

Indivisibility of relative class numbers of totally imaginary quadratic extensions and vanishing of these relative Iwasawa invariants.

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Abstract

In this paper, we consider relative class numbers of totally imaginary quadratic extensions of totally real number fields (called CM quadratic extensions). In particular, for a fixed totally real number field F which is Galois over \mathbb{Q} and prime number p satisfying some conditions, we obtain a lower bound of the number of CM quadratic extensions over F whose relative class numbers are indivisible by p . Using Friedman's criterion, we also have a similar result on vanishing of relative μ -, λ -invariant. To prove them, we use Hilbert modular forms of half-integral weight, these diagonal restrictions, a Sturm-type theorem via diagonal restriction, and Chebotarev's density theorem.

1 Introduction.

For a totally real number field F , we consider indivisibility of relative class numbers of totally imaginary quadratic extensions of F (*i.e.*, CM quadratic extensions). Here the relative class number $h^-(K)$ of the CM field K is the positive

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integer defined as the quotient of the class number $h(K)$ of K by the class number $h(K^+)$ of the maximal totally real subfield K^+ . In this paper, under the assumption that the extension F/\mathbb{Q} is Galois and prime p satisfies some condition, we show a sufficient condition to give a lower bound of the number of CM quadratic extensions whose relative class number are relatively prime to p . Adding a decomposition condition to indivisibility, and applying Friedman's criterion (Lemma 4), we have a similar condition to give a lower bound of the number of them whose relative Iwasawa λ -, μ -invariants vanish.

In the case of $F = \mathbb{Q}$, Gauss (for $p = 2$) and Hartung [6] (for odd prime p) proved that there are infinitely many imaginary quadratic fields whose class number is relatively prime to p . Considering a decomposition condition of p with the result of Hartung, Horie [7] showed that there are infinitely many imaginary quadratic fields whose Iwasawa λ -, μ -invariants vanish for odd primes. In the case of general totally real number field F , Naito proved the following result. For prime p , let $n(p)$ be the maximal integer n with $[F(\zeta_{p^n}) : F] \leq 2$, where ζ_{p^n} is a primitive p^n -th root. We set $w_F = 2^{n(2)+1} \prod_{p:\text{odd prime}} p^{n(p)}$. Then Serre showed that $w_F \zeta_F(-1)$ is an integer. Naito showed that for prime $p \nmid w_F \zeta_F(-1)$, there are infinitely many CM quadratic extensions of F whose relative Iwasawa λ -, μ -invariants vanish. Naito used Shimizu's trace formula and p -adic Galois representations.

As a next step, we should consider the asymptotic behavior of the number of such CM quadratic extensions. We remark that the trace formula arguments by Hartung, Horie and Naito cannot give any estimates of the number of such CM quadratic extensions. For $F = \mathbb{Q}$, Cohen-Lenstra's heuristics [3] predict that for fixed prime p , the probability of imaginary quadratic fields whose class number is prime to p is

$$\prod_{n=1}^{\infty} (1 - p^{-n}) = 1 - p^{-1} - p^{-2} + p^{-5} + p^{-7} - \dots .$$

For $p = 3$, the following results are known. Davenport and Heilbronn [4] obtained a density of imaginary quadratic fields whose class number is prime to 3. For any general totally real number field F and $p = 3$, Horie and Kimura [8] obtained a limit inferior of the number of CM quadratic extensions of F whose relative Iwasawa λ - and μ -invariants vanish.

For prime $p > 3$, Kohnen and Ono [9] obtained a lower bound of the number of imaginary quadratic fields whose class numbers are relatively prime to p . They used elliptic modular forms of half-integral weights, and Sturm's theorem. Mod-

ifying this method, Byeon [2] gave a lower bound of the number of imaginary quadratic fields vanishing of Iwasawa λ -invariants.

Generalizing the method of Kohnen-Ono and Byeon, we show indivisibility of relative class numbers of CM quadratic extensions over fixed totally real Galois number field F . Moreover, such indivisibility and decomposition condition of prime ideals of F on p at the CM extension of F imply vanishing of the relative Iwasawa invariants by Friedman's criterion. We express the simple form of main theorem:

Theorem (Restatement of Corollary 1). *Let F be a totally real number field such that F/\mathbb{Q} is Galois. Then, for sufficiently large p*

$$\#\{\text{CM quad. ext. } K/F \mid p \nmid h^-(K/F), |N_{F/\mathbb{Q}}(D(K/F))| < X\} \gg \frac{X^{\frac{1}{2g}}}{\log X}.$$

For CM quadratic extension K/F , K_∞ and F_∞ are the cyclotomic \mathbb{Z}_p -extension of K and F respectively. Let $\lambda_p(K), \mu_p(K)$ and $\lambda_p(F), \mu_p(F)$ be the Iwasawa invariants of the \mathbb{Z}_p -extensions. We set

$$\begin{aligned}\lambda_p^-(K) &= \lambda_p(K) - \lambda_p(F), \\ \mu_p^-(K) &= \mu_p(K) - \mu_p(F)\end{aligned}$$

and are called them relative Iwasawa λ -, μ -invariants of K/F . For the vanishing of relative Iwasawa λ -, μ -invariants, we prove a similar result.

Theorem (Restatement of Theorem 2). *Let $g = [F : \mathbb{Q}]$ and $D(F)$ be the discriminant of F/\mathbb{Q} . Let p be a prime such that $g \leq [F(\zeta_p) : F]/2^{\text{ord}_2([F(\zeta_p) : F])}$ and $p > 2g + 1$. We set*

$$A = \frac{gp^2 D(F)}{8} \prod_{d \mid pD(F), d:\text{prime}} \left(1 + \frac{1}{d}\right).$$

If there is a prime number $q > (A/g)^g$ such that

$$\sum_{\substack{\xi \in 2^{-1}\mathcal{O}_{F,+}, \chi_i(\xi) = \epsilon_i \ (i=1,2,\dots,r) \\ \text{Tr}(\xi) = qg/2, (q\mathcal{O}_F, \xi\mathcal{O}_F) \neq 1}} \beta(\xi) \frac{2^g h^-(F(\sqrt{-2\xi}))}{Q_{F(\sqrt{-2\xi})} w_{F(\sqrt{-2\xi})}} \not\equiv 0 \pmod{p},$$

then

$$\begin{aligned}\#\{\text{CM quad. ext. } K/F \mid \lambda_p^-(K) = \mu_p^-(K) = 0, |N_{F/\mathbb{Q}}(D(K/F))| < X\} \\ \gg_{F,p} \frac{X^{\frac{1}{2g}}}{\log X}.\end{aligned}$$

Here the symbols $\beta(\xi)$, $Q_{F(\sqrt{-2\xi})}$, $w_{F(\sqrt{-2\xi})}$ are defined in §. 2.

Our main tools are Hilbert modular forms of half-integral weights, diagonal restrictions of them, the Sturm-type theorem modified via diagonal restrictions, and Chebotarev’s density theorem. Kohnen and Ono used the cube of the theta function, because class numbers of imaginary quadratic fields appear in their Fourier coefficients. Instead of the cube of theta, we use the Eisenstein series whose Fourier coefficients include relative class numbers introduced by Shimura [13]. In our argument, the assumption that F/\mathbb{Q} is Galois is unavoidable. Because, we have to control the prime ℓ in the proof of Theorem 1 as totally splitting at F/\mathbb{Q} . To avoid the assumption, we should prove the theorem without diagonal restriction. To do that, we should use a Hilbert modular version of Sturm’s theorem proved by the author [17]. But it seems difficult to prove indivisibility by any similar method because the bound in [17] is slightly large.

This article is organized as follows. In §.2 we recall some facts on Hilbert modular forms of half-integral weight and introduce tools for the proof of theorems, *e.g.*, some operators acting on modular forms and the Sturm-type theorem via diagonal restriction. In §.3 we prove the indivisibility of relative class numbers and vanishing of relative Iwasawa invariants.

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2 Preliminaries.

2.1 Hilbert modular forms of half-integral weight.

In this section, we review Hilbert modular forms of half-integral weight. We use the terminology in [13]. This terminology is slightly different from [14]. But in the parallel weight case, the difference is only the factor of automorphy (and also “Nebentypus” character).

Let F be a totally real number field, $g = [F : \mathbb{Q}]$, \mathfrak{d}_F be the different ideal of F/\mathbb{Q} , and $D(F)$ be the discriminant of F/\mathbb{Q} . Let \mathfrak{a} and \mathfrak{f} be the set of the archimedean places and the non-archimedean places of F respectively, and $\mathbb{A}_F = \mathbb{A}$ be the adèle ring of F . For $a \in \mathbb{A}_F$, $a_{\mathfrak{a}}$ and $a_{\mathfrak{f}}$ are the archimedean part of a and the non-archimedean part of a respectively. We set $G_F = SL_2(F)$, $G_v = SL_2(F_v)$ for $v \in \mathfrak{a} \cup \mathfrak{f}$, and $G_{\mathbb{A}} = SL_2(\mathbb{A})$. For $\alpha \in G_{\mathbb{A}}$, $a_{\alpha}, b_{\alpha}, c_{\alpha}, d_{\alpha} \in \mathbb{A}_F$ mean that

$$\alpha = \begin{pmatrix} a_{\alpha} & b_{\alpha} \\ c_{\alpha} & d_{\alpha} \end{pmatrix}.$$

Let $\mathfrak{H} = \mathcal{H}^g$ be the g -tuple product of the upper-half plane. For $\xi \in F$, we set $\mathbf{e}(\xi z) = e^{2\pi\sqrt{-1}Tr(\xi z)}$, where $Tr(\xi z) = \sum_{v \in \mathfrak{a}} \xi_v z_v$. We also set for $\xi = (\xi_v)_{v \in \mathfrak{a}} \in F^{\mathfrak{a}}$, $\mathbf{e}_{\mathfrak{a}}(\xi) = e^{2\pi\sqrt{-1}\sum_{v \in \mathfrak{a}} \xi_v}$. For prime number p , $v \in \mathfrak{f}$ such that $v|p$ and $\xi \in F_v$, we set

$$\mathbf{e}_v(\xi) = \exp(2\pi\sqrt{-1}Tr_{F_v/\mathbb{Q}_p}(\text{fractional part of } \xi)).$$

For $\xi \in \mathbb{A}_{\mathfrak{f}}$, we define $\mathbf{e}_{\mathfrak{f}}(\xi) = \prod_{v \in \mathfrak{f}} \mathbf{e}_v(\xi_v)$. For $\xi \in \mathbb{A}_F$, we set $\mathbf{e}_{\mathbb{A}}(\xi) = \mathbf{e}_{\mathfrak{a}}(\xi)\mathbf{e}_{\mathfrak{f}}(\xi)$.

Let $M_{\mathbb{A}}$ be the metaplectic group of Weil [18] and $\text{pr} : M_{\mathbb{A}} \rightarrow G_{\mathbb{A}}$ be the projection. We set

$$\begin{aligned} \Omega_{\mathbb{A}} &= \{g \in G_{\mathbb{A}} \mid c_g \neq 0\}, \\ P_{\mathbb{A}} &= \{g \in G_{\mathbb{A}} \mid c_g = 0\}. \end{aligned}$$

Then there are splitting homomorphisms

$$\begin{aligned} r &: G_F \rightarrow M_{\mathbb{A}}, \\ r_P &: P_{\mathbb{A}} \rightarrow M_{\mathbb{A}}, \\ r_{\Omega} &: \Omega_{\mathbb{A}} \rightarrow M_{\mathbb{A}}. \end{aligned}$$

For fractional ideals $\mathfrak{b}, \mathfrak{c}$, we set $D[\mathfrak{b}, \mathfrak{c}] = SO_2(\mathbb{R})^{\mathfrak{a}} \prod_{v \in \mathfrak{f}} D_v[\mathfrak{b}, \mathfrak{c}]$, where

$$D_v[\mathfrak{b}, \mathfrak{c}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F_v) \mid a, d \in \mathcal{O}_{F_v}, b \in (2\mathfrak{d}^{-1})_v, c \in (2^{-1}\mathfrak{c}\mathfrak{d})_v \right\}$$

for $v \in \mathfrak{f}$ and $D_0(\mathfrak{c}) = D[\mathcal{O}_F, \mathfrak{c}]$. We set $\Gamma[\mathfrak{b}, \mathfrak{c}] = D[\mathfrak{b}, \mathfrak{c}] \cap G_F$ and $\Gamma_0(\mathfrak{c}) = \Gamma[\mathcal{O}_F, \mathfrak{c}]$. Then

$$\Gamma_0(\mathfrak{c}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F) \mid a, d \in \mathcal{O}_F, b \in 2\mathfrak{d}^{-1}, c \in 2^{-1}\mathfrak{c}\mathfrak{d} \right\}.$$

To define a factor of automorphy, we use the following theta series:

$$\theta(z) = \sum_{\xi \in \mathcal{O}_F} \mathbf{e}(\xi^2 z/2).$$

We define the factor of automorphy $h(\gamma, z)$ as follows:

$$h(\gamma, z) = \theta(\gamma z)/\theta(z) \text{ for } \gamma \in \Gamma_0(4\mathcal{O}_F).$$

The factor $h(\gamma, z)$ satisfies

$$\begin{aligned} h(\gamma, z)^2 &= \text{sgn}(N_{F/\mathbb{Q}}(d_\gamma)) \theta^*(d_\gamma \mathfrak{a}_\gamma^{-1}) j(\gamma, z) \\ &= \text{sgn}(N_{F/\mathbb{Q}}(d_\gamma)) \theta^*(d_\gamma \mathcal{O}_F) j(\gamma, z), \end{aligned}$$

where θ is the Hecke character associated with the extension $F(\sqrt{-1})/F$, θ^* is the ideal character corresponding to θ , and

$$\mathfrak{a}_\gamma = c_\gamma \mathfrak{d}_F^{-1} + d_\gamma \mathcal{O}_F.$$

We introduce the group

$$\mathcal{G}_F = \{(\alpha, \phi_\alpha(z)) \mid \alpha \in G_F, \exists t \in \mathbb{T} \text{ such that } \phi_\alpha(z)^2 = tJ(\alpha, z)\},$$

where $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and

$$J(\alpha, z) = \prod_{v \in \mathfrak{a}} (c_v z_v + d_v) \text{ for } \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The group law is defined as

$$(\alpha, \phi_\alpha(z))(\beta, \phi_\beta(z)) = (\alpha\beta, \phi_\alpha(\beta z)\phi_\beta(z)).$$

Then we have the injection $\Gamma_0(4\mathcal{O}_F) \rightarrow \mathcal{G}_F: \gamma \mapsto (\gamma, h(\gamma, z))$. We regard $\Gamma_0(4\mathcal{O}_F)$ as a subgroup of \mathcal{G}_F

For $\xi = (\alpha, \phi(z)) \in \mathcal{G}_F$ and $k \in \mathbb{Z}$, we set

$$f|_k[\xi](z) = f(\alpha z)\phi(z)^{-k}.$$

Let ψ be a Hecke character whose conductor divides \mathfrak{c} and $k \in \mathbb{Z}$. Then Hilbert modular form f of parallel weight $(k/2, k/2, \dots, k/2)$, level $\Gamma_0(\mathfrak{c})$, and ‘‘Neben-typus’’ character ψ is defined to be

$$f|_k[(\gamma, h(\gamma, z))](z) = \psi(d_\gamma)f \text{ for all } \gamma \in \Gamma_0(\mathfrak{c}).$$

$M_{k/2}(\Gamma_0(\mathfrak{c}), \psi)$ denotes the vector spaces of the forms of parallel weight $(k/2, \dots, k/2)$, level $\Gamma_0(\mathfrak{c})$ and character ψ .

To introduce some operators in next section, we should introduce adelic Hilbert modular forms. A function $g : M_{\mathbb{A}} \rightarrow \mathbb{C}$ is an adelic modular form of level $\prod_{v \in \mathfrak{f}} D_v(\mathfrak{c})$ if g satisfies the following conditions:

1. $g(r(a)\xi) = g(\xi)$ for all $a \in G_F$,
2. $g(\xi\gamma) = g(\xi)$ for all $\text{pr}(\gamma) \in \prod_{v \in \mathfrak{f}} D_v(\mathfrak{c})$,
3. $g(\xi k_\infty) = \psi(k_\infty)g(\xi)$ for all $\text{pr}(k_\infty) \in SO_2(\mathbb{R})^{\mathfrak{a}}$,

where $\text{pr} : M_{\mathbb{A}} \rightarrow G_{\mathbb{A}}$ is the projection.

Shimura [13] extended $h(\gamma, z)$ to a larger subgroup $\text{pr}^{-1}(P_{\mathbb{A}}D_0(4\mathcal{O}_F))$ of $M_{\mathbb{A}}$ for the correspondence between adelic and classical modular forms, denoted by $h(\xi, z)$ for $\text{pr}(\xi) \in P_{\mathbb{A}}D_0(4\mathcal{O}_F)$. For a classical Hilbert modular form f of weight $(k/2, \dots, k/2)$ and character ψ , we can define the adelic form by

$$g(\xi) = f(\text{pr}(\xi)_{\mathfrak{a}}(\mathbf{i}))h(\xi, \mathbf{i})^{-k}\psi(d_{\text{pr}(\xi)})^{-1}, \text{ for } \text{pr}(\xi) \in P_{\mathbb{A}}C.$$

Then we set $g = f_{\mathbb{A}}$.

The adelic function $f_{\mathbb{A}}$ has a Fourier expansion:

$$f_{\mathbb{A}} \left(r_P \begin{pmatrix} t & s \\ 0 & t^{-1} \end{pmatrix} \right) = \sum_{\xi \in F} \lambda(\xi, t\mathcal{O}_F) \mathbf{e}_{\mathbb{A}}(\xi t s / 2) \mathbf{e}_{\mathfrak{a}}(\xi t_{\mathfrak{a}}^2 \mathbf{i} / 2).$$

for $t \in \mathbb{A}_F^{\times}, s \in \mathbb{A}_F$. Here for $t \in \mathbb{A}_F^{\times}$, $t\mathcal{O}_F$ is the ideal of F determined by $(t\mathcal{O}_F)_v = t_v\mathcal{O}_{F_v}$ for all $v \in \mathfrak{f}$, and $\lambda(\xi, t\mathcal{O}_F)$ satisfies the following conditions:

1. $\lambda(\xi, t\mathcal{O}_F) \neq 0$ only when $\xi \in t^{-2}\mathcal{O}_F$ and $\xi \gg 0$,
2. $\lambda(a^2\xi, t\mathcal{O}_F) = a^k\psi(a)\lambda(\xi, at\mathcal{O}_F)$ for all $a \in F^\times$.

Here $\xi \gg 0$ means that ξ is totally positive. In particular, if we set $z = s_{\mathbf{a}}t_{\mathbf{a}} + t_{\mathbf{a}}^2\mathbf{i}$ and $s_{\mathbf{f}} = (0, \dots, 0)$, $f(z)$ has the Fourier expansion:

$$f(z) = \sum_{\xi \in F} \lambda(\xi, \mathcal{O}_F) \mathbf{e}_{\mathbf{a}}(\xi z/2) = \sum_{\xi \in 2^{-1}\mathcal{O}_{F,+} \cup \{0\}} a_{\xi} \mathbf{e}_{\mathbf{a}}(\xi z).$$

2.2 Some operators and quadratic twists.

For a prime ideal \mathfrak{l} , Achimescu and Saha defined U_{Γ} -operator in the case of class number one [1, Lem. 4.4]. For prime number ℓ , to construct U_{ℓ} -operator for the case of higher class numbers, we should use the adelic forms. Here, we only consider the situation which we need, *i.e.* ℓ is unramified at F . Let $\ell\mathcal{O}_F = \mathfrak{l}_1\mathfrak{l}_2 \cdots \mathfrak{l}_r$ be a prime ideal factorization.

For a Hilbert modular form f of weight $(k/2, k/2, \dots, k/2) \in \mathbb{Z}_{\geq 0}^{\mathbf{a}}$, level $\Gamma_0(\mathfrak{c})$ and character ψ , we set $g(z) = f(z/\ell)$. Then $g(z)$ is a Hilbert modular form of same weight, level $\Gamma[\ell\mathcal{O}_F, \mathfrak{c}]$, and character ψ . We remark that

$$\begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix}^{-1} \Gamma_0(\mathfrak{c}) \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} = \Gamma[\ell\mathcal{O}_F, \mathfrak{c}].$$

Then by [15, Prop. 1.1], $g_{\mathbb{A}}$ has the Fourier expansion

$$g_{\mathbb{A}} \left(r_P \begin{pmatrix} t & s \\ 0 & t^{-1} \end{pmatrix} \right) = \sum_{\xi \in F} \lambda(\xi, t\mathcal{O}_F) \mathbf{e}_{\mathbb{A}}(\xi t s/2\ell) \mathbf{e}_{\mathbf{a}}(\xi t_{\mathbf{a}}^2 \mathbf{i}/2\ell).$$

For prime ideal $\mathfrak{l}|\ell\mathcal{O}_F$, we set

$$\tilde{g}_{\mathbb{A}}(z) = \sum_{j \in \mathcal{O}_{F_1}/\mathfrak{l}} g_{\mathbb{A}} \left(\xi r_P \begin{pmatrix} 1 & 2j\delta_{\mathfrak{l}} \\ 0 & 1 \end{pmatrix}^{-1} \right).$$

Then, by [15, Prop. 1.1],

$$\tilde{g}_{\mathbb{A}}(z) = \sum_{\xi \in F} \lambda(\xi, t\mathcal{O}_F) \mathbf{e}_{\mathbb{A}}(\xi t s/2\ell) \mathbf{e}_{\mathbf{a}}(\xi t_{\mathbf{a}}^2 \mathbf{i}/2\ell) \sum_{j \in \mathcal{O}_{F_1}/\mathfrak{l}} \mathbf{e}_{\mathfrak{l}}(-\xi j \delta_{\mathfrak{l}}^{-1} t^2/\ell),$$

and the last sum is

$$\sum_{j \in \mathcal{O}_{F_l}/\mathfrak{l}} \mathbf{e}_l(-\xi j \delta_l^{-1} t^2 / \ell) = \begin{cases} \#(\mathcal{O}_{F_l}/\mathfrak{l}) = N_{F/\mathbb{Q}}(\mathfrak{l}) & \text{if } \xi \delta_l^{-1} t^2 / \ell \in \mathfrak{d}_{F_l}^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Because

$$\xi \delta_l^{-1} t^2 / \ell \in \mathfrak{d}_{F_l}^{-1} \iff \xi \in t_l^{-2} \mathfrak{l},$$

we have

$$\tilde{g}_\mathbb{A}(z) = N_{F/\mathbb{Q}}(\mathfrak{l}) \sum_{\xi \in \mathfrak{l}} \lambda(\xi, t \mathcal{O}_F) \mathbf{e}_\mathbb{A}(\xi t s / 2\ell) \mathbf{e}_\mathbf{a}(\xi t_\mathbf{a}^2 i / 2\ell).$$

Let \tilde{g} be the classical form corresponding to $\tilde{g}_\mathbb{A}$. Then we can show that $\tilde{g} \in M_k(\Gamma_0(\mathfrak{c}\mathfrak{l}), \chi \epsilon_l)$ similarly as [1, Lem. 4.4]. Here ϵ_l is the character defined as

$$\epsilon_l(\xi) = \left(\frac{F(\sqrt{-\xi})/F}{\mathfrak{l}} \right).$$

We remark that in our case, we use the fact that

$$D_0(\mathfrak{c})_l = \bigsqcup_{j \in \mathcal{O}_{F_l}/\mathfrak{l}} D[\mathfrak{l}, \mathfrak{c}]_l r_P \begin{pmatrix} 1 & 2j \delta_l^{-1} \\ 0 & 1 \end{pmatrix},$$

and, for $\alpha \in D_0(\mathfrak{c})$, there are $\alpha' \in D[\mathfrak{l}, \mathfrak{c}]$ and $j' \in \mathcal{O}_{F_l}/\mathfrak{l}$ such that

$$r_P \begin{pmatrix} 1 & 2j \delta_l^{-1} \\ 0 & 1 \end{pmatrix} \alpha = \alpha' r_P \begin{pmatrix} 1 & 2j' \delta_l^{-1} \\ 0 & 1 \end{pmatrix}.$$

We define the U_ℓ -operator as the composition of the above correspondences:

$$U_\ell : \begin{array}{ccccc} M_{k/2}(\Gamma_0(\mathfrak{c}), \chi) & \rightarrow & M_{k/2}(\Gamma[\mathfrak{l}\mathcal{O}_F, \mathfrak{c}], \chi) & \rightarrow & M_{k/2}(\Gamma_0(\ell\mathfrak{c}), \chi \epsilon_\ell) \\ f & \mapsto & g & \mapsto & \tilde{g}. \end{array}$$

V_ℓ -operator is defined by Shimura [14, Proposition 3.2] as $f(z) \mapsto f(\ell z)$. Then, V_ℓ maps $M_{k/2}(\Gamma_0(\mathfrak{c}), \psi)$ to $M_{k/2}(\Gamma_0(\ell\mathfrak{c}), \psi \epsilon_\ell)$.

We introduce twisted modular forms by a quadratic Hecke character. For the case of $F = \mathbb{Q}$, this is defined in [12, Lem. 3.6]. In our case, by applying a similar construction for adelic forms they can be defined. Let \mathfrak{r} be an integral ideal in \mathcal{O}_F ,

χ be a quadratic Hecke character whose conductor is \mathfrak{r} . For the Hilbert modular form f of weight k , level $\Gamma_0(\mathfrak{c})$ and character ψ , and its Fourier expansion

$$f(z) = \sum_{\xi \in F} \lambda(\xi, \mathcal{O}_F) \mathbf{e}_{\mathbf{a}}(\xi z/2),$$

we can define the twist of f by $\chi = \prod_{v|\mathfrak{r}} \chi_v = \prod_{i=1}^m \chi_{v_i}$ denoted by $f \otimes \chi$ as $h(z)$ corresponding to

$$\begin{aligned} h_{\mathbb{A}} &= (\cdots (((f_{\mathbb{A}} \otimes \chi_{v_1}) \otimes \chi_{v_2}) \otimes \cdots) \otimes \chi_{v_m}), \\ f_{\mathbb{A}} \otimes \chi_v(\xi) &= G(\bar{\chi}_v)^{-1} \sum_{x \in \mathcal{O}_F/\mathfrak{r}_v} \bar{\chi}_v(x) f_{\mathbb{A}}(\xi r_P \left(\begin{pmatrix} 1 & x/\pi_v^{e_v} \\ 0 & 1 \end{pmatrix} \right)), \end{aligned}$$

where $G(\chi_v)$ is the Gauss sum of χ_v , π_v is a uniformizer of F_v , and $\mathfrak{r}_v = \pi_v^{e_v} \mathcal{O}_{F_v}$. Then the Fourier expansion of $f \otimes \chi$ is

$$(f \otimes \chi)(z) = \sum_{\xi \in F} \chi(\xi) \lambda(\xi, \mathcal{O}_F) \mathbf{e}_{\mathbf{a}}(\xi z/2).$$

Then $f \otimes \chi \in M_k(\Gamma_0(\mathfrak{cm}), \psi\chi^2)$ for $\mathfrak{m} = \text{l.c.m.}(\mathfrak{c}, \mathfrak{n}, \mathfrak{r}^2)$, where \mathfrak{n} are the conductor of ψ .

2.3 Diagonal restriction.

In this section, we consider a diagonal restriction of Hilbert modular forms of parallel weight $(k/2, \dots, k/2)$ to elliptic modular forms $kg/2$, where $g = [F : \mathbb{Q}]$.

Let \mathfrak{c} be an ideal in \mathcal{O}_F and C be the positive integer $C\mathbb{Z} = \mathfrak{c} \cap \mathbb{Z}$. Then the group

$$\Gamma'_0(C) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Q}) \mid a, d \in \mathbb{Z}, b \in 2\mathbb{Z}, c \in 2^{-1}CD(F)\mathbb{Z} \right\}$$

is a subgroup of $\Gamma_0(\mathfrak{c})$. This congruence subgroup $\Gamma'_0(C)$ is isomorphic to a standard congruence subgroup

$$\Gamma_0(CD(F)) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \in CD(F)\mathbb{Z} \right\}$$

as follows: For $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$,

$$\Gamma'_0(C) \ni \gamma' \mapsto \alpha \gamma' \alpha^{-1} \in \Gamma_0(CD(F)).$$

We define the diagonal restriction D of Hilbert modular forms are defined by

$$\begin{aligned} D : M_{k/2}(\Gamma_0(\mathfrak{c}), \chi) &\rightarrow M_{kg/2}(\Gamma_0(CD(F)), \chi) \\ f(z_1, \dots, z_g) &\mapsto f(2z, 2z, \dots, 2z), \end{aligned}$$

where χ is regarded as a Dirichlet character modulo $CD(F)$.

2.4 Certain Eisenstein series.

Let F be a totally real number field. We review an Eisenstein series of weight $(3/2, \dots, 3/2)$, denoted by \overline{E}' . The Eisenstein series was constructed by Shimura [13, Prop. 6.3].

Lemma 1 (Shimura). *The Eisenstein series \overline{E}' is a form of parallel weight $(3/2, \dots, 3/2)$, level $\Gamma_0(\mathfrak{c})$, and character ψ , and its Fourier expansion is as follows:*

$$\overline{E}' = a_0 + \sum_{\xi \in 2^{-1}\mathcal{O}_{F,+}} \beta(\xi) \frac{2^g h^-(F(\sqrt{-2\xi}))}{Q_{F(\sqrt{-2\xi})} w_{F(\sqrt{-2\xi})}} \mathbf{e}(\xi z),$$

where

$$\beta(\xi) = \sum_{\mathfrak{a}, \mathfrak{b}} \mu(\mathfrak{a}) \left(\frac{F(\sqrt{-2\xi})/F}{\mathfrak{a}} \right) N_{F/\mathbb{Q}}(\mathfrak{b}),$$

the pair $(\mathfrak{a}, \mathfrak{b})$ runs the all integral ideals relatively prime to $2\mathcal{O}_F$ such that $(\mathfrak{a}\mathfrak{b})^2 | 2\xi\mathcal{O}_F$, μ is the Möbius function, $Q_{F(\sqrt{-2\xi})} \in \{\pm 1\}$ is the Hasse index of $F(\sqrt{-2\xi})$, and $w_{F(\sqrt{-2\xi})}$ is the number of roots of unity in $F(\sqrt{-2\xi})$.

Remark 1. If $p - 1 > 2g$, then $p \nmid w_{F(\sqrt{-2\xi})}$. Indeed, if a primitive p -th root of unity ζ_p is in $F(\sqrt{-2\xi})$, then $\mathbb{Q}(\zeta_p) \subset F(\sqrt{-2\xi})$, so $2g = [F(\zeta_p) : \mathbb{Q}] \geq [\mathbb{Q}(\zeta_p) : \mathbb{Q}] = p - 1$. Thus for prime $p > 2g + 1$, the all coefficient a_ξ ($\xi \neq 0$) is p -integral. More precisely, $p \nmid w_{F(\sqrt{-2\xi})}$ if $p \nmid w_F$ for w_F in §.1.

2.5 Sturm-type theorem via diagonal restriction.

Here, we introduce a Sturm-type theorem via the diagonal restriction. More precisely, we give the criterion of vanishing of the image of diagonal restriction of Hilbert modular forms.

Lemma 2. *Let p be a prime number, $k \in \mathbb{Z}_{\geq 0}$, \mathfrak{c} an ideal of \mathcal{O}_F such that $4\mathcal{O}_F | \mathfrak{c}$, ψ a Hecke character whose conductor divides \mathfrak{c} , and $f \in M_{k/2}(\Gamma_0(\mathfrak{c}), \psi)$. We set its Fourier expansion as*

$$f = a_0 + \sum_{\xi \in 2^{-1}\mathcal{O}_{F,+}} a_\xi q^\xi.$$

Assume that $a_\xi \in \mathbb{Z}_{(p)}$ for all ξ , where $\mathbb{Z}_{(p)}$ is the set of p -integers.

If $D(f) \not\equiv 0 \pmod{p}$, then there is $\xi \in 2^{-1}\mathcal{O}_{F,+} \cup \{0\}$ such that

$$\text{Tr}_{F/\mathbb{Q}}(\xi) < \frac{kg}{24} CD(F) \prod_{d|CD(F), d:\text{prime}} \left(1 + \frac{1}{d}\right) = \kappa,$$

and $a_\xi \not\equiv 0 \pmod{p}$. Here C is the positive integer such that $C\mathbb{Z} = \mathfrak{c} \cap \mathbb{Z}$ and for $a \in \mathbb{Z}_{(p)}$, $a \equiv 0 \pmod{p}$ means $a \in p\mathbb{Z}_{(p)}$.

Proof. To show that, we apply Sturm's theorem [16, Th. 1] (for a more general statement, cf. [11, Th. 2.58]) for the elliptic modular forms to the diagonal restriction of f . The diagonal restriction $D(f)$ is

$$D(f)(z) = a_0 + \sum_{n=1}^{\infty} \left(\sum_{\xi \in 2^{-1}\mathcal{O}_{F,+}, \text{Tr}(\xi)=n/2} a_\xi \right) \mathbf{e}_a(nz).$$

Thus, by Sturm's theorem [16, Th. 1], if $D(f) \not\equiv 0 \pmod{p}$, then there is an integer $0 \leq n \leq \kappa$ such that

$$\sum_{\xi \in 2^{-1}\mathcal{O}_{F,+} \cup \{0\}, \text{Tr}(\xi)=n/2} a_\xi \not\equiv 0 \pmod{p}.$$

Thus there is an element $\xi \in 2^{-1}\mathcal{O}_{F,+} \cup \{0\}$ such that $\text{Tr}(\xi) \leq \kappa$ and $a_\xi \not\equiv 0 \pmod{p}$. \square

3 Proofs.

In this section, we show the main theorem on indivisibility of relative class numbers of CM quadratic extensions of fixed totally real number field which is Galois over \mathbb{Q} . We also give a result on vanishing of relative Iwasawa invariants and an example of the result.

3.1 Indivisibility of relative class numbers.

The main theorem in this article is the following:

Theorem 1. *Let $g = [F : \mathbb{Q}]$, $D(F)$ be the discriminant of F/\mathbb{Q} , p be a prime such that $g \leq M(p)2^{-\text{ord}_2(M(p))}$ and $p > 2g + 1$, r a positive integer, $\epsilon_1, \epsilon_2, \dots, \epsilon_r \in \{0, \pm 1\}$ such that $\epsilon_i \neq 0$ for some i , $\chi_1, \chi_2, \dots, \chi_r$ be quadratic Hecke characters of F whose conductor is integral ideal $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_r$ respectively. We set the positive integer N as $N\mathbb{Z} = \mathcal{N}_1\mathcal{N}_2 \cdots \mathcal{N}_r \cap \mathbb{Z}$ and set*

$$A = \frac{gN^2D(F)}{8} \prod_{d|ND(F), d:\text{prime}} \left(1 + \frac{1}{d}\right).$$

If there is a prime number $q > (A/g)^g$ such that

$$\sum_{\substack{\xi \in 2^{-1}\mathcal{O}_{F,+}, \chi_i(\xi) = \epsilon_i, i=1,2,\dots,r \\ \text{Tr}(\xi) = qg/2, (q\mathcal{O}_F, \xi\mathcal{O}_F) \neq 1}} \beta(\xi) \frac{2^g h^-(F(\sqrt{-2\xi}))}{Q_{F(\sqrt{-2\xi})} w_{F(\sqrt{-2\xi})}} \not\equiv 0 \pmod{p},$$

then

$$\# \left\{ K = F(\sqrt{-2\xi}) \mid \begin{array}{l} p \nmid h^-(K/F), |N_{F/\mathbb{Q}}(D(K/F))| < X, \\ \chi_i(\xi) = \epsilon_i \text{ for } i = 1, 2, \dots, r \end{array} \right\} \gg_{F,p} \frac{X^{\frac{1}{2g}}}{\log X}.$$

Proof. To show Theorem 1, we use the Eisenstein series $\overline{E}' = \sum_{\xi} a_{\xi} q^{\xi}$ reviewed in section 2.4. For $\epsilon_i \in \{\pm 1\}$, we set

$$\overline{E}'_i(z) = \frac{\overline{E}' \otimes \chi_i + \epsilon_i (\overline{E}' \otimes \chi_i) \otimes \chi_i}{2} = \sum_{\substack{\xi \in 2^{-1}\mathcal{O}_{F,+}, \\ \chi_i(\xi) = \epsilon_i}} a_{\xi} q^{\xi}.$$

While if $\epsilon_i = 0$, we set

$$\overline{E}'_i(z) = \overline{E}' - (\overline{E}' \otimes \chi_i) \otimes \chi_i = a_0 + \sum_{\substack{\xi \in 2^{-1}\mathcal{O}_{F,+}, \\ \chi_i(\xi) = 0}} a_{\xi} q^{\xi}.$$

Then $\overline{E}'_i \in M_{\frac{3}{2}}(\Gamma_0(4\mathcal{N}_i^2), \psi')$. Iterating this process for $\chi_1, \chi_2, \dots, \chi_r$, we have

$$\tilde{E}'_i(z) = \sum_{\substack{\xi \in 2^{-1}\mathcal{O}_{F,+}, \\ \chi_i(\xi) = \epsilon_i, i=1,2,\dots,r}} a_\xi q^\xi \in M_{\frac{3}{2}}(\Gamma_0(4\mathcal{N}_1^2 \mathcal{N}_2^2 \cdots \mathcal{N}_r^2), \psi'').$$

Because we assume that $\epsilon_i \neq 0$ for some i , the constant term a_0 is removed.

Let ℓ be a prime number such that ℓ completely splits at F , and set prime ideal factorization $\mathfrak{l}_1 \cdots \mathfrak{l}_g = \ell \mathcal{O}_F$. We introduce shifted modular forms as follows:

$$\begin{aligned} D(U_\ell \tilde{E}') &= \sum_{n=1}^{\infty} \left(\sum_{\text{Tr}(\xi)=n/2} a_{\ell\xi} \right) q^n, \\ D(V_\ell \tilde{E}') &= \sum_{n=1}^{\infty} \left(\sum_{\text{Tr}(\xi)=n/2} a_\xi \right) q^{\ell n}, \\ G &= U_\ell D(U_\ell \tilde{E}') - D(U_\ell^2 \tilde{E}') \\ &= \sum_{n=1}^{\infty} \left(\sum_{\substack{\text{Tr}(\xi)=\ell n/2, \\ \ell \mathcal{O}_F \nmid \xi \mathcal{O}_F}} a_{\ell\xi} \right) q^n. \end{aligned}$$

Then $D(U_\ell \tilde{E}'), D(V_\ell \tilde{E}'), G \in M_{\frac{3}{2}}(\Gamma_0(CN^2\ell), \psi''')$.

We set

$$S = \left\{ D_\xi \subset \mathcal{O}_F \mid \begin{array}{l} D_\xi = D(F(\sqrt{-2\xi})/F), \xi \in 2^{-1}\mathcal{O}_{F,+}, \epsilon(\xi) = -1, \\ \text{Tr}(\xi) < A, \text{ or } \text{Tr}(\xi) = gq/2, \text{ s.t. } (q\mathcal{O}_F, \xi\mathcal{O}_F) = 1 \end{array} \right\}.$$

Now we may take ℓ satisfying the following conditions:

- (1) $\ell^g - (\ell + 2)^g \not\equiv 0 \pmod{p}$,
- (2) $\left(\frac{F(\sqrt{-2\xi})/F}{\mathfrak{l}_i} \right) = -1$, for $\forall i, \forall \xi \in S$,
- (3) $\left(\frac{F(\sqrt{-2\xi})/F}{\mathfrak{l}_i} \right) = 1$, for $\forall i, \forall \xi$ s.t. $\text{Tr}(\xi) = gq/2$, and $D_\xi \notin S$.

Lemma 3.

$$\#\{\text{prime number } \ell < X \mid \ell \text{ satisfies (1), (2), (3)}\} \sim \frac{X}{\log X}$$

Proof. To apply Chebotarev's density theorem, it is sufficient to construct an appropriate Galois element for a Galois extension over \mathbb{Q} . Let $\{\xi \mid D(F(\sqrt{-2\xi})/F) \in S\} = \{\xi_1, \dots, \xi_s\}$ and $\{\xi \mid \text{Tr}(\xi) = qg/2, D(F(\sqrt{-2\xi})/F) \notin S\} = \{\xi'_1, \dots, \xi'_t\}$. Then we set the CM field K as the composite field of $F(\sqrt{-2\xi_1}), \dots, F(\sqrt{-2\xi_t})$ and set the CM field K' as the composite field of K and $F(\sqrt{-2\xi'_1}), \dots, F(\sqrt{-2\xi'_t})$. We also set CM field K'' as the composite field of K' and $F(\zeta_p)$ and set K''_{gal} as the Galois closure of K'' over \mathbb{Q} .

We may find prime number ℓ such that ℓ splits completely at F/\mathbb{Q} , all prime ideal $\mathfrak{l}|\ell$ is inert at $F(\sqrt{-2\xi_i})/F$ for all $i = 1, \dots, s$ and splits at $F(\sqrt{-2\xi'_i})/F$ for all $i = 1, \dots, t$, and ℓ satisfies (1). For the Frobenius element $\sigma = \text{Frob}_\ell \in \text{Gal}(K''_{gal}/\mathbb{Q})$, because ℓ splits completely at F/\mathbb{Q} , restriction $\sigma|_F$ to F is id_F . Thus we may regard $\sigma \in \text{Gal}(K''_{gal}/F)$. Then K''/F is Galois, so we set $\sigma' = \sigma|_{K''} \in \text{Gal}(K''/F)$. Then by the assumption of q in Theorem 1, we can take ℓ such that the prime ideal $\mathfrak{l}|\ell$ of F splits completely at the maximal totally real subfield K_+ of K and is inert at K/K_+ and splits completely at K'/K . Finally, we need to add the property (1). The polynomial $f(x) = (x+2)^g - x^g$ is of degree $g-1$. Thus $f(x)$ has at most $g-1$ solutions in \mathbb{F}_p . While by the assumption of p and the fact $[K' : F]$ is 2-power, $[K'' : K'] \geq g$. Thus we can take $f(\ell) \not\equiv 0 \pmod{p}$. By the Chebotarev density theorem, we have the above estimate. \square

We set

$$H = \sum_{n=1}^{\infty} b_n q^n = D(U_\ell \tilde{E}') - V_\ell G - (\ell + 2)^g D(V_\ell \tilde{E}')$$

Then $H \in M_{\frac{3}{2}}(\Gamma_0(CN^2\ell^2), \psi\epsilon^2\phi)$ and $H \not\equiv 0 \pmod{p}$ by the assumption. Indeed, for $1 \leq n < \ell$,

$$\begin{aligned} b_{\ell n} &= \sum_{\text{Tr}(\xi)=\ell n} a_{\ell\xi} - \sum_{\substack{\text{Tr}(\xi)=\ell n, \\ \ell\mathcal{O}_F \nmid \xi\mathcal{O}_F}} a_{\ell\xi} - (\ell + 2)^g \sum_{\text{Tr}(\xi)=n} a_\xi, \\ &= \sum_{\text{Tr}(\xi)=n} a_{\ell^2\xi} + \sum_{\substack{\text{Tr}(\xi)=\ell n, \\ \ell\mathcal{O}_F \nmid \xi\mathcal{O}_F}} a_{\ell\xi} - \sum_{\substack{\text{Tr}(\xi)=\ell n, \\ \ell\mathcal{O}_F \nmid \xi\mathcal{O}_F}} a_{\ell\xi} - (\ell + 2)^g \sum_{\text{Tr}(\xi)=n} a_\xi, \\ &= \sum_{\text{Tr}(\xi)=n} (a_{\ell^2\xi} - (\ell + 2)^g a_\xi). \end{aligned}$$

If we take ℓ as sufficiently large, then $\mathfrak{l}_i \nmid \xi\mathcal{O}_F$ for every $i = 1, \dots, g$ and $\text{Tr}(\xi) \leq$

$gg/2$. Indeed, if $\mathfrak{l}_i | \xi \mathcal{O}_F$, by arithmetic and geometric means, we have

$$\ell = N(\mathfrak{l}_i) \leq N(\xi) \leq \left(\frac{\text{Tr}(\xi)}{g} \right)^g.$$

Thus

$$\text{Tr}(\xi) < g\ell^{\frac{1}{g}} \implies \mathfrak{l}_i \nmid \xi \mathcal{O}_F$$

In our case, we may take the prime ℓ satisfying $n_0 < g\ell^{\frac{1}{g}}$. Then,

$$b_{\ell n_0} = (\ell^g - (\ell + 2)^g) \sum_{\substack{\text{Tr}(\xi)=n_0, \\ \chi_i(\xi)=\epsilon_i, i=1,2,\dots,r \\ \xi \in 2^{-1}\mathcal{O}_{F,+}, D_\xi \notin S}} a_\xi,$$

and, by the assumption, $H \not\equiv 0 \pmod{p}$ if we take $n_0 = gg/2$. Therefore we have

$$D(U_\ell \tilde{E}') - (\ell + 2)^g D(V_\ell \tilde{E}') \not\equiv 0 \text{ or } V_\ell G \not\equiv 0 \pmod{p}.$$

Because

$$V_\ell G \not\equiv 0 \pmod{p} \iff G \not\equiv 0 \pmod{p},$$

in any way, by Lemma 2, there is an element ξ such that $\ell \mathcal{O}_F \nmid \xi \mathcal{O}_F$, $\text{Tr}(\xi) < \kappa(4N^2\ell) < A\ell$, and $a_{\ell\xi} \not\equiv 0 \pmod{p}$. We remark that, by arithmetic means and geometric means,

$$N(\xi) \leq \frac{\text{Tr}(\xi)^g}{g^g} < \frac{A^g \ell^g}{g^g}.$$

and that

$$\mathfrak{l}_i \mid d(F(\sqrt{-2\ell\xi})/F) \mid 4(2\ell\xi)\mathcal{O}_F$$

for some $i = 1, \dots, g$. Thus we have

$$\ell < |N_{F/\mathbb{Q}}(d(F(\sqrt{-2\ell\xi})/F))| < A'' \ell^{2g}, \quad (1)$$

for a positive constant A'' independent of ℓ .

Let $\ell_{i_1} < \dots < \ell_{i_{2g+1}}$ be prime numbers as the above, ξ_{i_j} be the constant associated to ℓ_{i_j} , and $K_{i_j} = F(\sqrt{-2\ell_{i_j}\xi_{i_j}})$. If $K_{i_1} = \dots = K_{i_{2g+1}}$, we have

$$\ell_{i_1} \cdots \ell_{i_{2g+1}} \mid N(d(K_{i_1}/F)).$$

Therefore, inequality (1) implies

$$\ell_{i_1}^{2g+1} < \ell_{i_1} \cdots \ell_{i_{2g+1}} < |N(d(K_{i_1}/F))| < A'' \ell_{i_1}^{2g}.$$

This equality is true only for $\ell_{i_1} < A''$. Thus for a sufficiently large prime ℓ_{i_1} , $K_{i_1}, \dots, K_{i_{2g+1}}$ are same at most $2g$. By this argument and Lemma 3, we conclude the proof. \square

As a corollary of the proof of Theorem 1, we can prove the following simple statement for sufficiently large primes.

Corollary 1. *Let F be a totally real number extension of finite Galois over \mathbb{Q} . Then, for sufficiently large p*

$$\#\{\text{CM quad. ext. } K/F \mid p \nmid h^-(K/F), |N_{F/\mathbb{Q}}(D(K/F))| < X\} \gg \frac{X^{\frac{1}{2g}}}{\log X}.$$

Proof. If we take $r = 1$ and a Hecke character $\chi = \chi_1$ whose conductor \mathcal{N}_1 is independent of p , A is independent of p . So we may take a prime q and fix it. Then the assumption

$$\sum_{\substack{\xi \in 2^{-1}\mathcal{O}_{F,+}, \chi_i(\xi) = \epsilon_i, i=1,2,\dots,r \\ \text{Tr}(\xi) = qg/2, (q\mathcal{O}_F, \xi\mathcal{O}_F) \neq 1}} \beta(\xi) \frac{2^g h^-(F(\sqrt{-2\xi}))}{Q_{F(\sqrt{-2\xi})} w_{F(\sqrt{-2\xi})}} \not\equiv 0 \pmod{p},$$

holds for sufficiently large prime p because the LHS is positive. While the assumption $g \leq M(p)2^{-\text{ord}_2(M(p))}$ is sufficient condition to admit the condition (1) in Lemma 3. However the condition (1) also holds for sufficiently large p . Indeed, for sufficiently large p , $K' \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$ because K' and its discriminant $D(K'/\mathbb{Q})$ is independent of p . Thus we have

$$[K'' : K'] = [K' \cdot \mathbb{Q}(\zeta_p) : K'] = [\mathbb{Q}(\zeta_p) : K' \cap \mathbb{Q}(\zeta_p)] = [\mathbb{Q}(\zeta_p) : \mathbb{Q}] = p - 1.$$

Therefore for sufficiently large p , we have $[K'' : K'] \geq g$ and the condition (1) holds. This completes the proof of corollary. \square

3.2 Vanishing of relative Iwasawa invariants.

If we take a certain character as the quadratic character χ in the statement of Theorem 1, we can investigate vanishing of relative Iwasawa invariants.

Friedman [5, Criterion 1.0] showed vanishing criterion of relative Iwasawa invariants of CM fields.

Lemma 4 (Friedman's criterion). *Let K be a CM field, K^+ the maximal totally real subfield of K , and p an odd prime. Then the followings are equivalent:*

1. $\lambda_p^-(K) = \mu_p^-(K) = 0$,
2. $p \nmid h^-(K)$ and there is no prime ideal $\mathfrak{p} | p\mathcal{O}_F$ of K^+ splitting at K/K^+ .

For prime number $p \geq 3$ and its prime ideal factorization $p\mathcal{O}_F = (\mathfrak{p}_1 \cdots \mathfrak{p}_r)^e$, we set the character χ_i as quadratic residue symbol:

$$\chi_i(\xi) = \left(\frac{F(\sqrt{-2\xi})/F}{\mathfrak{p}_i} \right).$$

Taking all χ_i as the character and $\epsilon_i \in \{-1, 0\}$ for $i = 1, 2, \dots, r$ or adding an auxiliary character if $\epsilon_i = 0$ for all i , we have the following theorem:

Theorem 2. *Let $g = [F : \mathbb{Q}]$ and $D(F)$ be the discriminant of F/\mathbb{Q} . Let p be a prime such that $g \leq [F(\zeta_p) : F]/2^{\text{ord}_2([F(\zeta_p):F])}$ and $p > 2g + 1$. We set*

$$A = \frac{gp^2 D(F)}{8} \prod_{d|pD(F), d:\text{prime}} \left(1 + \frac{1}{d} \right).$$

If there is a prime number $q > (A/g)^g$ such that

$$\sum_{\substack{\xi \in 2^{-1}\mathcal{O}_{F,+}, \chi_i(\xi) = \epsilon_i \ (i=1,2,\dots,r) \\ \text{Tr}(\xi) = qg/2, (q\mathcal{O}_F, \xi\mathcal{O}_F) \neq 1}} \beta(\xi) \frac{2^g h^-(F(\sqrt{-2\xi}))}{Q_{F(\sqrt{-2\xi})} w_{F(\sqrt{-2\xi})}} \not\equiv 0 \pmod{p},$$

then

$$\#\{\text{CM quad. ext. } K/F \mid \lambda_p^-(K) = \mu_p^-(K) = 0, |N_{F/\mathbb{Q}}(D(K/F))| < X\} \gg_{F,p} \frac{X^{\frac{1}{2g}}}{\log X}.$$

Remark 2. We cannot prove the similar result to Corollary 1 for Theorem 2. Indeed, in the case of Theorem 2 the constant A depends on p . Thus even if we take sufficiently large prime p , we cannot ensure the existence of the summation indivisible by p .

We give a simple example on Theorem 2 for an exceptional prime number of Naito's result [10].

Example 1. Let $F = \mathbb{Q}(\sqrt{44})$, $p = 7$. (For real quadratic fields, the exceptional primes are Fermat primes.) We note that $p|w_F\zeta_F(-1)$, i.e., this case is an exceptional case of Naito's theorem [10]. Then $A = 1008$ and $(A/g)^g = 254016$. As the prime q , we choose $q = 254027$ which is inert at F . Then we have

$$\begin{aligned} \sum_{\substack{\xi \in 2^{-1}\mathcal{O}_{F,+}, \chi_i(\xi) = -1 \ (i=1,2) \\ \text{Tr}(\xi) = qg/2, (q\mathcal{O}_F, \xi\mathcal{O}_F) \neq 1}} \beta(\xi) \frac{2^g h^-(F(\sqrt{-2\xi}))}{Q_{F(\sqrt{-2\xi})} w_{F(\sqrt{-2\xi})}} &= \frac{2^g h^-(F(\sqrt{-q}))}{Q_{F(\sqrt{-q})} w_{F(\sqrt{-q})}} \\ &= u \times 27686 \equiv u \times 1 \not\equiv 0 \pmod{7}, \end{aligned}$$

where u is a p -unit. Thus we have

$$\begin{aligned} \#\{\text{CM quad. ext. } K/\mathbb{Q}(\sqrt{5}) \mid \lambda_{11}^-(K) = \mu_{11}^-(K) = 0, \\ |N(D(K/\mathbb{Q}(\sqrt{5})))| < X\} \gg \frac{X^{\frac{1}{4}}}{\log X}. \end{aligned}$$

References

- [1] S. Achimescu and A. Saha. Hilbert modular forms of weight 1/2 and theta functions. *J. Number Theory*, 128(12):3037–3062, 2008.
- [2] D. Byeon. A note on basic iwasawa λ -invariants of imaginary quadratic fields and congruence of modular forms. *Acta Arith*, 89(3):295–299, 1999.
- [3] H. Cohen and H. Lenstra Jr. Heuristics on class groups of number fields. In *Number Theory Noordwijkerhout 1983*, pages 33–62. Springer, 1984.
- [4] H. Davenport and H. Heilbronn. On the density of discriminants of cubic fields. ii. *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.*, pages 405–420, 1971.
- [5] E. Friedman. Iwasawa invariants. *Math. Ann.*, 271(1):13–30, 1985.
- [6] P. Hartung. Proof of the existence of infinitely many imaginary quadratic fields whose class number is not divisible by 3. *J. Number Theory*, 6(4):276–278, 1974.
- [7] K. Horie. A note on basic Iwasawa λ -invariants of imaginary quadratic fields. *Invent. Math.*, 88(1):31–38, 1987.

- [8] K. Horie and I. Kimura. On quadratic extensions of number fields and iwasawa invariants for basic \mathbb{Z}_3 -extensions. *J. Math. Soc. Japan*, 51(2):387–402, 1999.
- [9] W. Kohlen and K. Ono. Indivisibility of class numbers of imaginary quadratic fields and orders of Tate-Shafarevich groups. *Invent. Math.*, 135:387–398, 1999.
- [10] H. Naito. Indivisibility of class numbers of totally imaginary quadratic extensions and their Iwasawa invariants. *Journal of the Mathematical Society of Japan*, 43(1):185–194, 1991.
- [11] K. Ono. *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q -series*. National Science Foundation, 2004.
- [12] G. Shimura. On modular forms of half integral weight. *Ann. of Math.*, 97(3):440–481, 1973.
- [13] G. Shimura. On Eisenstein series of half-integral weight. *Duke Math. J.*, 52(2):281–314, 1985.
- [14] G. Shimura. On Hilbert modular forms of half-integral weight. *Duke Math. J.*, 55(4):765–838, 1987.
- [15] G. Shimura. On the fourier coefficients of hilbert modular forms of half-integral weight. *Duke Math. J.*, 71(2):501–557, 1993.
- [16] J. Sturm. On the congruence of modular forms. In *Number theory*, volume 1240 of *Lect. Notes Math.*, pages 275–280, New York, 1984-1985, 1987. Springer, Berlin.
- [17] Y. Takai. An analogue of Sturm’s theorem for Hilbert modular forms. to appear in *Algebra & Number Theory*.
- [18] A. Weil. Sur certains groupes d’opérateurs unitaires. *Acta Math.*, 111(1):143–211, 1964.