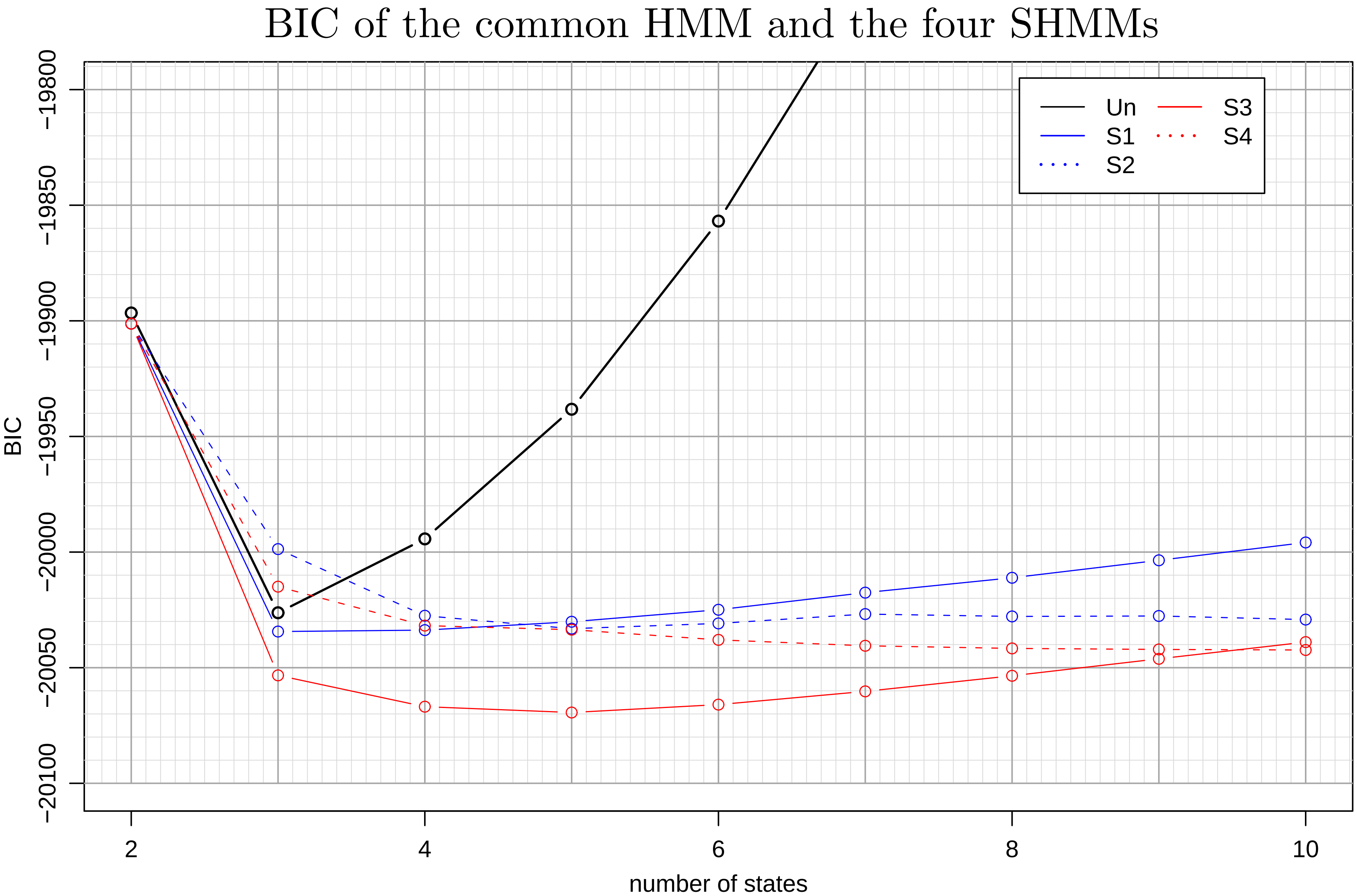
Data: Daily returns from the DJIA, 31.12.93-11.10.2005 (3072 observations)

Models: 5 HMMs, normal conditional distributions with mean zero

- 'Un': HMM without structure (m^2)
- 'S1': TPM Type I, variances unrestricted ($1 + m$)
- 'S2': TPM Type I, variances equidistant (3)
- 'S3': TPM Type II, variances unrestricted ($1 + m$)
- 'S4': TPM Type II, variances equidistant (3)


AIC of the common HMM and the four SHMMs
Un
S1
S2
S3
S4
−19900
−19950
−20000
−20050
−20100
AIC
2
4
6
8
10
Number of states

3. Application - Daily Return Series



3. Application - Daily Return Series

Underlying state sequence: states represent different levels of risk

Inference about the underlying states $\Rightarrow$ Viterbi-algorithm

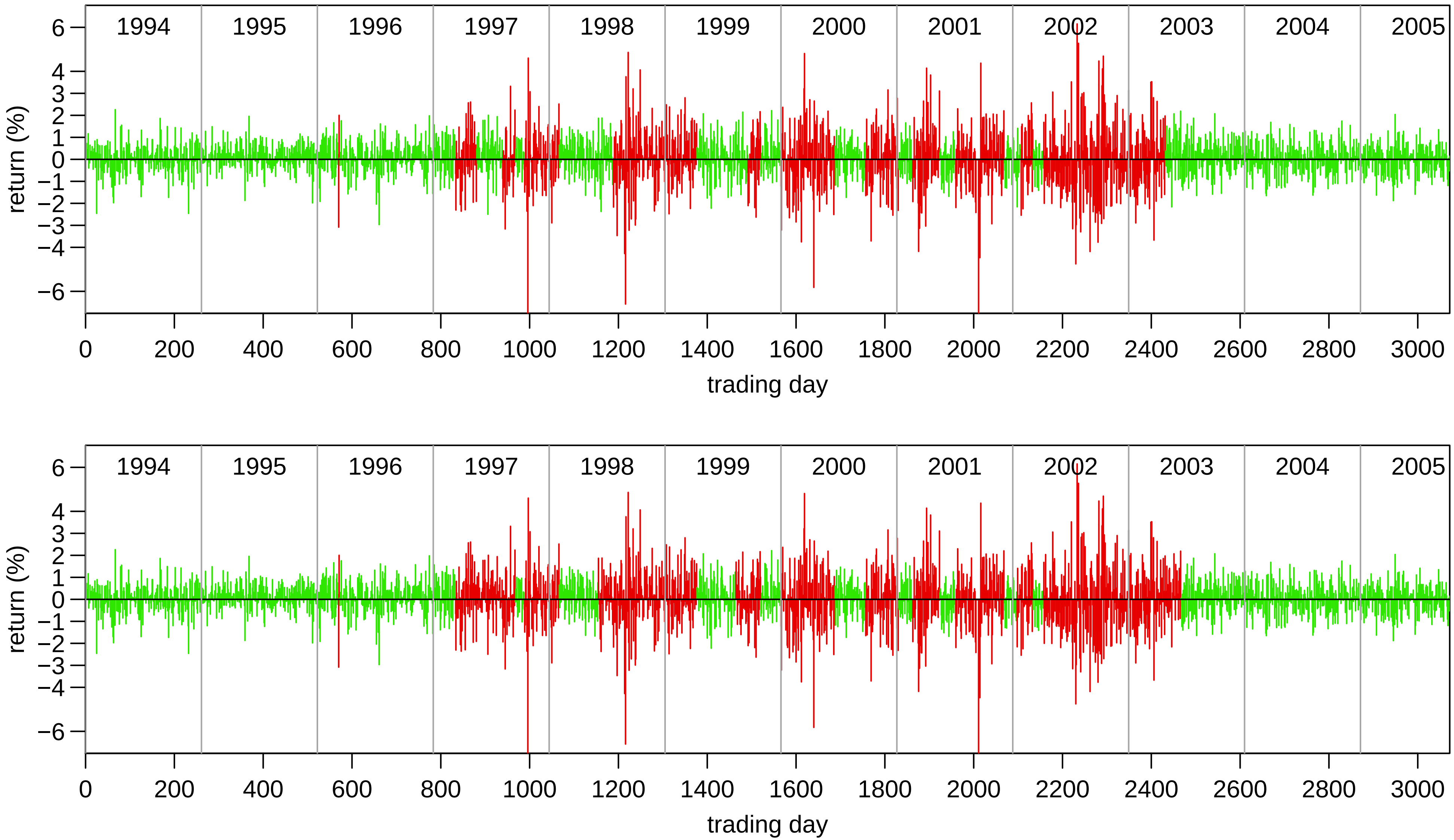
Determine the sequence of states $(i_1^*, \ldots, i_T^*)$, which maximizes the conditional probability

$$(i_1^*, \ldots, i_T^*) = \underset{i_1,\ldots,i_T \in \{1,\ldots,m\}}{\operatorname{argmax}} P(S_1 = i_1, \ldots, S_T = i_T \,|\, X_1^T = x_1^T),$$

where $X_1^T := X_1, \ldots, X_T$

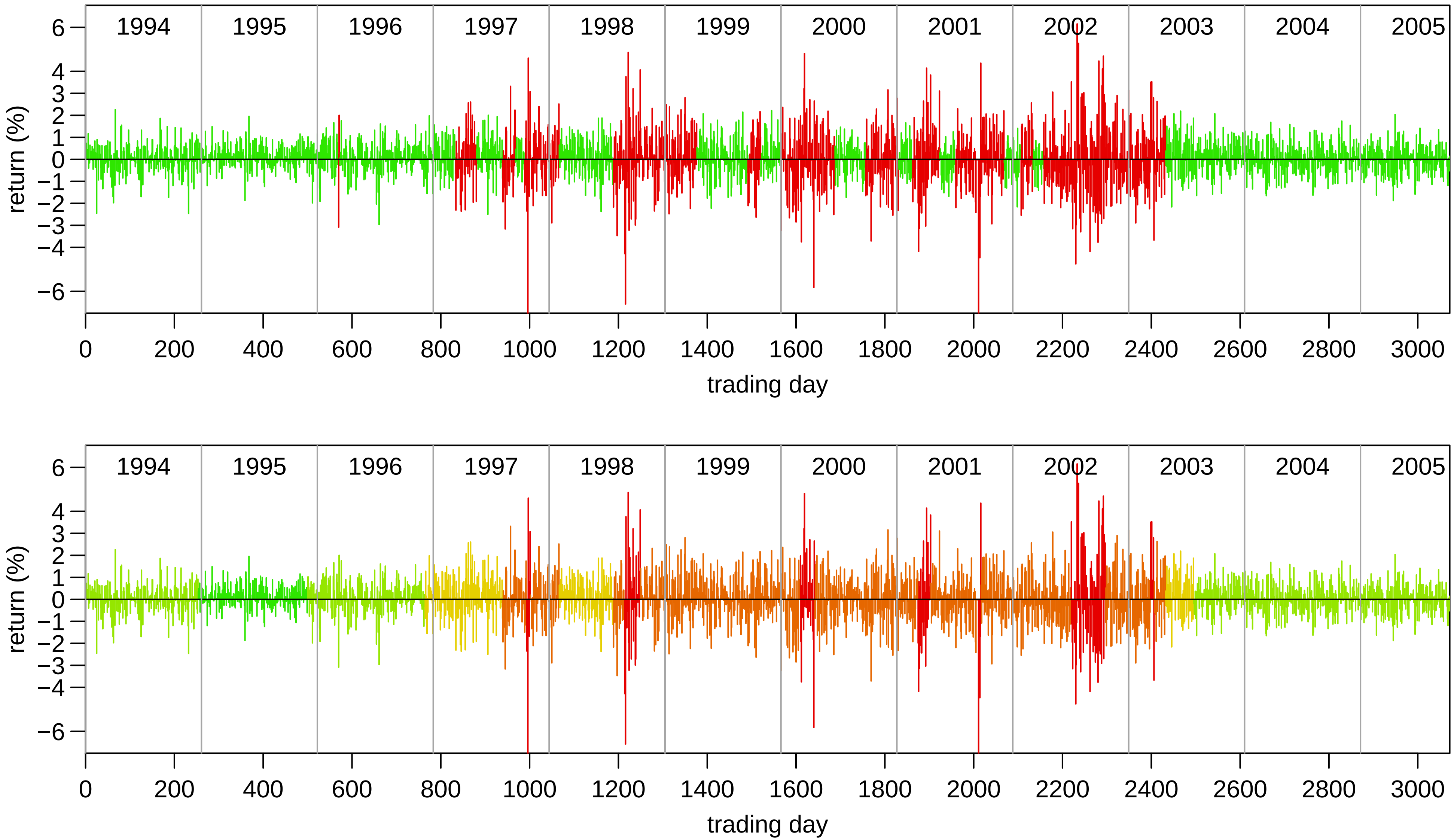
3. Application - Daily Return Series

Daily return series and Viterbi-path, HMM (2 states) and SHMM (2 states)



3. Application - Daily Return Series

Daily return series and Viterbi-path, HMM (2 states) and SHMM (5 states)



3. Application - Daily Return Series

Comparison: HMM - SHMM

- SHMM yields a lower AIC/BIC
- SHMM allows for more precise inference about underlying states
  $\Rightarrow$ Risk structure of daily returns is described better
- Fitting SHMMs is significantly faster than fitting a HMM (with same number of states)

4. Application - Asset Allocation

Underlying state sequence represents different levels of risk

Idea: Viterbi-based investment strategies possible?

Setup: Investment in the DJIA, 2-state-(S)HMM

- High volatility: 100% money market
- Low volatility: 100% DJIA
- Benchmark: DJIA index
- Risk-free interest rate 2%

4. Application - Asset Allocation

In-sample results

	Index	HMM	SHMM
Return (%, p.a.)	7.04	9.89	7.85
s.d. (%, p.a.)	16.5	9.14	8.49
Sharpe-ratio (%, p.a.)	0.37	0.86	0.7

Explanation: Volatile period has also low return

- s.d. 23.7%, Return -5.6% for HMM
- s.d. 22.7%, Return 0.1% for SHMM
- All strong declines fall into the volatile period

Remark: Sharp-ratio $:= \dfrac{E(\text{excess returns})}{s.d.(\text{excess returns})}$

4. Application - Asset Allocation

"Out-of-sample" results

	Index	HMM	SHMM
Return (%, p.a.)	7.04	7.91	6.86
s.d. (%, p.a.)	16.5	9.37	8.69
Sharpe-ratio (%, p.a.)	0.37	0.65	0.58

Explanation: Volatile period still has low return

- s.d. 23.4%, Return -0.4% for HMM
- s.d. 22.5%, Return 2.3% for SHMM

Out-of-sample results SHMM

- Returns DJIA 2. Jan. 1976 - 19. Feb. 2007
- Fit 2-state-SHMM in a sliding window of 2000 observations
- Identify state according to 2 different strategies

  Strategy A - 1. Estimate Viterbi-path for first window

  2. State at time t: $\max_i P(S_t = i \,|\, X_t = x_t, S_{t-1} = s_{t-1})$

  Strategy B - 1. Estimate Viterbi-paths for all windows

  2. Smooth results

	Index	Strategy A	Strategy B
Sharpe-ratio (%, p.a.)	0.41	0.56	0.71

4. Application - Asset Allocation

Conclusion

$\Rightarrow$ Simple strategy with promising results

Open:

- Real out-of-sample forecasts for HMM
- Transaction costs, “finer” strategies
- Weekly/Monthly Returns with states representing also trends

Thank you!