MODELS FOR STOCHASTIC MORTALITY WITH PARAMETER UNCERTAINTY

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Plan

- Introduction
- Approaches to modelling mortality improvements
- A two-factor model for stochastic mortality
- Application
  - The survivor index
- Adding in a cohort effect
- Conclusions

The facts about mortality:

- Life expectancy is increasing.
- Future development of life expectancy is uncertain.

"Longevity risk"

Longevity Risk = the risk that aggregate future mortality rates are lower than anticipated

Focus here: Mortality rates above age 60

STOCHASTIC MORTALITY

n lives, probability p of survival, N survivors

- Unsystematic mortality risk:

  $\Rightarrow N|p \sim$ Binomial(n, p)

  $\Rightarrow$ risk is diversifiable, $N/n \longrightarrow p$ as $n \to \infty$

- Systematic mortality risk:

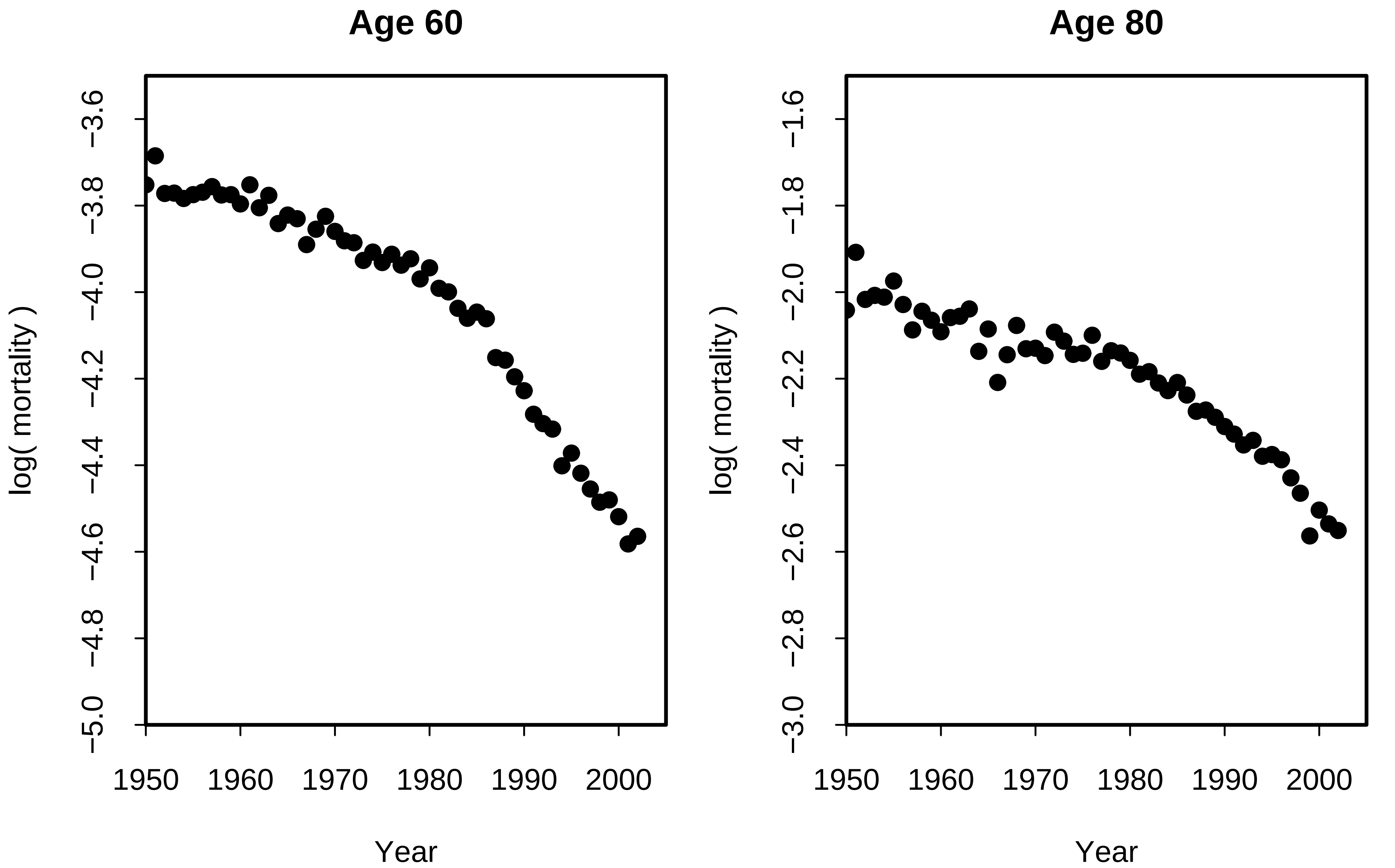
  $\Rightarrow p$ is uncertain

  $\Rightarrow$ risk associated with p is not diversifiable

Where is stochastic mortality relevant?

- Risk management in general
- Pension plans: what level of reserves?
- Life insurance contracts with embedded options.
- Pricing and hedging longevity-linked securities.

England and Wales log mortality rates 1950-2002



Stochastic Models

Different approaches to modelling

- Lee-Carter
- P-splines
- Parametric, time-series models
- Market models
- Age-Period-Cohort extensions

Stochastic Models

Limited historical data $\Rightarrow$

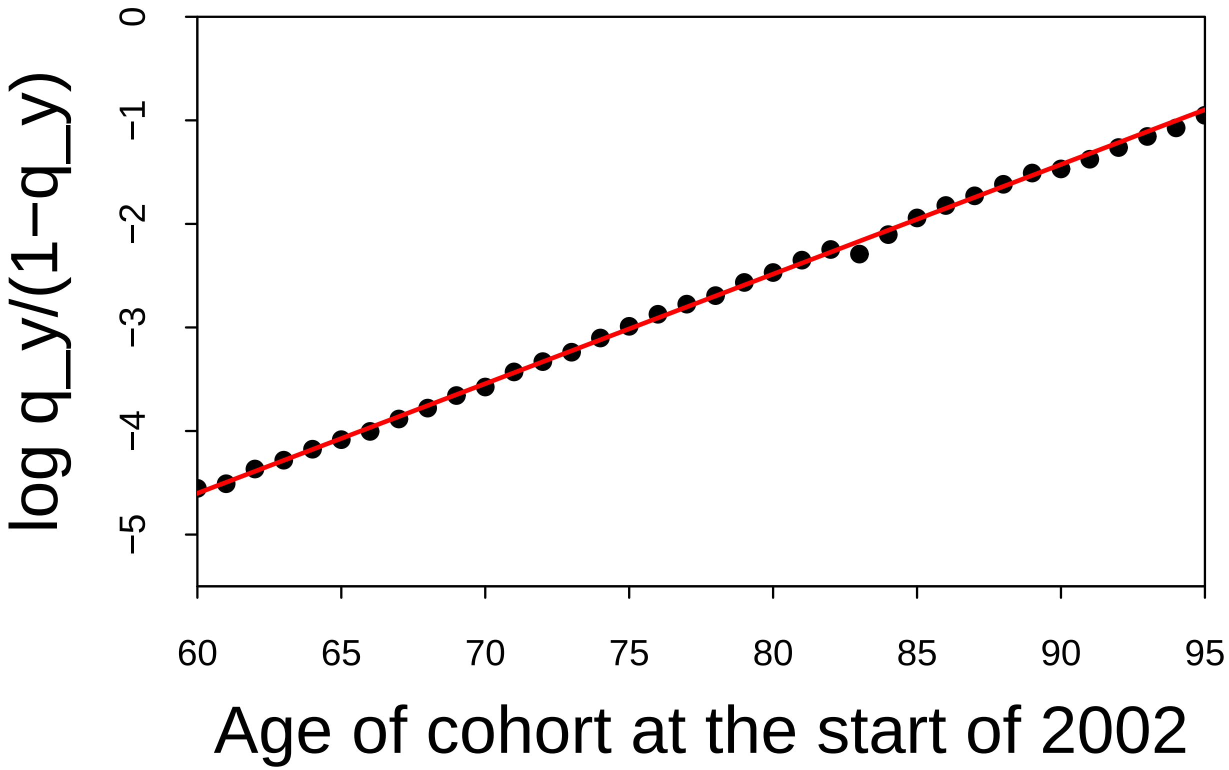
- No single model is 'the right one'

  limited data $\Rightarrow$ Model risk

- Even with the right model

  limited data $\Rightarrow$ Parameter risk

Case study: England and Wales males, age 60-95



q_y = mortality rate at age y in 2002

Data suggests $\log q_y/(1 - q_y)$ is linear

PARAMETRIC TIME-SERIES MODELS

- $x =$ age at time t
- $t - x =$ approximate year of birth
- $q(t, x)$ Mortality rates for the year t to $t + 1$ for individuals aged x at t:
- $N =$ number of factors

PARAMETRIC TIME-SERIES MODELS

General class of models

$$\text{logit}\, q(t,x) = \sum_{i=1}^{N} \beta_x^{(i)} \kappa_t^{(i)} \gamma_{t-x}^{(i)}$$

"Parametric" $\Rightarrow$ $\beta_x^{(i)}$ is a simple function of x

OR

$$q(t,x) = \frac{\exp\left(\sum_{i=1}^{N} \beta_x^{(i)} \kappa_t^{(i)} \gamma_{t-x}^{(i)}\right)}{1 + \exp\left(\sum_{i=1}^{N} \beta_x^{(i)} \kappa_t^{(i)} \gamma_{t-x}^{(i)}\right)}$$

Estimation

- Data: Deaths $D(t,x)$, Exposures $E(t,x)$

  $\Rightarrow$ Crude death rates $\hat{m}(t,x) = D(t,x)/E(t,x)$

- Underlying $m(t,x) = -\log[1-q(t,x)]$

  (by assumption)

- $D(t,x) \sim$ independent Poisson$\Big(m(t,x)E(t,x)\Big)$

- Maximum likelihood $\Rightarrow \hat{\beta}_x^{(i)}$, $\hat{\kappa}_t^{(i)}$ and $\hat{\gamma}_{t-x}^{(i)}$

TWO PARAMETRIC TIME-SERIES MODELS

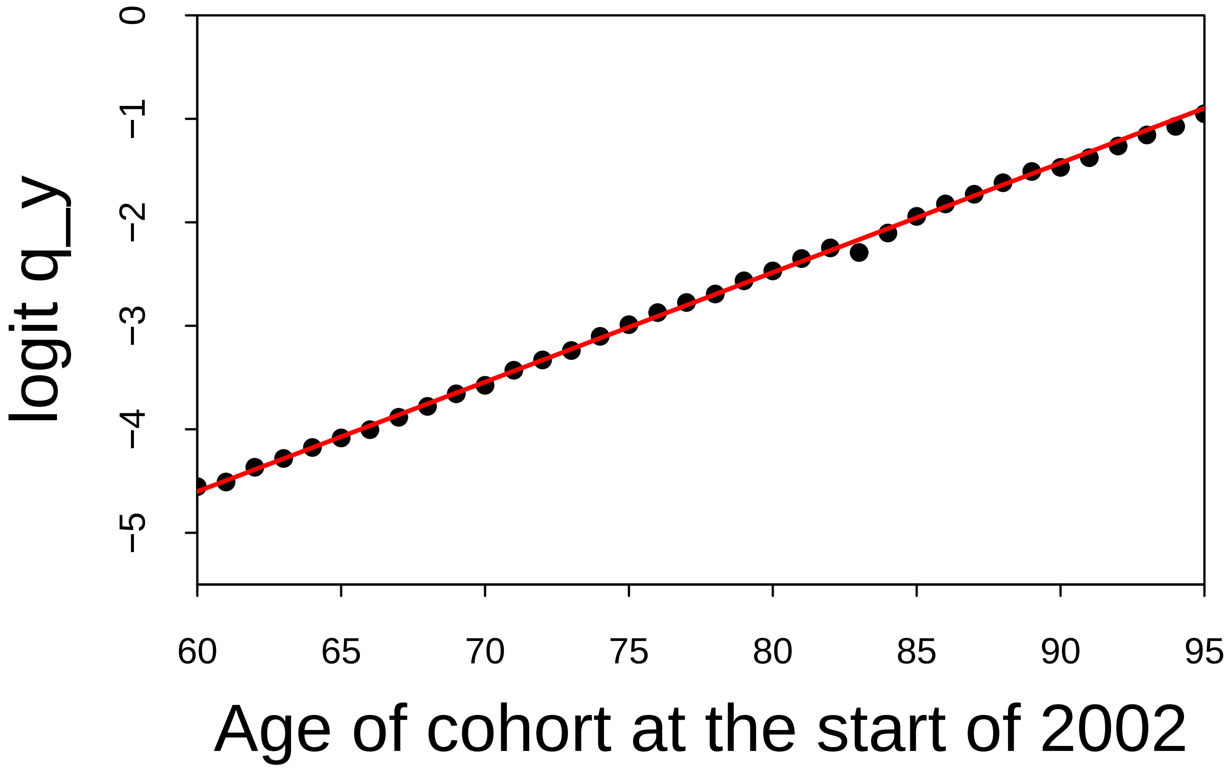
Model 1 (Age-Period model):

$$\text{logit}\, q(t,x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x})$$

Model 2 (Age-Period-Cohort model):

$$\begin{aligned}\text{logit}\, q(t,x) \;=\; & \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) \\ & +\kappa_t^{(3)}\left[(x - \bar{x})^2 - \sigma_x^2\right] \\ & +\gamma_{t-x}^{(4)}\end{aligned}$$

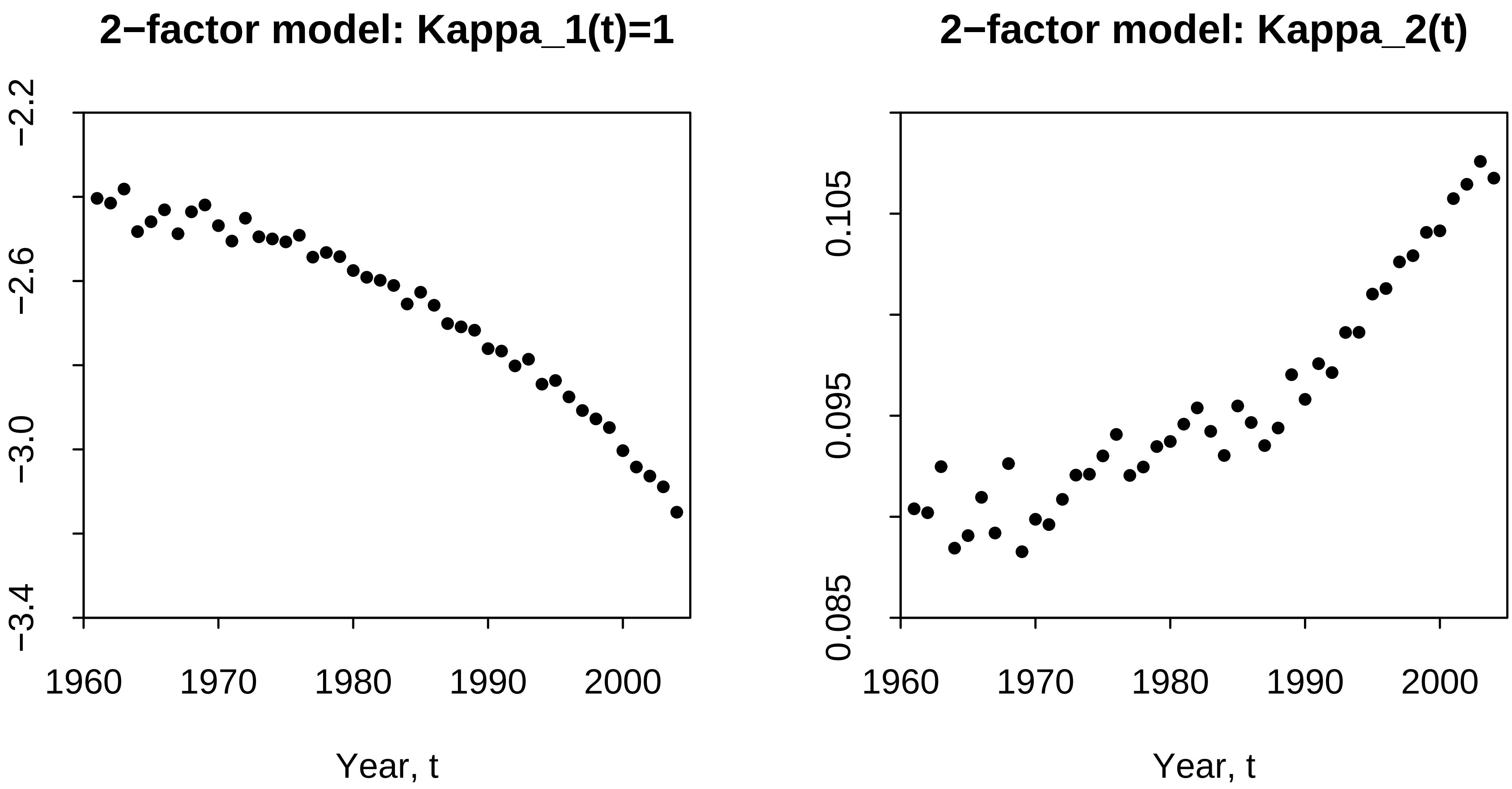
Model 1: Case study – England and Wales males



$\kappa_t^{(1)} \Rightarrow$ level

$\kappa_t^{(2)} \Rightarrow$ slope

Model 1


2−factor model: Kappa_1(t)=1
−2.2
−2.6
−3.0
−3.4
1960
1970
1980
1990
2000
Year, t
2−factor model: Kappa_2(t)
0.105
0.095
0.085
1960
1970
1980
1990
2000
Year, t

$\kappa_t = (\kappa_t^{(1)}, \kappa_t^{(2)})'$

Model: Random walk with drift

$$\kappa_{t+1} - \kappa_t = \mu + CZ(t+1)$$

- $\mu = (\mu_1, \mu_2)' =$ drift
- $V = CC' =$ variance-covariance matrix
- Estimate μ and V
- Quantify parameter uncertainty in μ and V

WHY 2 FACTORS? (i.e. $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$)

Data suggest changes in *underlying* mortality rates are not perfectly correlated across ages.

1 factor (e.g. most Lee-Carter-based models)
$\Rightarrow$ changes over time in the $q(t, x)$ are perfectly correlated.

Bayesian approqch to parameter uncertainty

- Jeffreys prior $p(\mu, V) \propto |V|^{-3/2}$.
- Data: vector $D(t) = \kappa_t - \kappa_{t-1}$ for $t = 1, \ldots, n$
- MLE's: $\hat{\mu}$ and $\hat{V}$.
- Posterior:

$$V^{-1}|D \sim \text{Wishart}(n-1, n^{-1}\hat{V}^{-1})$$

$$\mu|V, D \sim MVN(\hat{\mu}, n^{-1}V)$$

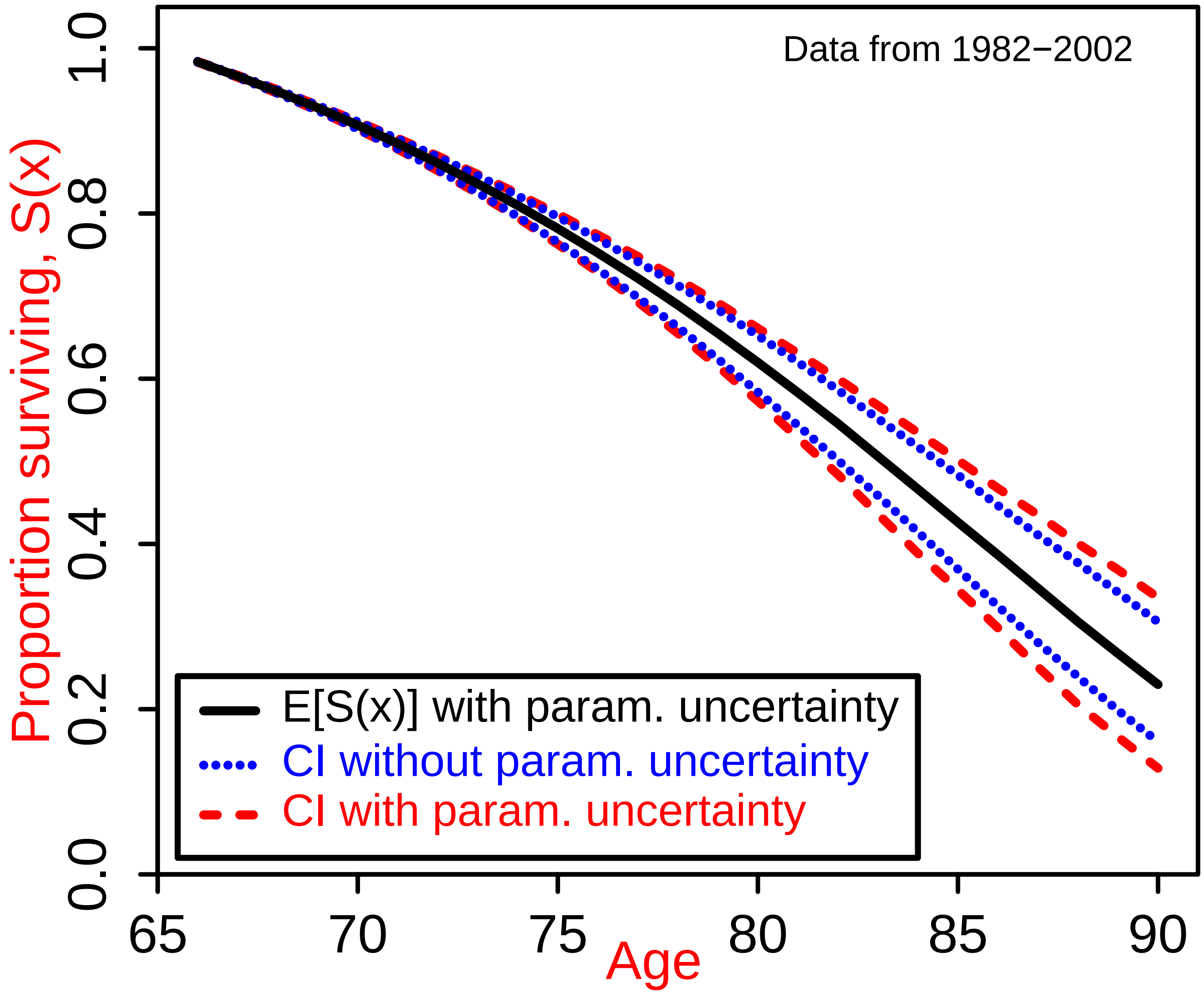
Application: cohort survivorship

- Cohort: Age x at time $t = 0$
- $S(t, x) =$ survivor index at t

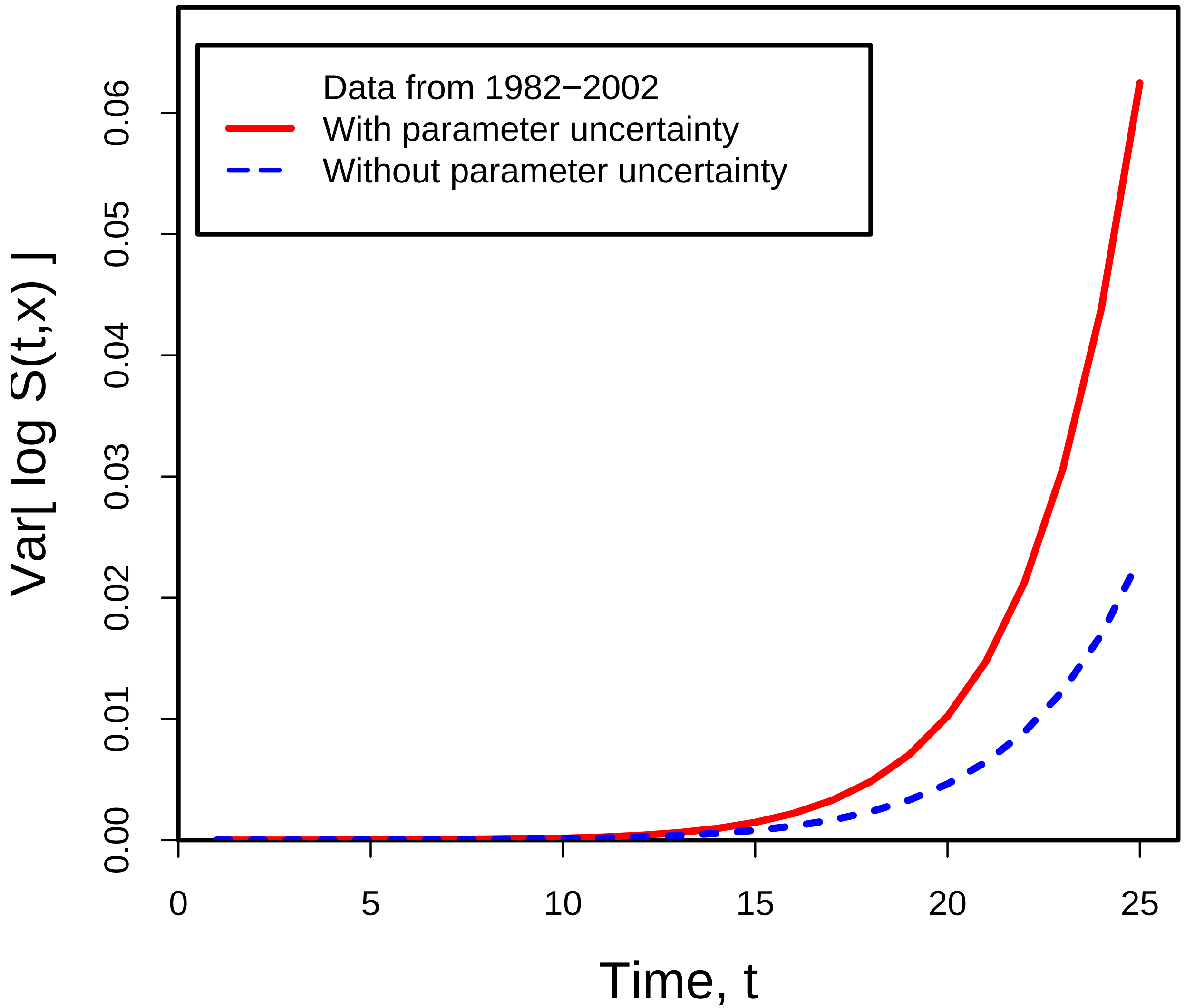
  proportion surviving from time 0 to time t

$$S(t, x) \; = \; (1 - q(0, x)) \times (1 - q(1, x + 1) \times \ldots$$
$$\ldots \times (1 - q(t - 1, x + t - 1))$$

90% Confidence Interval (CI) for Cohort Survivorship



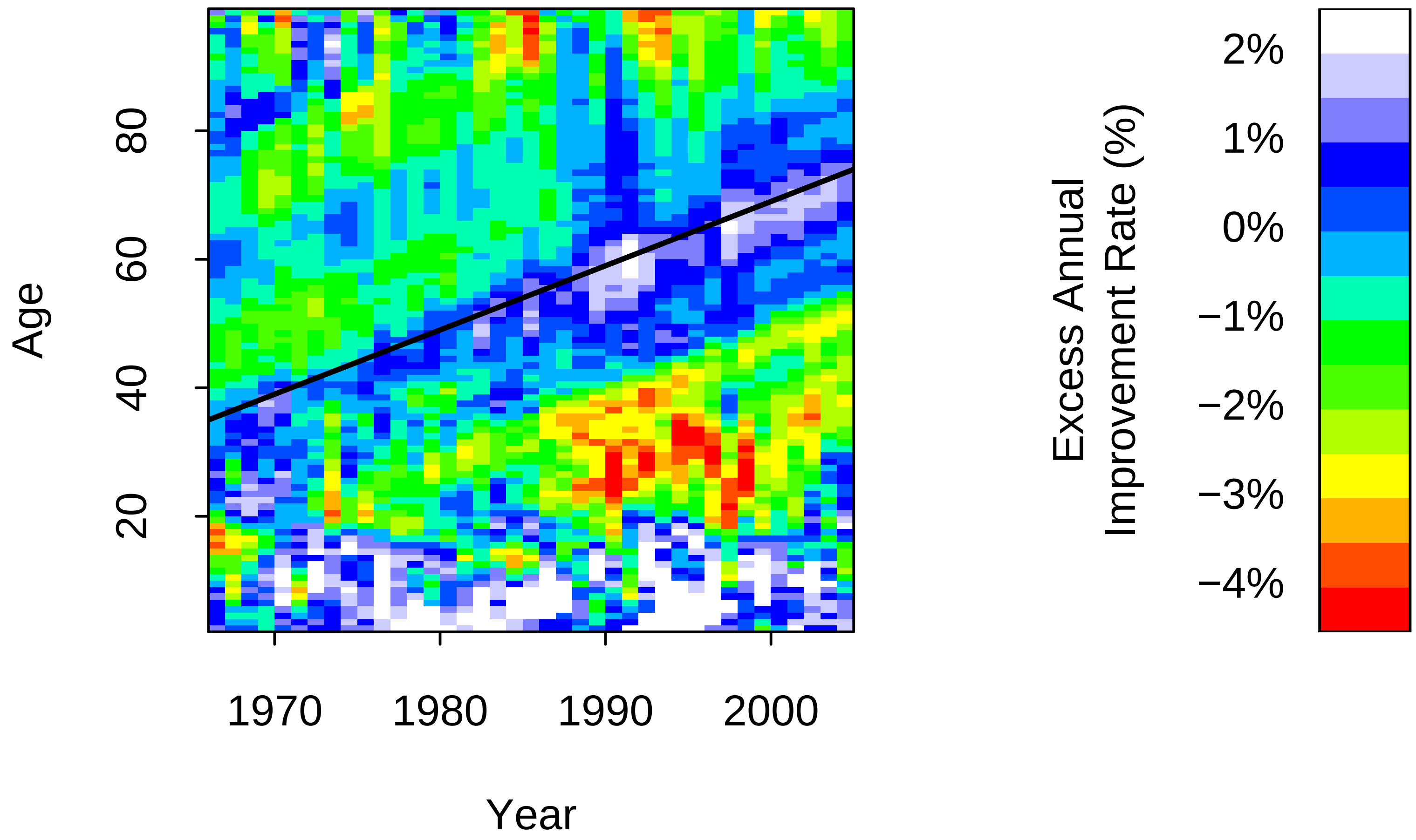
$Var[\log S(t,x)]$ for $x = 65$



Cohort Survivorship: General Conclusions

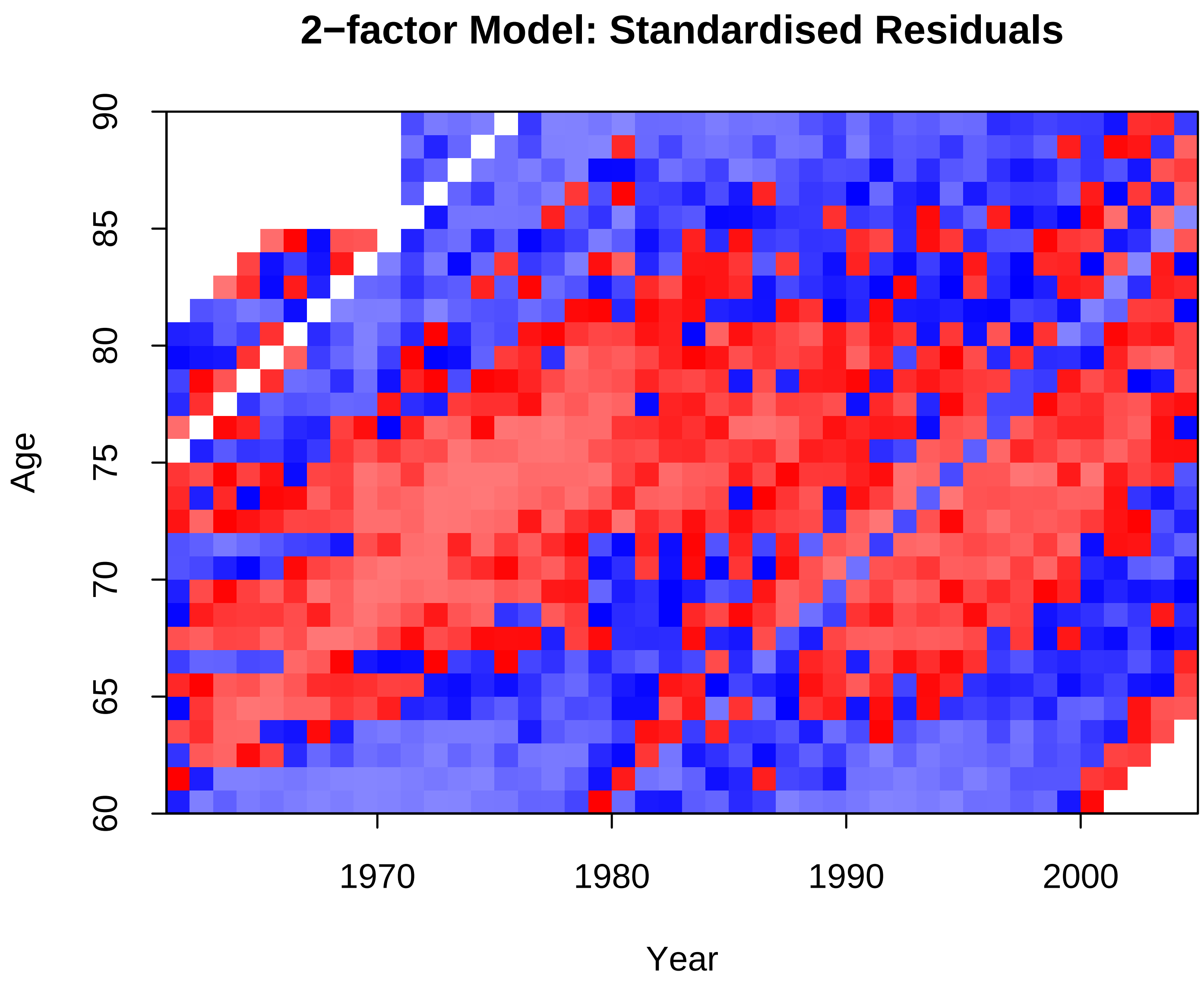
- Less than 10 years:
  - Systematic risk not significant
- Over 10 years
  - Systematic risk becomes more and more significant over time
- Over 20 years
  - Model and parameter risk begin to dominate

The cohort effect: England and Wales



Mortality improvement relative to calendar year average.

The Cohort Effect



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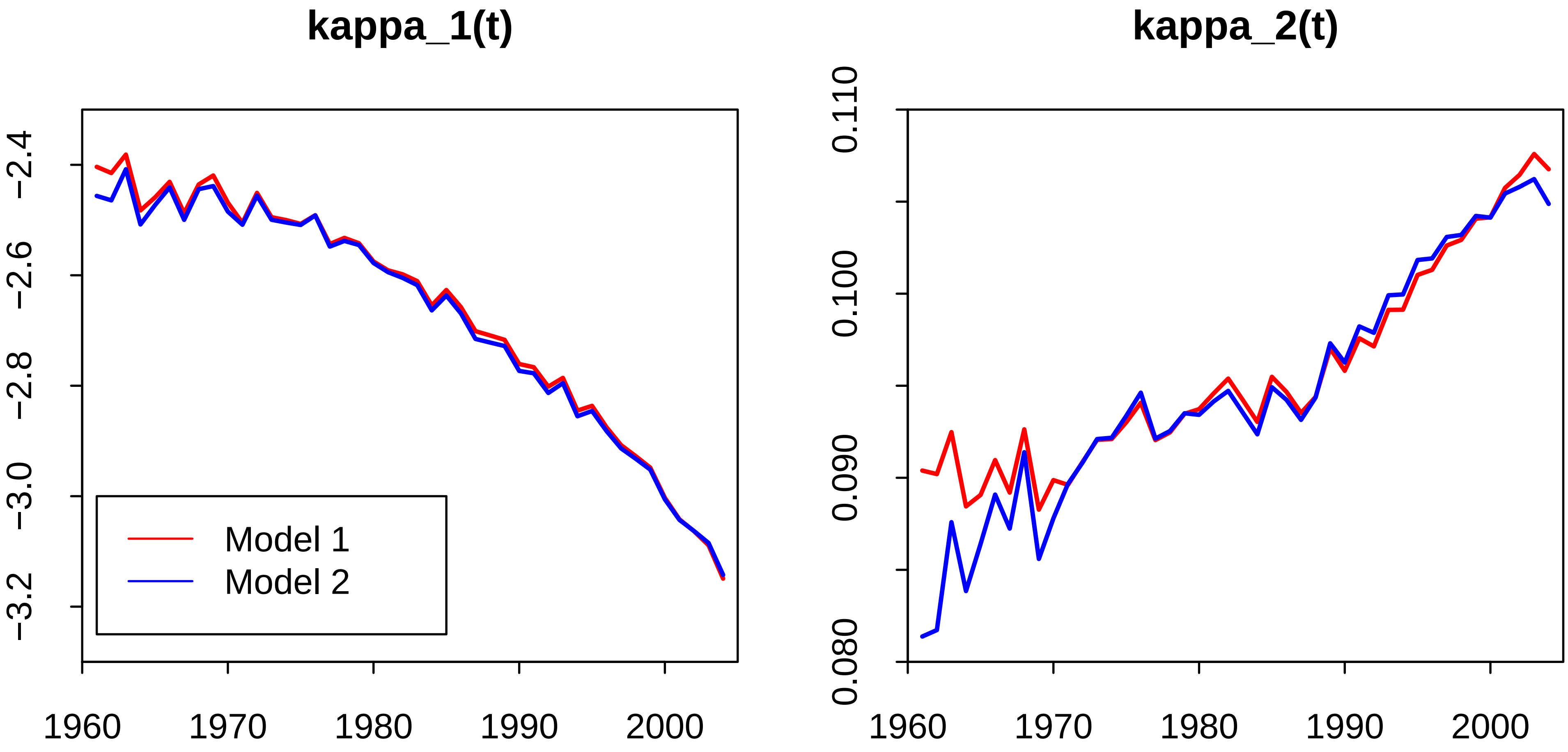
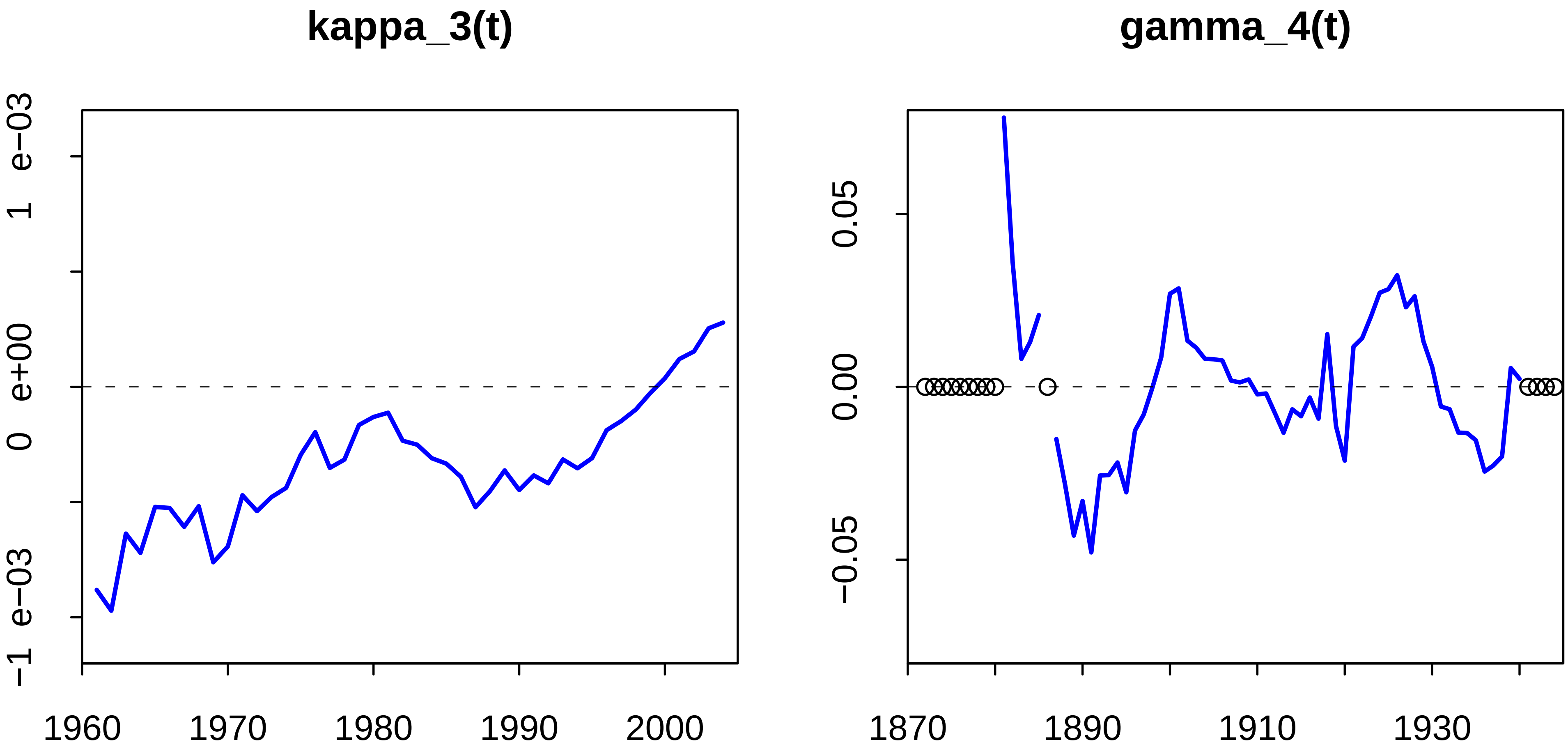
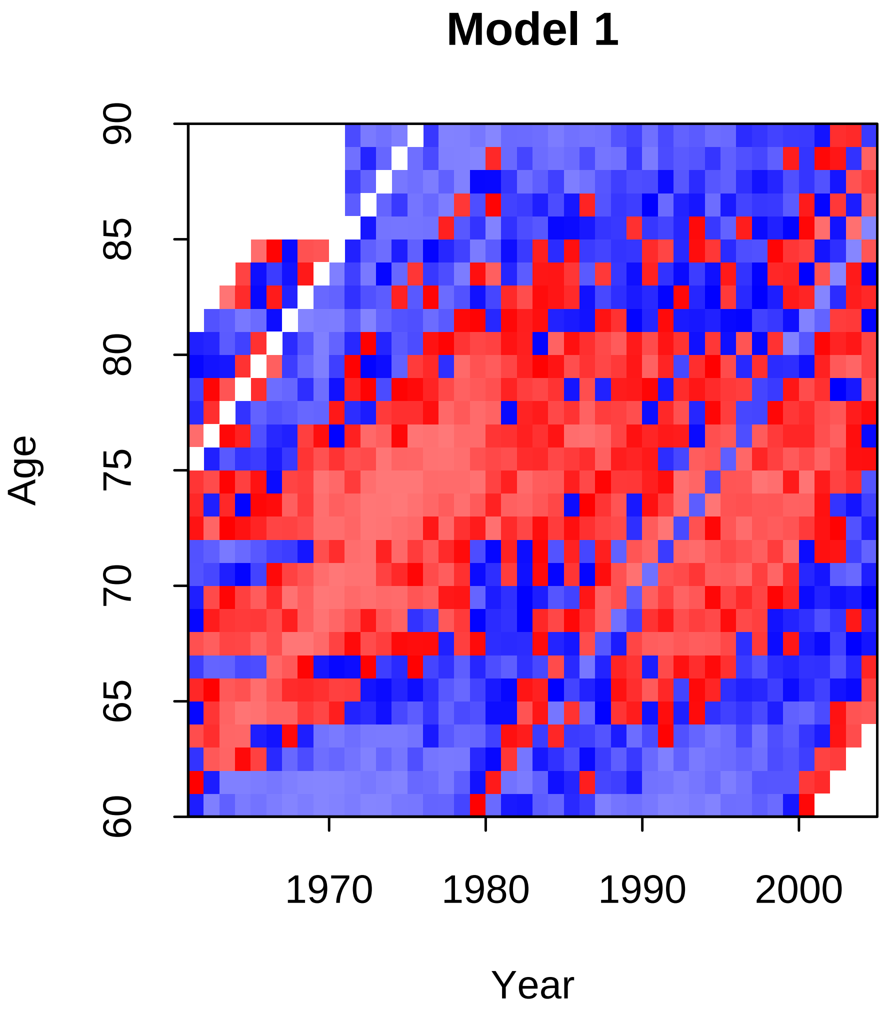
Model 1 versus Model 2

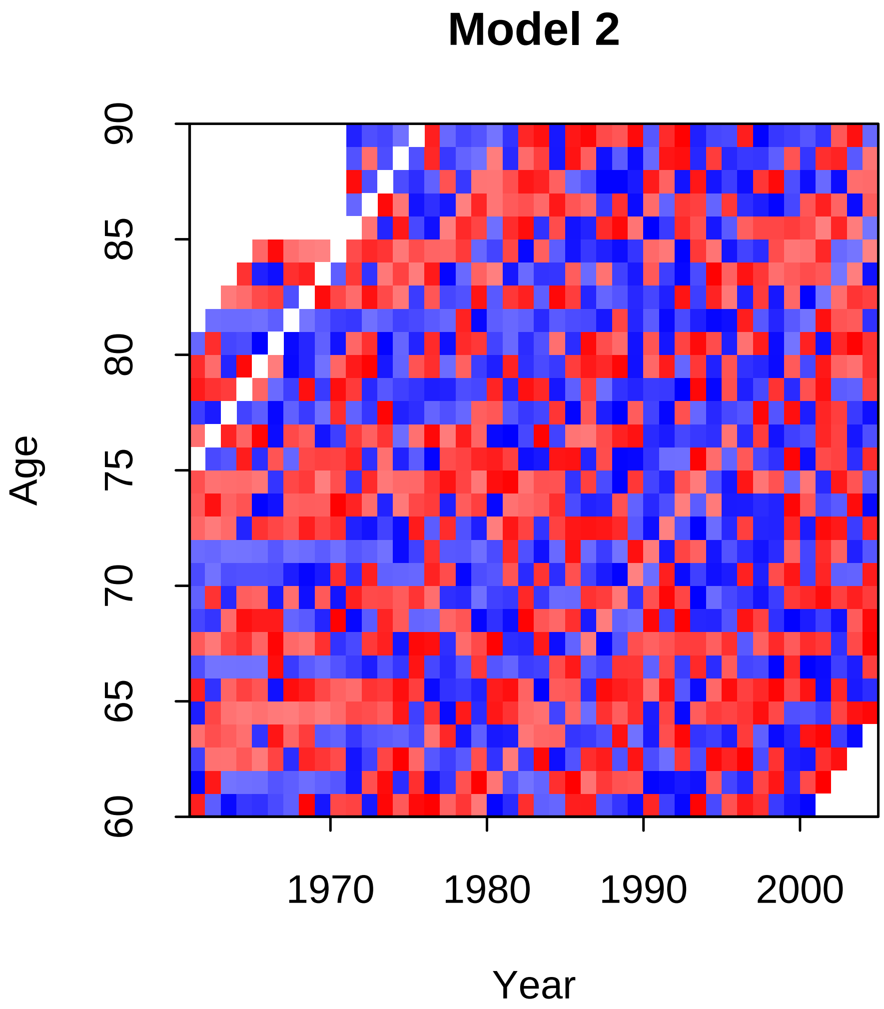


Model 2: extra factors

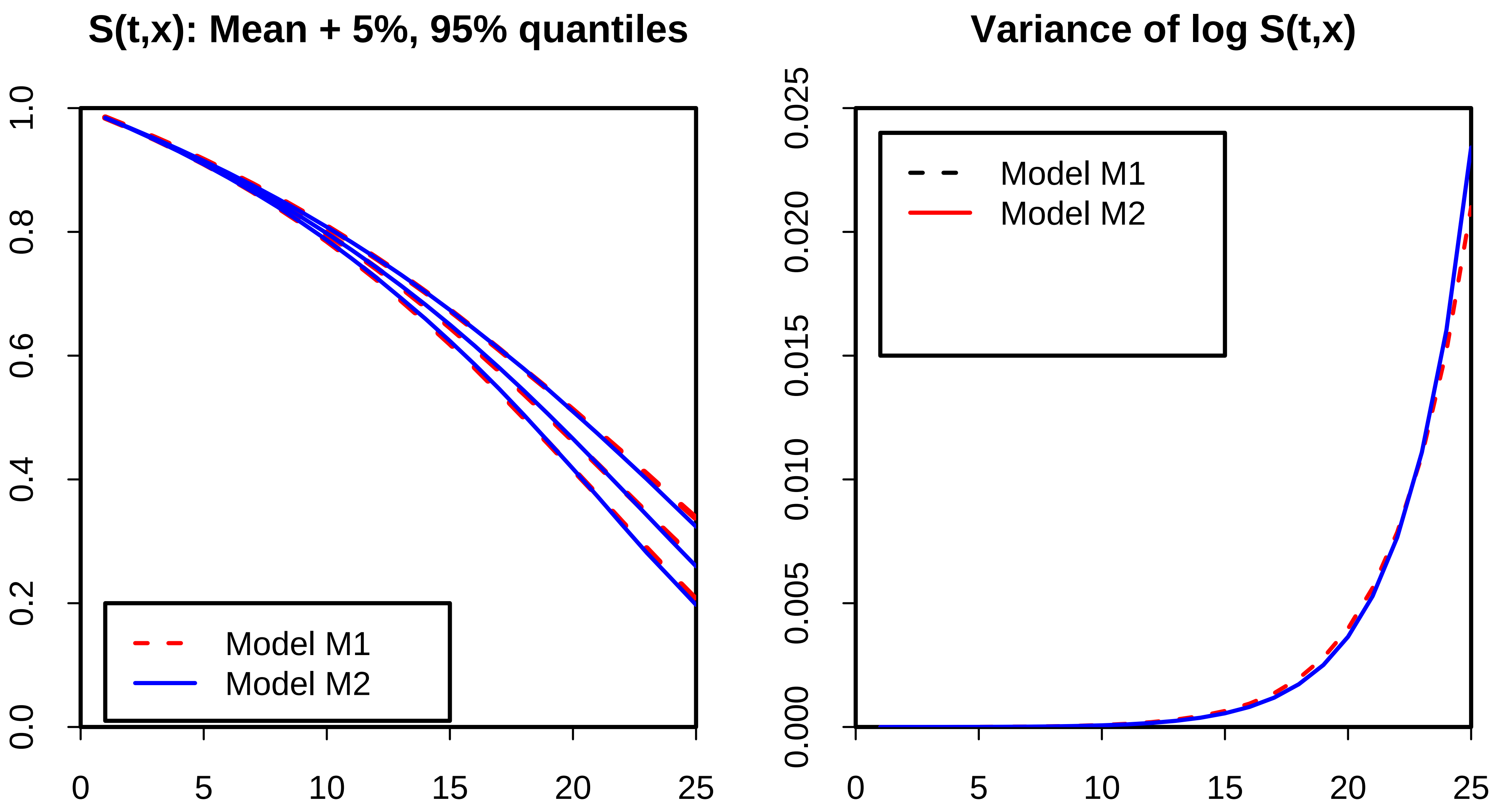


Standardised residuals





Survivor index projections



4% Annuity Values

	Model 1	Model 2	
		$\gamma_{1944}^{(4)} = -0.0398$	$\gamma_{1944}^{(4)} = 0.0402$
$x = 60$	13.472	13.557	13.350
$x = 65$	11.449	11.451	
$x = 70$	9.325	9.354	
$x = 75$	7.220	7.240	

Conclusions 1

- Stochastic models important for
  - risk measurement and management
  - valuing life policies with option characteristics
- Two models out of many possibilities
- Significant longevity risk in the medium/long term

Conclusions 2

- Parameter risk is important
- Model risk might be important
- The significance of longevity risk varies from one problem to the next:
  - In absolute terms
  - As a percentage of the total risk

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