Operational Risk Modeling and Quantification

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Agenda

- Loss Distribution Approach for OR
- Challenges. Is it possible to model/quantify OR?
- OR Frequency and Severity
- OR Insurance and point process
- Estimation with Truncated Data
- Dependence between risks
- Distribution tails: EVT, mixtures

Advanced Measurement Approach: bottom-up Loss Distribution Approach)

Internal Loss Data

(scaling, bias)

External Loss Data

(scaling, bias)

Expert Opinions

(business judgments on

#losses and $ranges)

Insurance

Dependence Factors

Control/Risk Indicator

Risk Frequency and
Severity distributions

Monte Carlo
simulations

Risk annual loss distributions

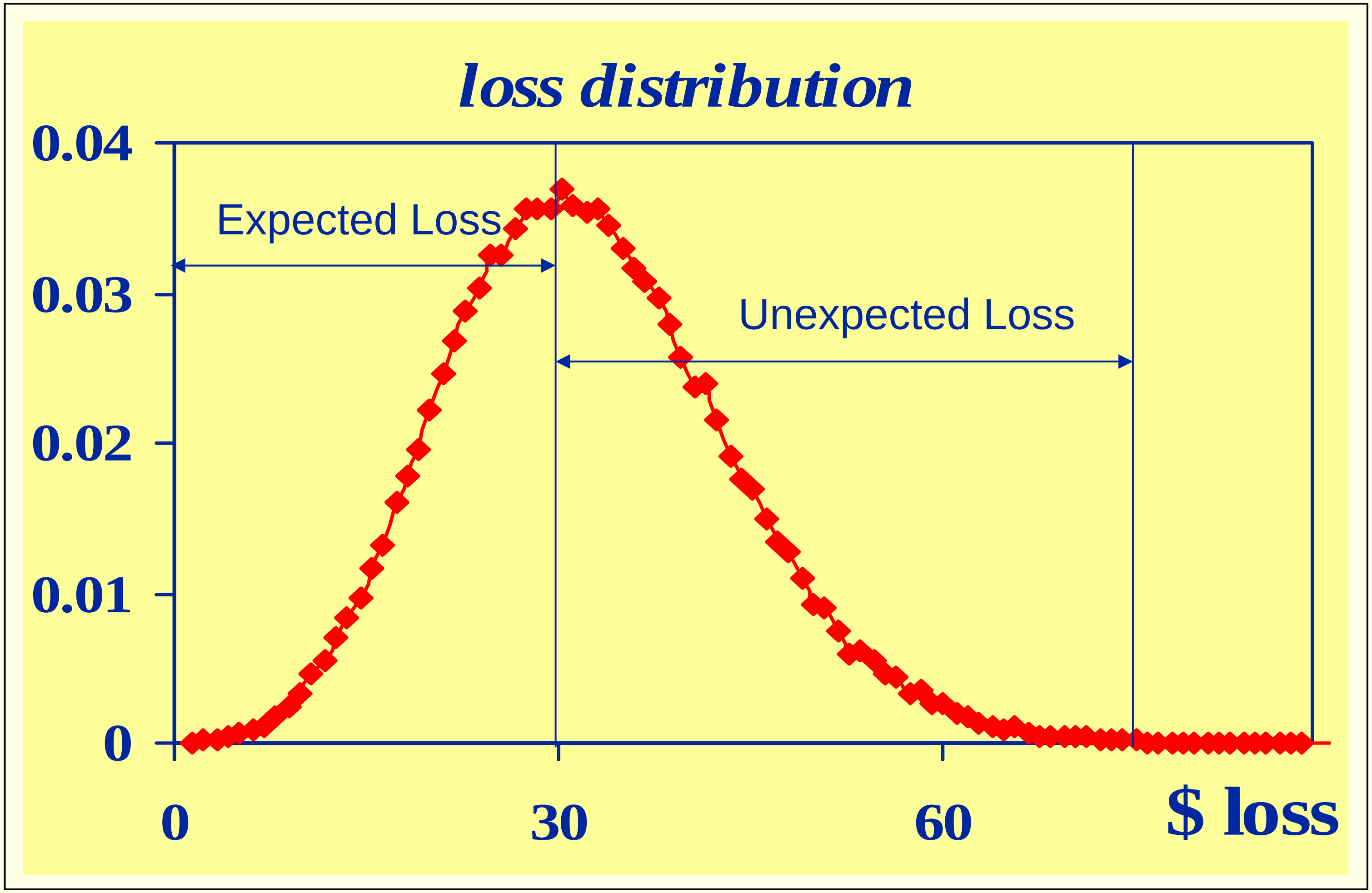
Total annual loss distribution (over all risks)

Annual Capital Charge (Expected and Unexpected losses)

Capital Sensitivities, what if scenarios

◆ Annual Capital Charge (≥100,000 MC simulations)

unexpected loss=VaR-Expected Loss; Pr [Loss<=VaR]=0.999



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Challenges; Tools

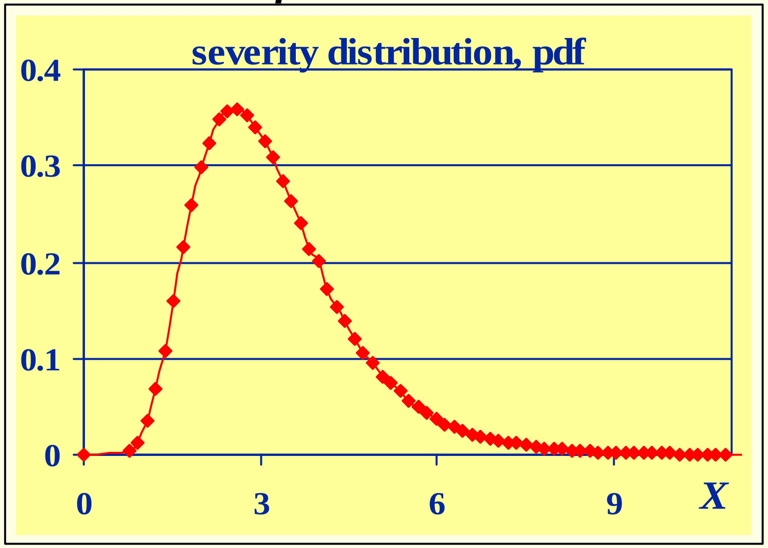
- Definition, identification, measurement, monitoring, indicators/controls
- Data Truncation: known threshold, stochastic threshold, unknown threshold
- Limited Data: mixing internal and external data via credibility theory, Bayesian techniques
- External data relevance: scaling
- Expert opinions
- Data sufficiency: capital charge accuracy
- Correlation between risks and its estimation: copula, common process
- Company indicators: regression/factorial analysis
- OR insurance: point processes
- Non-Gaussian distributions, Fat tails: EVT, mixed distributions, splices
- VaR pitfalls: coherent risk measures, expected shortfall
- Calculation Capital charge and its sensitivities: Monte Carlo simulation (Computing time/RAM), Fast Fourier Transform

Loss Distribution Approach: key parameters

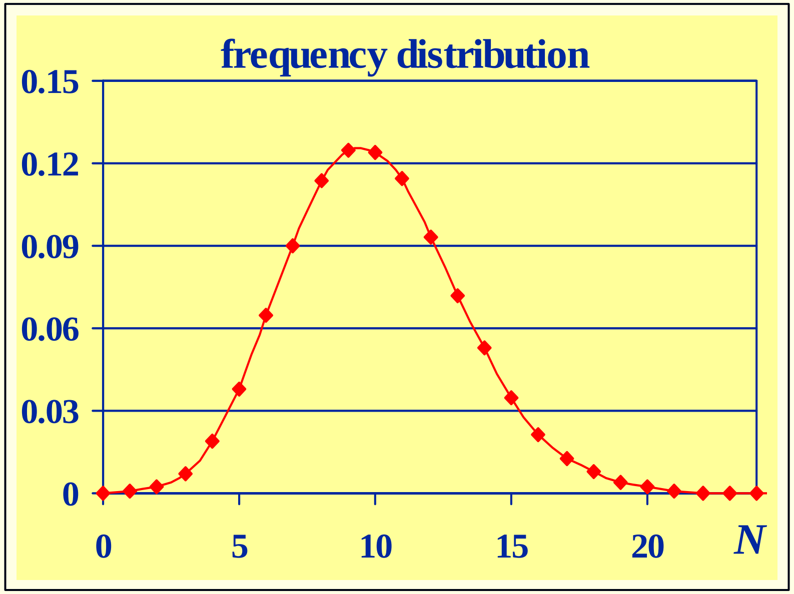
- Annual frequency of events
- Loss severity of the event
- Insurance against losses
- Correlation between risks

Single Risk

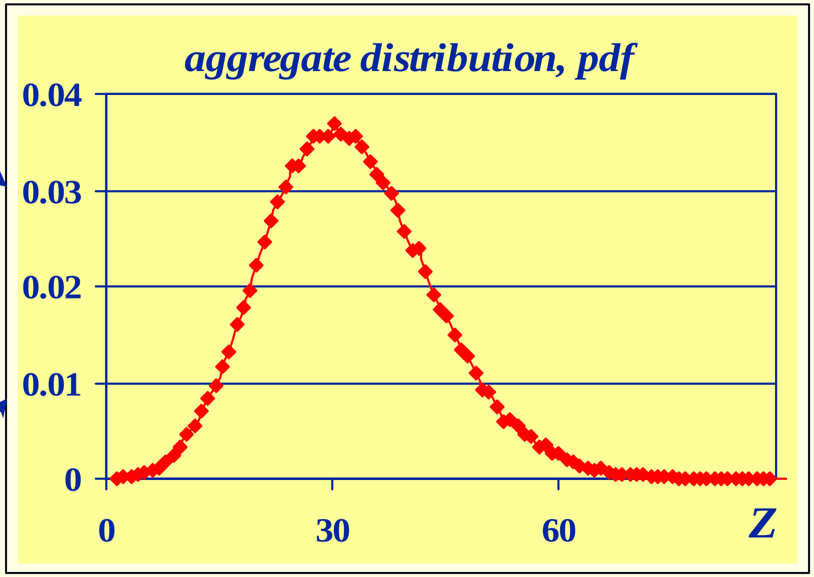
loss of the event, X



annual number of events, N



$$annual\ loss, Z = \sum_{i=1}^{N} X_i$$



- **Severity Distributions *f(X)*:**

  e.g. LogNormal, Gamma, Weibull

- **Frequency distributions *P(N)*:**

  e.g. Poisson, Negative Binomial, Binomial

- **Annual loss**

$$Z = \sum_{i=1}^{N} X_i$$

- **Assumptions**

$$N \text{ and } X_i \ (i = 1, \ldots, N) \text{ are independent}$$

$$X_i \text{ and } X_j \ (i \neq j) \text{ are independent}$$

Distribution of Z : semi-analytic, Monte Carlo simulations

Insurance for Operational Risks

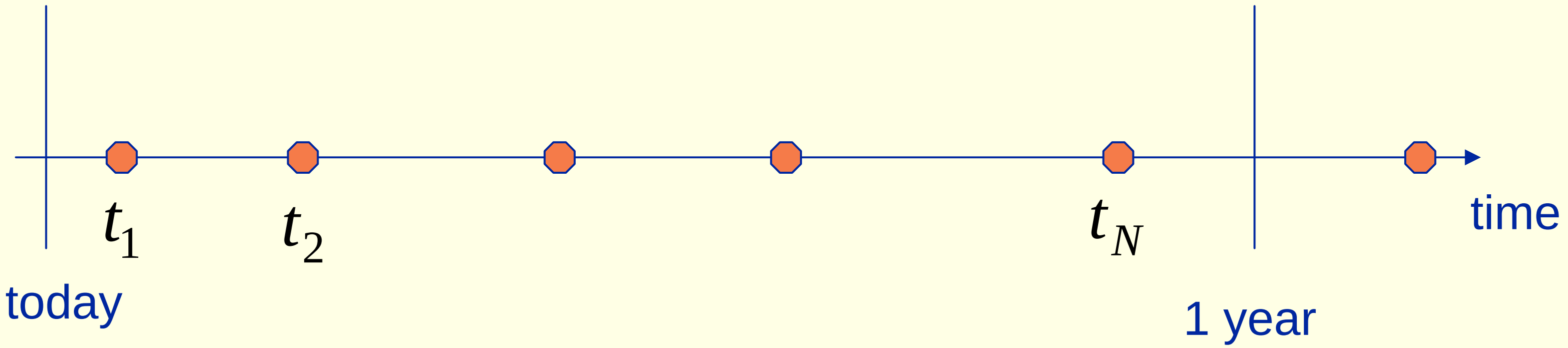
- Insurance: probability of coverage, insurer default, cover limit, excess, regulatory cap
- Modelling of loss event times is required instead of event frequency to address OR insurance.
- point process: $0 < t_1 < \ldots < t_N \leq 1 < t_{N+1} < \ldots$

  e.g. homogeneous Poisson process

$$\delta t = t_{i+1} - t_i \sim \text{Exponential}(\lambda),\ N \sim \text{Poisson}(\lambda)$$

non-homogeneous Poisson: $\lambda(t)$

doubly stochastic Poisson: $\lambda \sim \text{Gamma}(.) \Rightarrow N \Rightarrow \text{NegativeBinomial}(.)$



Data Truncation models

Known constant truncation level $X_i > L, i = 1,2,\ldots.$

Known variable truncation level $X_i > L_i, i = 1,2,\ldots.$

Unknown truncation level

Stochastic truncation level $L \sim g(.)$

Known threshold

- Known constant truncated level, i.e. loss data $X_i \geq L, i = 1,2,....$

  Untrunctaed severity distribution $f(X), X \geq 0$

  Truncated severity distribution

$$f(X \mid X > L) = \frac{f(X)}{\Pr[X > L]}; \quad X \geq L; \quad \Pr[X > L] = \int_L^\infty f(X)dX$$

- Severity pdf fit via e.g. Maximum Likelihood Method

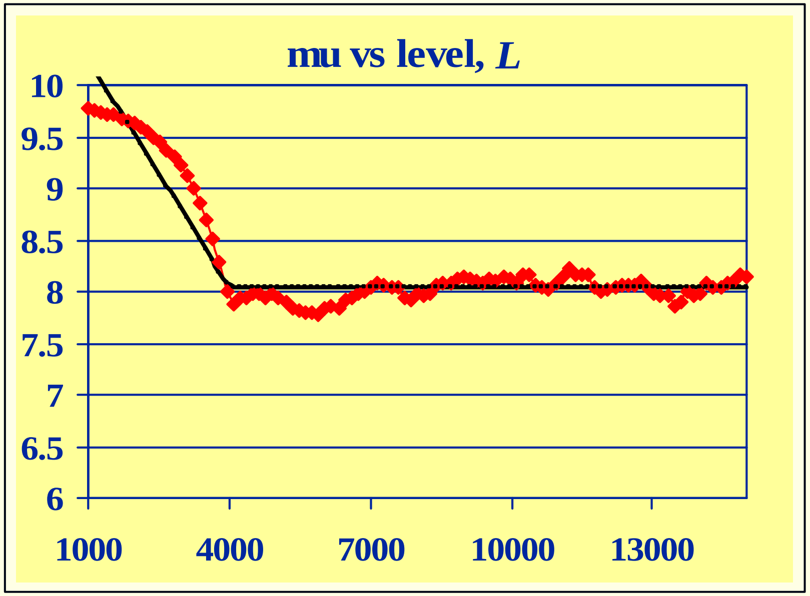
$$\Psi(\alpha_1,...,\alpha_N) = \prod_{i=1}^{K} \frac{f(X_i)}{\Pr[X > L]}$$

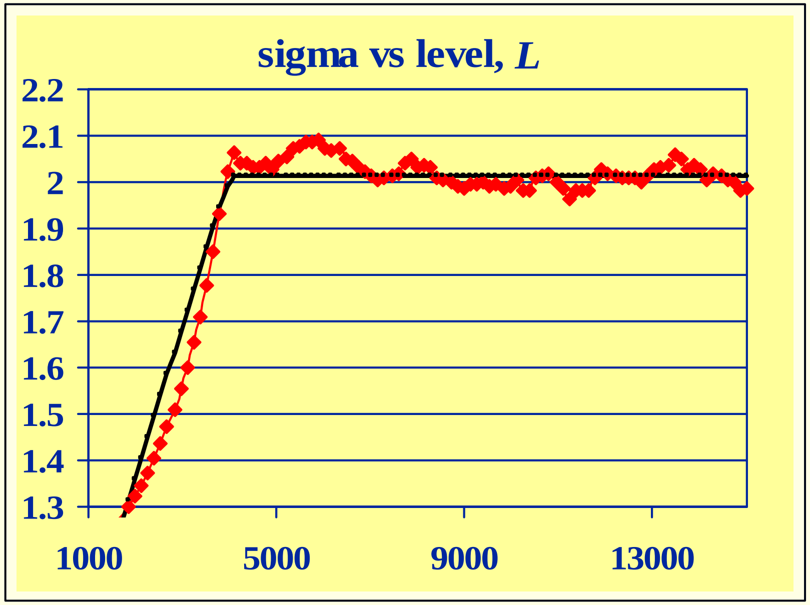
- Frequency adjustment $N^{(true)} \approx N^{obs} / \Pr[X > L]$

Unknown truncation level

- L is an extra parameter in likelihood function
- L is unknown: $\mu(L) = \begin{cases} \alpha \times L + \beta; & L < \gamma \\ \alpha \times \gamma + \beta; & L \geq \gamma \end{cases}$ $\min_{\alpha,\beta,\gamma} \sum_i [\mu^{obs}(L_i) - \mu(L_i)]^2$

Example: *X*~LogNormal(mu=8, sigma=2), *L*=4000
estimates: mu≈8.05, sigma≈2.01, *L*≈4001





Stochastic truncation level

Severity distribution of losses $f(X),\ X \geq 0$

reported losses $X_i > L, i = 1,\dots,K$ **where** $L \sim g(.)$

conditional pdf $f(X \mid L = a) = 1_{X>a} \dfrac{f(X)}{\Pr[X > a]};\quad X \geq a;\quad \Pr[X > a] = \int_a^\infty f(y)dy$

marginal distribution of reported losses

$$\tilde{f}(X) = \int_0^\infty f(X \mid L = a)g(a)da = f(X)\int_0^X \frac{g(a)}{\Pr[X > a]}da \qquad \Psi = \prod_i \tilde{f}(X_i)$$

$$N_{true} = N_{obs} / \Pr[X > L];\ \Pr[X > L] = \int_0^\infty g(L)dL \int_L^\infty f(X)dX$$

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Dependence between risks

- Diversification: $C(Z = R_1 + \ldots + R_n) < C(R_1) + \ldots + C(R_n)$
- VaR: $VaR_\alpha(Z) = F_Z^{-1}(\alpha) = \min\{z, F_Z(z) \geq \alpha\}$
- Conditional VaR (CVaR): $CVaR_\alpha(Z) = E[Z \mid Z > VaR_\alpha(Z)]$
- Dependence between frequencies
- Dependence between event point processes
- Dependence between severities
- Dependence between annual losses

$$total\ annual\ loss: Z = \sum_{k=1}^{K} Z_k = \sum_{i=1}^{N_1} X_i^{(1)} + \sum_{i=1}^{N_2} X_i^{(2)} + \ldots + \sum_{i=1}^{N_K} X_i^{(K)}$$

Basel Committee statement*:*

"Risk measures for different operational risk estimates must be added for purposes of calculating the regulatory minimum capital requirement. However, the bank may be permitted to use internally determined correlations in operational risk losses across individual operational risk estimates, provided it can demonstrate to a high degree of confidence and to the satisfaction of the national supervisor that its systems for determining correlations are sound, implemented with integrity, and take into account the uncertainty surrounding any such correlation estimates (particularly in periods of stress). The bank must validate its correlation assumptions."

Adding capitals=>perfect dependence between risks (too conservative)

Dependence between frequencies via copula

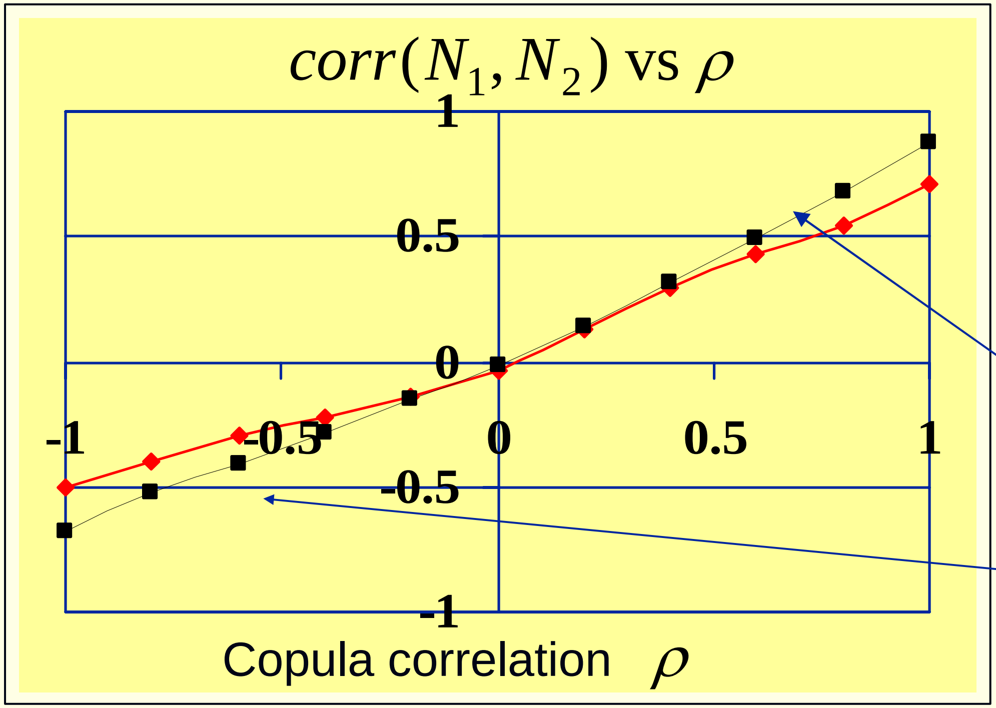
$$C(U_1, U_2, \ldots, U_K)), U_i \sim Uniform(0,1)$$

$$N_1 = F_1^{-1}[U_1], \ldots, N_K = F_K^{-1}[U_K]$$

e.g. Gaussian copula $C_\rho^{Ga}(u_1, \ldots, u_d) = F_N^{(\rho)}(F_N^{-1}(u_1), \ldots, F_N^{-1}(u_K))$

$$corr(N_i, N_j) \neq 0 \; if \;\; \rho \neq 0$$

Example



$$N_1 \sim Poisson(\lambda_1)$$

$$N_2 \sim Poisson(\lambda_2)$$

$$corr(N_1, N_2) \neq \rho$$

■ $\lambda_1 = 0.5, \lambda_2 = 1$

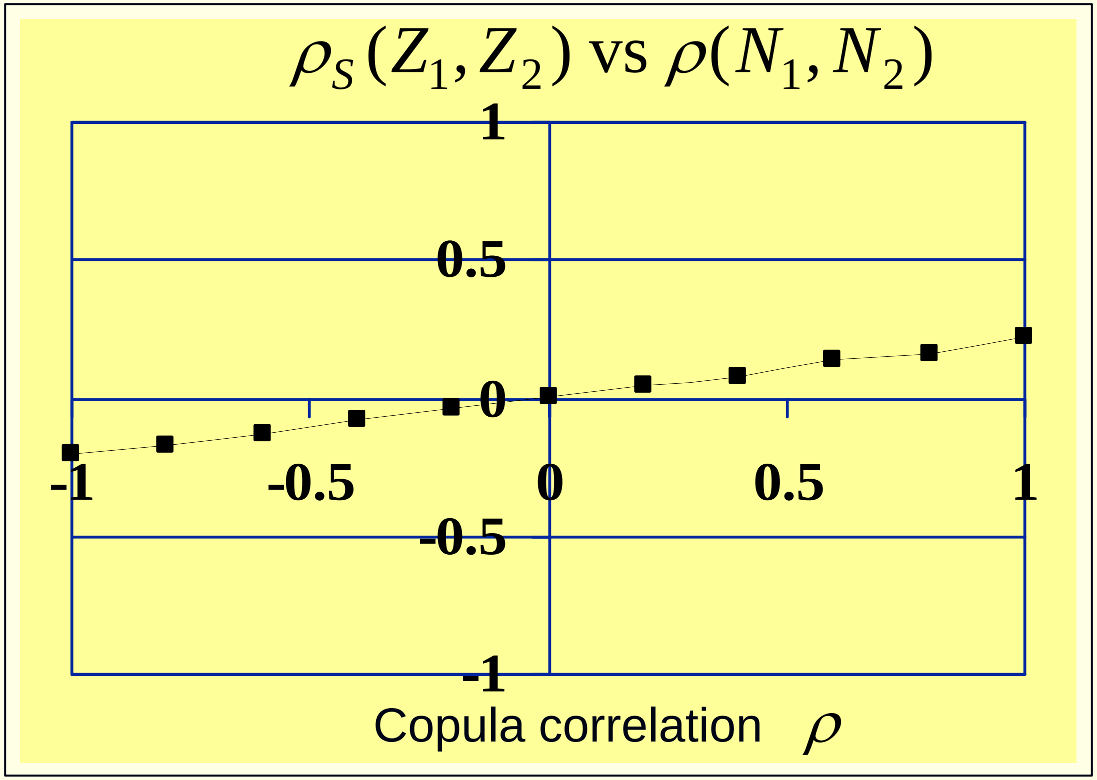
■ $\lambda_1 = 5, \lambda_2 = 10$

Dependence between frequencies=>Dependence between annual losses

Example

$$N_1 = F^{-1}_{Poisson}[U_1],\ N_2 = F^{-1}_{Poisson}[U_2] \qquad C(U_1,U_2) = C^{Ga}_{\rho}(U_1,U_2)$$

$$Z_1 = \sum_{i=1}^{N_1} X_i^{(1)} \quad Z_2 = \sum_{i=1}^{N_2} X_i^{(2)} \quad X^{(1)} \sim \text{LogNormal}(1,2),\ X^{(2)} \sim \text{LogNormal}(1,2),\ X^{(1)} \text{ ind } X^{(2)}$$



$$N_1 \sim Poisson(\lambda_1)$$

$$N_2 \sim Poisson(\lambda_2)$$

$$\lambda_1 = 5, \lambda_2 = 10$$

Dependence via common Poisson process

(Johnson, Kotz and Balakrishnan)

$$N_1(t) \sim \hat{N}_1(t) + N_C(t)$$

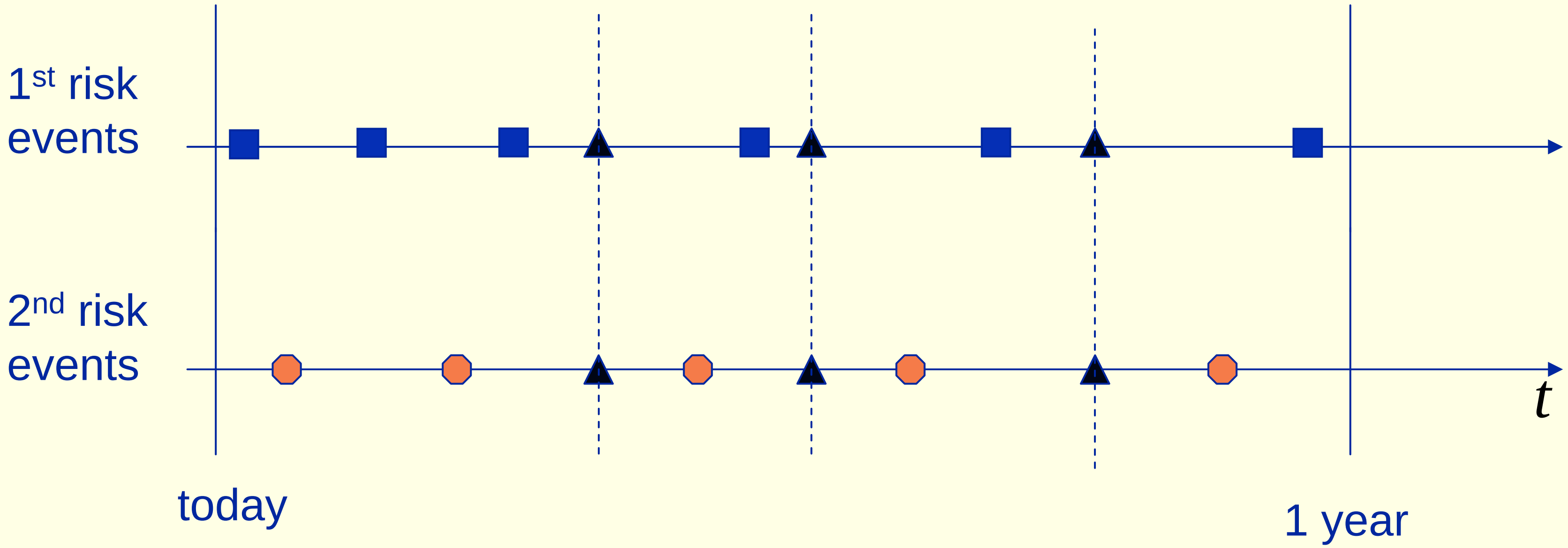
$$N_2(t) \sim \hat{N}_2(t) + N_C(t)$$

$$\hat{N}_1(t) \sim Poisson(\lambda_1); \hat{N}_2(t) \sim Poisson(\lambda_2); N_C(t) \sim Poisson(\lambda_C)$$

$$N_1(t) \sim Poisson(\lambda_1 + \lambda_C); N_2(t) \sim Poisson(\lambda_2 + \lambda_C); corr(N_1, N_2) = \lambda_C / \sqrt{(\lambda_1 + \lambda_C)(\lambda_2 + \lambda_C)}$$

positive dependence; constant covariance

extension $N_i(t) = \begin{cases} \hat{N}_i(t) + N_C(t) & \text{with prob } p_i \\ \hat{N}_i(t) & \text{with prob } 1 - p_i \end{cases}$ $\quad \text{cov}(N_i, N_k) = \lambda_C p_i p_k$



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Dependence between inter-arrival times

$$C(U_1,U_2,\ldots,U_K)),U_i \sim Uniform(0,1)$$

$$\tau^{(1)} = F_1^{-1}[U_1],\ldots.,\tau^{(K)} = F_K^{-1}[U_K]$$

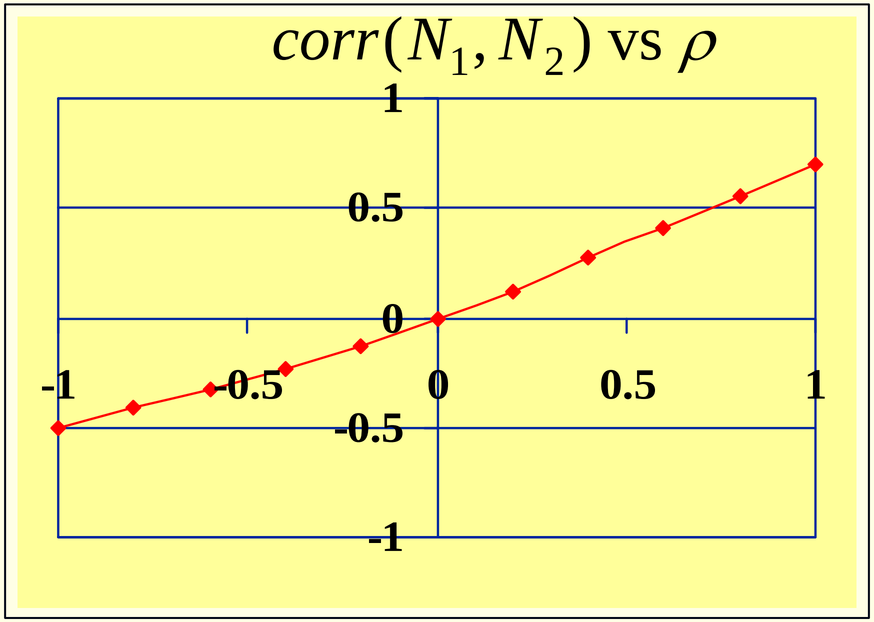
$$0 < t_1^{(i)} < \ldots < t_{N_i}^{(i)} \le 1 < t_{N_i+1}^{(i)} < \ldots$$

$$t_j^{(i)} = t_{j-1}^{(i)} + \tau_j^{(i)}; iid\ \tau_1^{(i)},\ldots,\tau_{N_i}^{(i)} \sim F_i^{-1}(.), i = 1,\ldots,K$$

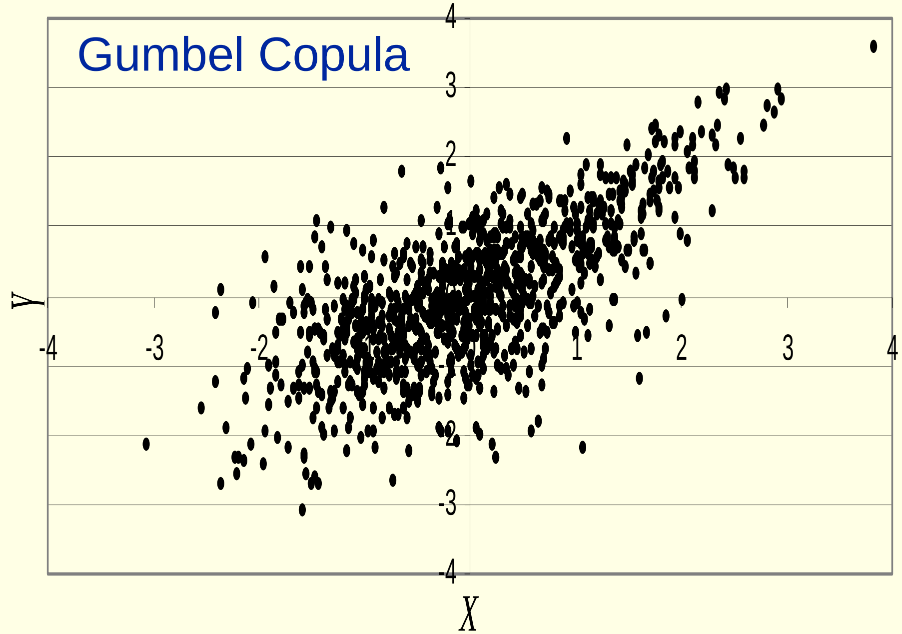
e.g. Gaussian copula $C_\rho^{Ga}(u_1,\ldots,u_d) = F_N^{(\rho)}(F_N^{-1}(u_1),\ldots,F_N^{-1}(u_K))$

$$\tau^{(i)} \sim Exp(\lambda_i);\ N_i \sim Poisson(\lambda_i);\ corr(N_i,N_j) \ne 0\ if\ \ \rho \ne 0;$$

Example: $\lambda_1 = 5, \lambda_2 = 10$

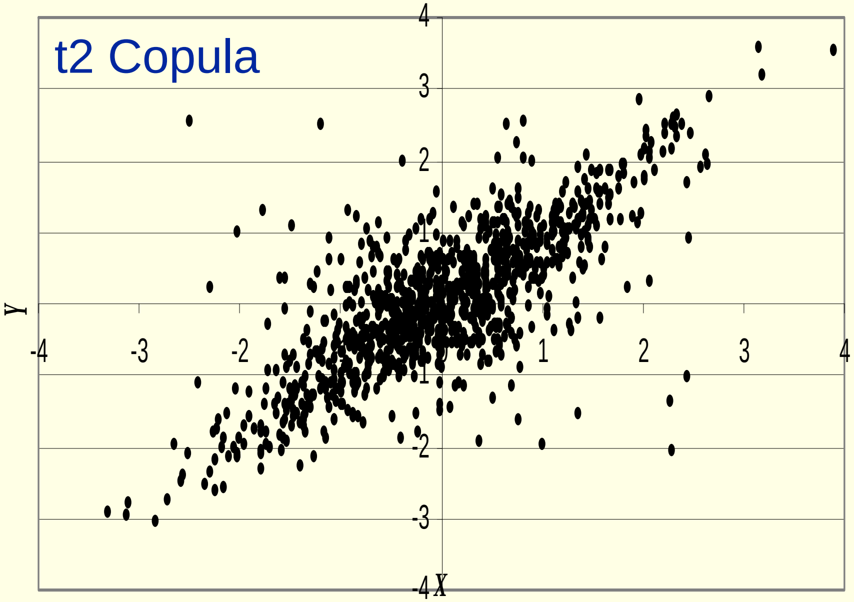


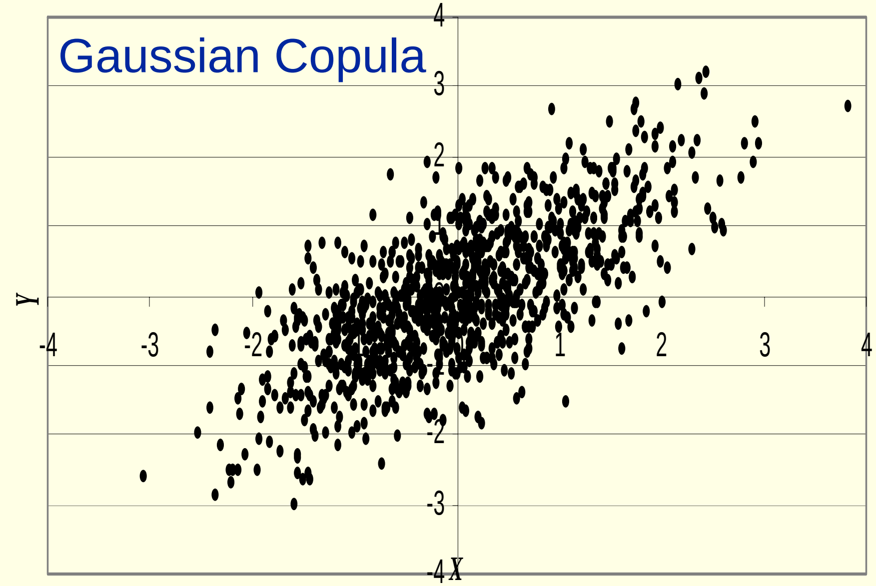
Dependence between severities via copula



$$X \sim Normal(0,1); Y \sim Normal(0,1)$$

$$corr(X, Y) = 0.7$$





Upper tail dependence

$$\lambda = \lim_{\alpha \to 1-} P[Y > F_2^{-1}(\alpha) \mid X > F_1^{-1}(\alpha)] = \lim_{\alpha \to 1-} \frac{1-2\alpha + C(\alpha,\alpha)}{1-\alpha}$$

***Gaussian* copula** $\lambda = 0$

$$C_\rho^{Ga}(u_1, u_2, \ldots, u_d) = F_N^{(\rho)}(F_N^{-1}(u_1), F_N^{-1}(u_2), \ldots, F_N^{-1}(u_d))$$

***t*-copula** $\lambda \geq 0$

$$C_\Sigma^{t_\nu}(u_1, u_2, \ldots, u_d) = F_{t_\nu}^{(\Sigma)}(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2), \ldots, t_\nu^{-1}(u_d))$$

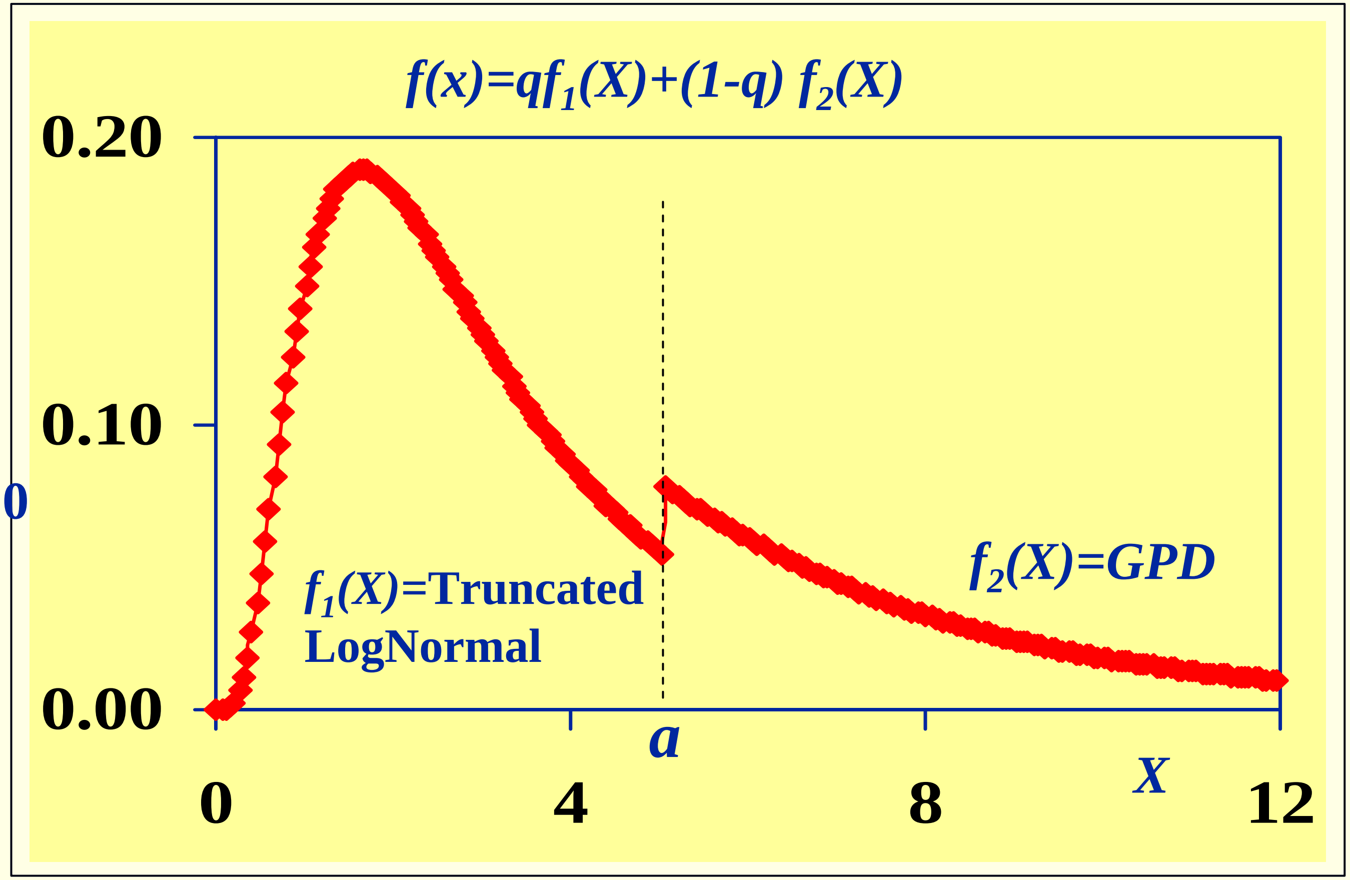
***Gumble* copula** $\lambda = 2 - 2^\beta$

$$C_\beta^{Gu}(u_1, \ldots, u_d) = \exp\left\{-\left[(-\ln u_1)^{1/\beta} + \ldots + (-\ln u_d)^{1/\beta}\right]^\beta\right\}, \quad 0 < \beta \leq 1$$

Severity Distribution Tail

Extreme Value Theory (peaks over thresholds) and splicing

Generalized Pareto Distribution (GPD) $H(x) = \begin{cases} 1-(1+\xi x/\beta)^{-1/\xi}; & \xi \neq 0 \\ 1-\exp[-x/\beta]; & \xi = 0 \end{cases}$



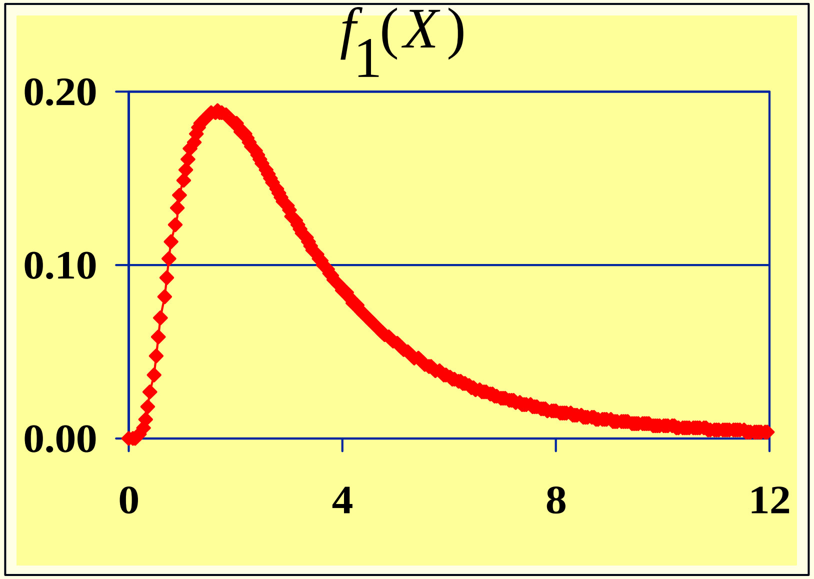
$f_1(X), 0 \le X < a$

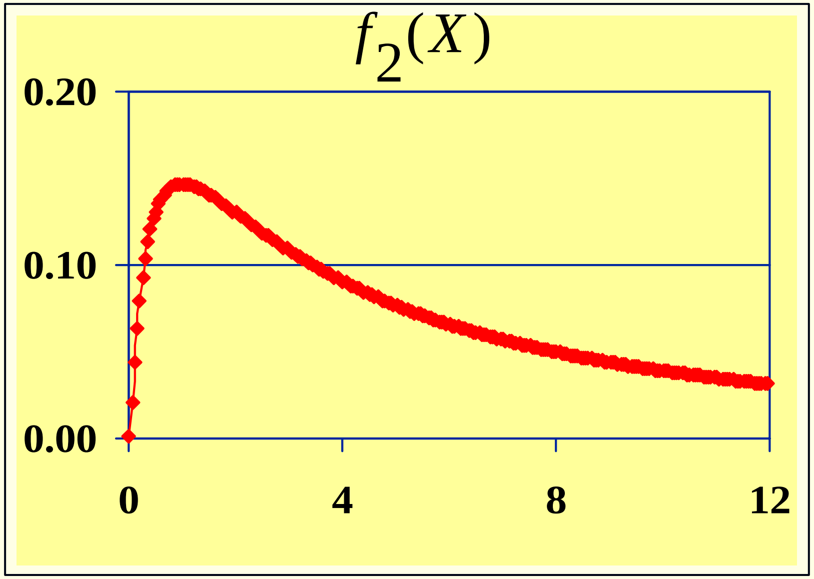
$f_2(X), X \ge a$

Mixture of distributions

$$f(X) = qf_1(X) + (1-q)f_2(X) \qquad f_1(X), 0 \le X < \infty; \quad f_2(X), 0 \le X < \infty$$

Maximum likelihood $\quad \Psi = \prod_{i=1}^{K} [qf_1(X_i) + (1-q)f_2(X_i)]$





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Capital Charge confidence interval

error in risk distribution parameters: Gaussian approx (ML), bootstrap

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uncertainty in capital charge: Monte Carlo

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Data sufficiency criteria

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Comments and suggestions are welcomed