# *Diophantine Analysis and Related Fields* 2018

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## Abstracts of the Talks

### Thursday 1 March

**Daniel Duverney** (Baggio Engineering School, France)

Title: **Irrationality of the sums of certain series of reciprocals of Fibonacci numbers**

Abstract: We show how to prove irrationality results by using expansions in Lambert series and the properties of arithmetical functions connected with the divisor function. The original ideas go back to Erdös in 1948. As an example, let $F_n$ be the sequence of the Fibonacci numbers, defined by $F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1}$ $(n \geq 1)$. We prove that the sum

$$T_h = \sum_{n=1}^{+\infty} \frac{1}{F_{n^h}}$$

is irrational for every positive integer $h$.

**Yuri Nesterenko** (Moscow State Univ.)

Title: **G-functions and algebraic independence**

Abstract: The talk will start with a short historical introduction to the theory of E and G functions of Siegel. We compare the main results proved in these two sections of the theory of transcendental numbers, we will describe the difficulties that arise in the study of the arithmetic properties of the values of G-functions. The main part of the talk will be devoted to proving the algebraic independence over the field of rational numbers of values of the Gaussian hypergeometric function with parameters 1/2, 1/2 and 1 in nonzero algebraic points. This function belongs to the class of G-functions and there exists only a finite set of G-functions, for which a similar property is proved. In a different form, this statement was proved in 1985 by G. Chudnovsky, who used the elliptic functions. Another proof was proposed in 1996 by Y. Andre. In the talk, we will discuss two more proofs of this theorem. One will use the properties of quasi-modular functions and the other will be very close to the classical proofs proposed in the theories of E and G functions. We also use an important G-functions property, the cancellation of factorials, which was predicted by Siegel and proved by Chudnovsky, and a parametrization of this hypergeometric function by quasi-modular functions.

**Federico Pellarin** (Univ. of Saint-Etienne)

Title: **From modular forms to algebraic independence**

Abstract: In 1996, Yu. Nesterenko published a famous result, asserting that if $0 < |q| < 1$, then at least three of the four complex numbers $q, E_2(q), E_4(q), E_6(q)$ are algebraically independent over the field of the rational numbers, where

$$E_{2k}(z) = 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1-q^n}$$

denotes the normalized Eisenstein series of weight $2k$ in their dependence of the parameter $q = e^{2\pi i z}$. The proof of this result is a tour de force in which deep properties of the functions $E_{2k}$ are combined and used in such a way that a new challenging

question ultimately raises, if other functions leading to a similar statement exist at all.

In this talk we are going to review the simplest bases of the theory of classical modular forms and then move to a different environment in which a class of modular forms in equal positive characteristic, introduced by D. Goss, plays a central role. While this class of functions presents an essential failure which forbids to achieve any proof of an analogous statement of Nesterenko's theorem (this was known since decades), we will present a new notion of $\infty$-adic families of these modular forms and a conjectural statement which seems much more affordable and could lead, hopefully, to a new algebraic independence result in line with Nesterenko's famous theorem.

**Hajime Kaneko** (Univ. of Tsukuba)

TITLE: **On the digital expansion of algebraic numbers** (a joint work with Makoto Kawashima (Osaka Univ.))

ABSTRACT: Many mathematicians have studied the uniformity of the base-$b$ expansion of real numbers. However, the base-$b$ expansion of algebraic irrational numbers is still mysterious. Borel conjectured for each integral base $b$ that any algebraic irrational number has uniform base-b expansion, which is still an open problem. In this talk, we discuss recent developments of the uniformity of digits in the base-$b$ expansion of algebraic irrational numbers. Moreover, we also discuss the beta expansion of algebraic numbers and its generalization.

**Eiji Miyanohara** (Waseda Univ.)

TITLE: **Transcendence of the series generated by a generalized Rudin-Shapiro sequence**

ABSTRACT: First, we generalize the Rudin-Shapiro sequence by means of counting digit pattern (This sequence is not necessarily an automatic sequence). Next, we give the simultaneous recursively word definition of the generalized Rudin-Shapiro sequence. We apply the transcendence measures criterion established by Adamczewski-Bugeaud (2011) to find that, the series generated by an arithmetic sub sequence of the non-periodic generalized Rudin-Shapiro sequence gives a Liouville number or an S-number or T-number. We also define two infinite matrix products from the simultaneous recursively word definition of the generalized Rudin-Shapiro sequence. We can regard the two infinite matrix products as generalizations of the infinite product studied by Tachiya(2007), Amou-Vaananen(2015, 2017). We study the transcendence of the power series generated by the two infinite matrix products by using the generalized Rudin-Shapiro sequence's properties. Finally, we introduce a generalization of k-regular sequence introduced by Allouche-Shallit(1992).

**Shin-ichi Yasutomi** (Toho Univ.)

TITLE: **Certain multidimension $p$-adic continued fraction algorithm and cubic numbers**

ABSTRACT: We propose certain multidimension $p$-adic continued fraction algorithm. We show that for every cubic number field $K$ with $K \subset \mathbb{Q}_p$ if $\{1, \alpha, \beta\}$ is linearly independent over $\mathbb{Q}$ for $\alpha, \beta \in K$, the $(\alpha, \beta)$ has an eventually periodic continued fraction expansion related to this algorithm.

## Friday 2 March

**Makoto Kawashima** (Osaka Univ.)

TITLE: **Linear independence of special values of logarithm revisited** (a joint work with Noriko Hirata-Kohno (Nihon Univ.))

ABSTRACT: In this talk, we give a linear independence measure of special values of logarithm by using the Hermite-Pade approximation of logarithms.

**Chieh-Yu Chang** (National Tsing Hua Univ.)

TITLE: **Periods, logarithms and multiple zeta values**

ABSTRACT: In this talk, we will first recall multiple polylogarithms that are generalization of the classical logarithm. Multiple zeta values are specializations of these special functions and have period interpretation in the picture of mixed Tate motives. We then introduce p-adic multiple zeta values initiated by Furusho in 2004. A classical conjecture asserts that p-adic multiple zeta values satisfy the same linear relations that their corresponding real-valued multiple zeta values satisfy. The main result presented in this talk is to prove a function field analogue of this conjecture (joint work with Y. Mishiba). Key ideas of the proof will be sketched.

**Pietro Corvaja** (Univ. of Udine)

TITLE: **Betti maps of sections of abelian schemes and functional transcendence**

ABSTRACT: Given a family of abelian varieties (i.e. an abelian variety over a function field) and a section (i.e. a functional point) one can express its abelian logarithm in terms of a basis for the periods, obtaining the so-called Betti map. Some results of Bertrand and André or the monodromy group for such maps can be interpreted as statement in functional transcendence. In a recent work with André and Zannier, we show that under some mild conditions the differential of the Betti maps has maximal rank. In the talk, we present some applications of this result, and an idea of its proof, which makes use of several results in functional transcendence.

**Yu Yasufuku** (Nihon Univ.)

TITLE: **Integral points in one- and two-parameter orbits**

ABSTRACT: Based on the analogy between orbits and abelian varieties, it is natural to 'guess' that there are not many integral points in orbits. However, this is a very difficult problem in general. In this talk, we introduce this problem and resolve some low-rank cases, using affine algebraic geometry, known results in dynamical Mordell–Lang conjecture, and/or a very deep Diophantine conjecture by Vojta.

**Kaoru Sano** (Kyoto Univ.)

TITLE: **The arithmetic degrees for self-morphisms on semi-abelian varieties (a joint work with Yohsuke Matsuzawa (Univ. of Tokyo))**

ABSTRACT: For a dominant rational self-map on a smooth projective variety, the dynamical degree measures the topological complexity of the map. On the other hand, Silverman defined the arithmetic degree which measures the asymptotic behavior of the Weil height of rational point under the iteration of the map. Silverman conjectured that if the forward orbit of a rational point is Zariski dense, the arithmetic degree of the map at the point is equal to the dynamical degree. In this talk, we explain the proof of this conjecture for dominant self-morphisms on semi-abelian varieties.

**Wade Hindes** (City Univ. of New York)

TITLE: **Prime divisors in orbits and Galois groups of iterates**

ABSTRACT: Given a global field $K$ and a polynomial $\phi \in K[x]$, we study two finiteness questions related to iteration of $\phi$: whether all but finitely many terms of an orbit of $\phi$ must possess a primitive prime divisor, and whether the Galois groups of iterates of $\phi$ must have finite index in their natural overgroup $\mathrm{Aut}(T_d)$, where $T_d$ is the infinite tree of iterated preimages of 0 under $\phi$. We focus particularly on the case where $K$ is a finite extension of $\mathbb{F}_p(t)$, where far less is known. We resolve the first question in the affirmative under relatively weak hypotheses; interestingly, the main step in our proof is to rule out "Riccati differential equations" in backwards orbits. We then apply our result on primitive prime divisors to produce a family of polynomials for which the second question has an affirmative answer; these are the

first non-isotrivial examples of such polynomials.

## Saturday 3 March

**Tomohiro Yamada** (Osaka Univ.)

TITLE: **A diophantine equation related to odd perfect numbers**

ABSTRACT: We shall show that, for given primes $l, p_1, p_2$ such that $p_1 \equiv p_2 \equiv 1 \pmod{l}$ and $\max\{l, p_1, p_2\}$ is sufficiently large, the diophantine equation $(q^l-1)/(q-1) = p_1 p_2^m$ has at most four integer solutions $q, m \geq 0$ with $q$ prime.

**Yuta Suzuki** (Nagoya Univ.)

TITLE: **On relatively prime amicable pairs**

ABSTRACT: An *amicable pair* is a pair of distinct positive integers each of which is the sum of the proper divisors of the other. Gmelin (1917) conjectured that there is no relatively prime amicable pairs and Artjuhov (1975) and Borho (1974) proved that for any fixed positive integer $K$, there are only finitely many relatively prime amicable pairs $(M, N)$ with $\omega(MN) = K$, where $\omega(n)$ denotes the number of the distinct prime factors of $n$. Recently, Pollack (2015) obtained an upper bound

$$MN < (2K)^{2^{K^2}}$$

for such amicable pairs. In this talk, we improve this upper bound to

$$MN < \frac{\pi^2}{6} 2^{4^K - 2 \cdot 2^K}$$

and generalize this bound to some class of general amicable tuples.

**Makoto Nagata** (Osaka Univ. of Pharmaceutical Sciences)

TITLE: **Stochastic models related to lattice points of a right-angled triangle: on the fluctuations in the number of points**

ABSTRACT: About one hundred years ago, Hardy-Littlewood, Hecke, Ostrowski obtained some results about the number of lattice points in a right-angled triangle [1]. We remodel this classical Diophantine problem into an object in probability theory. In this talk, we will introduce stochastic models related to their Diophantine problem. First, we consider a very simple stochastic model related to the classical problem. Under a standard condition in probability theory (i.e., independent and identically distributed random variables: IID RVs), we know that the fluctuation of this simple model is small. However, this model has no lattice points, that is, there is no object to be counted. So, the following questions arise. ・What is a natural stochastic model from the viewpoint of counting lattice points? ・Does the standard condition of probability theory give a good estimation? That is, do IID RVs bring small fluctuation to the stochastic model? ・As in the classical case, does its fluctuation depend on the slope of the hypotenuse of the right-angled triangle? We propose a stochastic model which is made of IID RVs. In a special case, the above questions are solved by the standard approach of probability theory. In the general cases, they are difficult. However, there is a circumstantial evidence. If we accept supposition, the fluctuation of this model does not seem to be small. Then, what is a stochastic model which has objects to be counted and whose fluctuation is small? Since it is a stochastic model, it has fluctuation. However, under a suitable dependency, its fluctuation should be small. Question: In order to the fluctuation be small, how much dependency does the model need? In other words, how much independence (in IID RVs) can we preserve in the model? As an answer, we show an example whose independence is halved.

[1] G. H. Hardy and J. E. Littlewood "Some problems of Diophantine approximation: the lattice points of a right-angled triangle" Proceedings of the London Math. Society, Ser.2, Vol.20 (1921)