α-SELFDECOMPOSABLE DISTRIBUTIONS, MILD ORNSTEIN-UHLENBECK TYPE PROCESSES AND QUASI-SELFSIMILAR ADDITIVE PROCESSES

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Abstract. Selfdecomposable distributions, stationary Ornstein-Uhlenbeck type processes and selfsimilar additive processes are closely related and have been studied deeply. In this paper, generalizations of these concepts are introduced and investigated, which are α-selfdecomposable distributions, mild Ornstein-Uhlenbeck type processes and quasi-selfsimilar additive processes.

1. Introduction

There are close relations among selfdecomposable distributions, stationary Ornstein-Uhlenbeck type processes and selfsimilar additive processes. Let $\mathscr{P}(\mathbb{R}^d)$ and $I(\mathbb{R}^d)$ be the class of all probability distributions on $\mathbb{R}^d$ and the class of all infinitely divisible distributions on $\mathbb{R}^d$, respectively, and let $I_{\log}(\mathbb{R}^d)$ be the totality of $\mu \in I(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} \log^+ |x|\mu(dx) < \infty$, where $|x|$ is the Euclidean norm of $x \in \mathbb{R}^d$ and $\log^+ |x| = (\log |x|) \vee 0$. A distribution $\mu \in \mathscr{P}(\mathbb{R}^d)$ is said to be selfdecomposable if for each $b > 1$ there exists $\rho_b \in \mathscr{P}(\mathbb{R}^d)$ satisfying $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}_b(z), z \in \mathbb{R}^d$, where $\widehat{\mu}(z), z \in \mathbb{R}^d$, stands for the characteristic function of $\mu \in \mathscr{P}(\mathbb{R}^d)$. Here μ and ρ_b automatically belong to $I(\mathbb{R}^d)$. We denote the totality of selfdecomposable distributions on $\mathbb{R}^d$ by $L(\mathbb{R}^d)$. Thus, $L(\mathbb{R}^d) \subset I(\mathbb{R}^d)$. A stochastic process $\{Z_t, t \in \mathbb{R}\}$ on $\mathbb{R}^d$ is said to be a stationary Ornstein-Uhlenbeck type process (OU type process) if

$$Z_t = e^{-Ht} \int_{-\infty}^{t} e^{Hu} X(du), \quad t \in \mathbb{R}, \tag{1.1}$$

where $H > 0$ and X is an $\mathbb{R}^d$-valued homogeneous independently scattered random measure (homogeneous i.s.r.m.) over $\mathbb{R}$ with $\mathcal{L}(X((0,1])) \in I_{\log}(\mathbb{R}^d)$. Here and in

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what follows, $\mathcal{L}$ means "the law of". This process is known to be an almost surely unique stationary solution of the Langevin equation

$$Z_t - Z_s = \int_s^t X(du) - H\int_s^t Z_u du, \quad -\infty < s \le t < \infty. \tag{1.2}$$

Furthermore, for every $H > 0$, a stochastic process $\{Y_t, t \ge 0\}$ on $\mathbb{R}^d$ is said to be H-selfsimilar if for any $a > 0$,

$$\{Y_{at}, t \ge 0\} \stackrel{\mathrm{d}}{=} \{a^H Y_t, t \ge 0\},$$

where $\stackrel{\mathrm{d}}{=}$ stands for equality in all finite-dimensional distributions. Also, a stochastic process $\{Y_t, t \ge 0\}$ on $\mathbb{R}^d$ is called an additive process if it is a stochastically continuous càdlàg process with independent increments satisfying $Y_0 = 0$ a.s. Then, the three concepts above are related as follows, (see, e.g., [8, 10, 11]). Stationary OU type processes $\{Z_t\}$ satisfy $\mathcal{L}(Z_t) = \mathcal{L}\left(\int_0^\infty e^{-Hu}X(du)\right) \in L(\mathbb{R}^d)$ for all $t \in \mathbb{R}$. Conversely, for any $\mu \in L(\mathbb{R}^d)$, there is a stationary OU type process $\{Z_t\}$ on $\mathbb{R}^d$ satisfying $\mu = \mathcal{L}(Z_t)$ for all $t \in \mathbb{R}$. Also, stationary OU type processes correspond to selfsimilar additive processes through the Lamperti transformation introduced in [6]. Namely, if $\{Z_t, t \in \mathbb{R}\}$ is a stationary OU type process (1.1), then $\{Y_t, t \ge 0\}$ defined by

$$Y_t = \begin{cases} t^H Z_{\log t}, & \text{for } t > 0, \\ 0, & \text{for } t = 0, \end{cases}$$

is a H-selfsimilar additive process on $\mathbb{R}^d$. Conversely, if $\{Y_t, t \ge 0\}$ is a selfsimilar additive process on $\mathbb{R}^d$, then $\{Z_t, t \in \mathbb{R}\}$, defined by $Z_t = e^{-Ht}Y_{e^t}$, is a stationary OU type process. Moreover, the following relations between selfsimilar additive processes and selfdecomposable distributions are known. Any marginal distribution of a selfsimilar additive process is selfdecomposable. Conversely for any $\mu \in L(\mathbb{R}^d)$ there is a selfsimilar additive process with μ as its distribution at time 1.

The purpose of this paper is to introduce and study generalizations of these three concepts. Selfdecomposable distributions and stationary OU type processes have already been generalized to α-selfdecomposable distributions and mild OU type processes, respectively, in [9] as in the following two definitions.

Definition 1.1. Let $\alpha \in \mathbb{R}$. We say that $\mu \in I(\mathbb{R}^d)$ is *α-selfdecomposable*, if for any $b > 1$, there exists $\rho_b \in I(\mathbb{R}^d)$ satisfying

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)^{b^\alpha}\widehat{\rho}_b(z), \quad z \in \mathbb{R}^d. \tag{1.3}$$

We denote the totality of α-selfdecomposable distributions on $\mathbb{R}^d$ by $L^{\langle\alpha\rangle}(\mathbb{R}^d)$.

Throughout this paper, when we consider the case $\alpha = 1$, we will need the following function q_μ. For $\mu = \mu_{(A,\nu,\gamma)} \in I(\mathbb{R}^d)$, define a nonrandom continuous function $q_\mu\colon (-\infty,1) \to \mathbb{R}^d$ by

$$q_\mu(t) := \begin{cases} \displaystyle\int_t^0 (1-u)^{-1}du \left(\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+(1-u)^{-2}|x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right), & t \le 0, \\ 0, & 0 < t < 1. \end{cases}$$

If a random variable M satisfies $\mathcal{L}(M) = \mu \in I(\mathbb{R}^d)$, then we may also write q_M for q_μ.

Let

$$I_\alpha(\mathbb{R}^d) = \left\{\mu \in I(\mathbb{R}^d)\colon \int_{\mathbb{R}^d} |x|^\alpha \mu(dx) < \infty\right\}, \quad \text{for } 0 < \alpha < 2,$$

$$I_\alpha^0(\mathbb{R}^d) = \left\{\mu \in I_\alpha(\mathbb{R}^d)\colon \int_{\mathbb{R}^d} x\mu(dx) = 0\right\}, \quad \text{for } 1 \le \alpha < 2,$$

$$I_1^*(\mathbb{R}^d) = \left\{\mu_{(A,\nu,\gamma)} \in I_1^0(\mathbb{R}^d)\colon \lim_{T\to\infty}\int_1^T t^{-1}dt \int_{|x|>t} x\nu(dx) \text{ exists in } \mathbb{R}^d\right\}.$$

Definition 1.2. Suppose that X is an $\mathbb{R}^d$-valued homogeneous i.s.r.m. over $\mathbb{R}$ with $\mathcal{L}(X((0,1])) = \mu$.

(i) Let $\alpha < 0$. Then, a stochastic process $\{Z_t\}$ on $\mathbb{R}^d$ is said to be a *mild OU type process generated by* (α, X) if

$$Z_t = (1-\alpha t)^{1/\alpha}\int_{1/\alpha}^t (1-\alpha u)^{-1/\alpha} X(du), \quad 1/\alpha < t < \infty. \tag{1.4}$$

(ii) Let $0 < \alpha < 1$ and $\mu \in I_\alpha(\mathbb{R}^d)$. Suppose that S_α is an $\mathbb{R}^d$-valued strictly α-stable random variable independent of X. Then, a stochastic process $\{Z_t\}$ on $\mathbb{R}^d$ is said to be a *mild OU type process generated by* (α, X) *associated with* S_α if

$$Z_t = (1-\alpha t)^{1/\alpha}\left\{S_\alpha + \int_{-\infty}^t (1-\alpha u)^{-1/\alpha} X(du)\right\}, \quad -\infty < t < 1/\alpha. \tag{1.5}$$

(iii) Let $\alpha = 1$ and $\mu \in I_1(\mathbb{R}^d)$. Suppose that S_1 is an $\mathbb{R}^d$-valued 1-stable random variable independent of X. Then, a stochastic process $\{Z_t\}$ on $\mathbb{R}^d$ is said to be a *mild OU type process generated by* $(1, X)$ *associated with* S_1 if

$$Z_t = (1-t)\left\{S_1 + \operatorname*{p\text{-}lim}_{s\downarrow-\infty}\left(\int_s^t (1-u)^{-1}X(du) - q_\mu(s)\right)\right\}, \quad -\infty < t < 1, \tag{1.6}$$

where p-lim means limit in probability.

(iv) Let $1 < \alpha < 2$ and $\mu \in I_\alpha^0(\mathbb{R}^d)$. Suppose that S_α is an $\mathbb{R}^d$-valued α-stable random variable independent of X. Then, a stochastic process $\{Z_t\}$ on $\mathbb{R}^d$ is said to be a *mild OU type process generated by* (α, X) *associated with* S_α if $\{Z_t\}$ has the same expression as that in (1.5).

When $\alpha < 0$, we also call $\{Z_t\}$ just an *α-mild OU type process* if $\{Z_t\}$ is a mild OU type process generated by (α, X) for some X. Similarly, when $\alpha > 0$, we call $\{Z_t\}$ an *α-mild OU type process associated with* S_α if $\{Z_t\}$ is a mild OU type process generated by (α, X) associated with S_α for some X which is independent of S_α and fulfills the moment condition above. This X is called a *background driving homogeneous i.s.r.m.* of $\{Z_t\}$.

We now introduce the concept of quasi-selfsimilarity of additive processes.

Definition 1.3. Let $\alpha \in \mathbb{R}$ and $\{Y_t, t \geq 0\}$ be an additive processes on $\mathbb{R}^d$. We call $\{Y_t\}$ a *broad-sense α-quasi-selfsimilar additive process* if for any $a > 0$, there exists a function $c_a(t)$ such that for each $t \geq 0$,

$$\mathcal{L}(Y_{at}) = \mathcal{L}\left(aY_t + c_a(t)\right)^{a^{-\alpha}}, \tag{1.7}$$

where $\mathcal{L}\left(aY_t + c_a(t)\right)^{a^{-\alpha}}$ is the distribution whose characteristic function is the $a^{-\alpha}$-th power of that of $\mathcal{L}\left(aY_t + c_a(t)\right)$. If we can take $c_a(t) \equiv 0$ for all $a > 0$, then we call $\{Y_t\}$ an *α-quasi-selfsimilar additive process.*

Note that this concept depends only on equality of distributions of two processes for each fixed time t, which is different from the definition of ordinary selfsimilarity.

As to the generalized concepts above, we give several remarks as follows.

Remark 1.4. (1) We have $L^{\langle 0\rangle}(\mathbb{R}^d) = L(\mathbb{R}^d)$. Also, $L^{\langle -1\rangle}(\mathbb{R}^d)$ is the class of all s-selfdecomposable distributions on $\mathbb{R}^d$, which is sometimes written as $U(\mathbb{R}^d)$ and was studied deeply by Jurek, (see, e.g., [1, 3, 4, 5]). Also, the classes $L^{\langle \alpha\rangle}(\mathbb{R}^d), \alpha \in \mathbb{R}$, and similar ones were already studied by several authors. For details on this history, see [9].

(2) Consider the following *Langevin type equation*, which is an extension of (1.2) with $H = 1$ from $\alpha = 0$ to any $-\infty < \alpha < 2$:

(1.8)

$$Z_t - Z_s = \int_s^t X(du) - \int_s^t (1-\alpha s)^{-1} Z_s ds, \quad \begin{cases} 1/\alpha < s \leq t < \infty, & \text{when } \alpha < 0, \\ -\infty < s \leq t < 1/\alpha, & \text{when } 0 < \alpha < 2, \end{cases}$$

where X is an $\mathbb{R}^d$-valued homogeneous i.s.r.m. over $\mathbb{R}$. A stochastic process $\{Z_t\}$ on $\mathbb{R}^d$ is said to be a *solution* of the Langevin type equation (1.8) if sample paths of $\{Z_t\}$ are right-continuous with left limits and $\{Z_t\}$ satisfies (1.8) almost surely. Then, the following hold, (see Lemma 4.8 of [8], Theorems 2.4 and 2.8 of [16], and Theorem 8.3 of [9]).

(i) Let $\alpha < 0$. Then, a mild OU type process $\{Z_t\}$ generated by (α, X) is an almost surely unique solution of (1.8) satisfying $\text{p-lim}_{t\downarrow 1/\alpha}(1-\alpha t)^{-1/\alpha}Z_t = 0$.

(ii) Let $0 < \alpha < 1$. Then the improper stochastic integral in the expression (1.5) of a mild OU type process $\{Z_t\}$ generated by (α, X) associated with S_α is definable if and only if $\mu \in I_\alpha(\mathbb{R}^d)$. Also, $\{Z_t\}$ is an almost surely unique solution of (1.8) satisfying $\text{p-lim}_{t\downarrow -\infty}(1-\alpha t)^{-1/\alpha}Z_t = S_\alpha$.

(iii) Let $\alpha = 1$. Then the limit in probability in the expression (1.6) of a mild OU type process $\{Z_t\}$ generated by $(1, X)$ associated with S_1 is definable if and only if $\mu \in I_1(\mathbb{R}^d)$. Also, $\{Z_t\}$ is an almost surely unique solution of (1.8) satisfying $\text{p-lim}_{t\downarrow -\infty}\{(1-t)^{-1}Z_t + q_\mu(t)\} = S_1$.

(iv) Let $1 < \alpha < 2$. Then the improper stochastic integral in the expression (1.5) of a mild OU type process $\{Z_t\}$ generated by (α, X) associated with S_α is definable if and only if $\mu \in I^0_\alpha(\mathbb{R}^d)$. Also, $\{Z_t\}$ is an almost surely unique solution of (1.8) satisfying $\text{p-lim}_{t\downarrow -\infty}(1-\alpha t)^{-1/\alpha}Z_t = S_\alpha$.

(3) The case $\alpha = 0$ is that of selfdecomposable distributions, stationary OU type processes and selfsimilar additive processes, which is well known. Let $\alpha \geq 2$ and let X be an $\mathbb{R}^d$-valued homogeneous i.s.r.m. over $\mathbb{R}$. Then the improper stochastic integral $\int_{-\infty}^t (1-\alpha u)^{-1/\alpha} X(du)$ is definable if and only if $X = 0$, due to Lemma 4.8 of [8] and Theorem 2.4 of [16]. That is why we assume $\alpha \in (-\infty, 0) \cup (0, 2)$ in Definition 1.2. Also, Maejima and Ueda [9] proved that $L^{\langle 2\rangle}(\mathbb{R}^d)$ is the class of all Gaussian distributions, and that $L^{\langle\alpha\rangle}(\mathbb{R}^d) = \{\delta_\gamma \colon \gamma \in \mathbb{R}^d\}$ for $\alpha > 2$. Thus we treat only the case $\alpha \in (-\infty, 0) \cup (0, 2)$ henceforth in this paper, unless otherwise stated.

Organization of this paper is as follows. In Section 2, main results of this paper are stated. In Section 3, we give some preliminaries for the proofs of them. In Section 4, we prove the results stated in Section 2.

2. Main results

In this section, we state the main results of this paper, which are the generalization of the relations among selfdecomposable distributions, stationary Ornstein-Uhlenbeck type processes and selfsimilar additive processes with $\alpha = 0$.

We first state the following, which is the relation between α-selfdecomposable distributions and mild OU type processes.

Theorem 2.1. (i) *Let $\alpha < 0$. Then $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ if and only if there exists an $\mathbb{R}^d$-valued homogeneous i.s.r.m. X over $\mathbb{R}$ such that the mild OU type process $\{Z_t\}$on $\mathbb{R}^d$ generated by (α, X) fulfills that for all $t \in (1/\alpha, \infty)$, $\mathcal{L}(Z_t)^{(1-\alpha t)^{-1}} = \mu$.*
(ii) *Let $0 < \alpha < 1$. Then $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ if and only if there exist an $\mathbb{R}^d$-valued homogeneous i.s.r.m. X over $\mathbb{R}$ with $\mathcal{L}\left(X((0,1])\right) \in I_\alpha(\mathbb{R}^d)$ and an $\mathbb{R}^d$-valued strictly α-stable random variable S_α independent of X such that the mild OU type process $\{Z_t\}$ on $\mathbb{R}^d$ generated by (α, X) associated with S_α fulfills that for all $t \in (-\infty, 1/\alpha)$, $\mathcal{L}(Z_t)^{(1-\alpha t)^{-1}} = \mu$.*
(iii) *Let $\alpha = 1$. Then $\mu \in L^{\langle 1\rangle}(\mathbb{R}^d)$ if and only if there exist an $\mathbb{R}^d$-valued homogeneous i.s.r.m. X over $\mathbb{R}$ with $\mathcal{L}\left(X((0,1])\right) \in I_1(\mathbb{R}^d)$, an $\mathbb{R}^d$-valued 1-stable random variable S_1 independent of X, and a nonrandom function $p\colon (-\infty, 1) \to \mathbb{R}^d$ satisfying $p(0) = 0$ such that the mild OU type process $\{Z_t\}$ on $\mathbb{R}^d$ generated by $(1, X)$ associated with S_1 fulfills that for all $t \in (-\infty, 1)$, $\mathcal{L}(Z_t - p(t))^{(1-t)^{-1}} = \mu$.*
(iv) *Let $1 < \alpha < 2$. Then $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ if and only if there exist an $\mathbb{R}^d$-valued homogeneous i.s.r.m. X over $\mathbb{R}$ with $\mathcal{L}\left(X((0,1])\right) \in I^0_\alpha(\mathbb{R}^d)$ and an $\mathbb{R}^d$-valued α-stable random variable S_α independent of X such that the mild OU type process $\{Z_t\}$ on $\mathbb{R}^d$ generated by (α, X) associated with S_α fulfills that for all $t \in (-\infty, 1/\alpha)$, $\mathcal{L}\left(Z_t - \left\{(1-\alpha t)^{1/\alpha} - (1-\alpha t)\right\} c\right)^{(1-\alpha t)^{-1}} = \mu$, where $c \in \mathbb{R}^d$ is a constant for which $\mathcal{L}(S_\alpha - c)$ is strictly α-stable.*

We next state the relation between mild OU type processes and quasi-selfsimilar additive processes.

Theorem 2.2. (I) (i) *Let $\alpha < 0$. If $\{Z_t, t \in (1/\alpha, \infty)\}$ is an α-mild OU type process on $\mathbb{R}^d$, then $\{Y_t, t \geq 0\}$ defined by*

$$Y_t = \begin{cases} tZ_{(1-t^{-\alpha})/\alpha}, & t > 0, \\ 0, & t = 0, \end{cases} \tag{2.1}$$

is an α-quasi-selfsimilar additive process on $\mathbb{R}^d$.
(ii) *Let $0 < \alpha < 1$ and let S_α be an $\mathbb{R}^d$-valued strictly α-stable random variable. If $\{Z_t, t \in (-\infty, 1/\alpha)\}$ is an α-mild OU type process on $\mathbb{R}^d$ associated with S_α, then $\{Y_t, t \geq 0\}$ defined by*

$$Y_t = \begin{cases} tZ_{(1-t^{-\alpha})/\alpha} - S_\alpha, & t > 0, \\ 0, & t = 0, \end{cases} \tag{2.2}$$

is an α-quasi-selfsimilar additive process on $\mathbb{R}^d$.

(iii) *Let $\alpha = 1$ and let S_1 be an $\mathbb{R}^d$-valued* 1*-stable random variable. If $\{Z_t, t \in (-\infty, 1)\}$ is a* 1*-mild OU type process on $\mathbb{R}^d$ associated with S_1, then $\{Y_t, t \geq 0\}$ defined by*

$$Y_t = \begin{cases} tZ_{1-t^{-1}} + q_{X((0,1])}\left(1 - t^{-1}\right) - S_1, & t > 0, \\ 0, & t = 0, \end{cases} \tag{2.3}$$

is a broad-sense 1*-quasi-selfsimilar natural additive process on $\mathbb{R}^d$, where X is the background driving $\mathbb{R}^d$-valued homogeneous i.s.r.m. of $\{Z_t\}$.* (*The definition of naturalness will be given in Section 3.*)

(iv) *Let $1 < \alpha < 2$ and let S_α be an $\mathbb{R}^d$-valued α-stable random variable. If $\{Z_t, t \in (-\infty, 1/\alpha)\}$ is an α-mild OU type process on $\mathbb{R}^d$ associated with S_α, then $\{Y_t, t \geq 0\}$ defined by* (2.2) *is an α-quasi-selfsimilar additive process on $\mathbb{R}^d$ satisfying $E[|Y_1|] < \infty$ and $E[Y_1] = 0$.*

(II) (i) *Let $\alpha < 0$. If $\{Y_t, t \geq 0\}$ is an α-quasi-selfsimilar additive process on $\mathbb{R}^d$, then $\{Z_t, t \in (1/\alpha, \infty)\}$ defined by*

$$Z_t = (1 - \alpha t)^{1/\alpha} Y_{(1-\alpha t)^{-1/\alpha}}$$

is an α-mild OU type process on $\mathbb{R}^d$.

(ii) *Let $0 < \alpha < 1$ and let S_α be an $\mathbb{R}^d$-valued strictly α-stable random variable. If $\{Y_t, t \geq 0\}$ is an α-quasi-selfsimilar additive process on $\mathbb{R}^d$, then $\{Z_t, t \in (-\infty, 1/\alpha)\}$ defined by*

$$Z_t = (1 - \alpha t)^{1/\alpha} \left\{ S_\alpha + Y_{(1-\alpha t)^{-1/\alpha}} \right\} \tag{2.4}$$

is an α-mild OU type process on $\mathbb{R}^d$ associated with S_α.

(iii) *Let $\alpha = 1$ and let S_1 be an $\mathbb{R}^d$-valued* 1*-stable random variable. If $\{Y_t, t \geq 0\}$ is a broad-sense* 1*-quasi-selfsimilar natural additive process on $\mathbb{R}^d$, then there exist an $\mathbb{R}^d$-valued homogeneous i.s.r.m. X over $\mathbb{R}$ with $\mathcal{L}(X((0,1])) \in I_1(\mathbb{R}^d)$ and a continuous function $f\colon [0, \infty) \to \mathbb{R}^d$ which is locally of bounded variation and satisfies $f(0) = 0$, such that $\{Z_t, t \in (-\infty, 1)\}$ defined by*

$$Z_t = (1 - t) \left\{ S_1 + Y_{(1-t)^{-1}} - q_{X((0,1])}(t) - f\left((1 - t)^{-1}\right) \right\}$$

is a mild OU type process on $\mathbb{R}^d$ generated by $(1, X)$ *associated with S_1.*

(iv) *Let $1 < \alpha < 2$ and let S_α be an $\mathbb{R}^d$-valued α-stable random variable. If $\{Y_t, t \geq 0\}$ is an α-quasi-selfsimilar additive process on $\mathbb{R}^d$ satisfying $E[|Y_1|] < \infty$ and $E[Y_1] = 0$, then $\{Z_t, t \in (-\infty, 1/\alpha)\}$ defined by* (2.4) *is an α-mild OU type process on $\mathbb{R}^d$ associated with S_α.*

We call the transformations in Theorem 2.2 between mild OU type processes and quasi-selfsimilar additive processes *Lamperti type transformations.*

We finally state the relation between quasi-selfsimilar additive processes and α-selfdecomposable distributions.

Theorem 2.3. (I) *Let $\alpha \in \mathbb{R}$. If $\{Y_t, t \geq 0\}$ be a broad-sense α-quasi-selfsimilar additive process on $\mathbb{R}^d$, then, for all $t \geq 0$,*

$$\mathcal{L}(Y_t) \in \begin{cases} L^{\langle\alpha\rangle}(\mathbb{R}^d), & \text{if } \alpha \in (-\infty, 0) \cup [2, \infty), \\ L^{\langle\alpha\rangle}(\mathbb{R}^d) \cap \mathcal{C}_\alpha(\mathbb{R}^d), & \text{if } \alpha \in (0, 2), \end{cases}$$

where $\mathcal{C}_\alpha(\mathbb{R}^d)$ is the totality of $\mu \in I(\mathbb{R}^d)$ whose Lévy measure ν (in (3.1) below) satisfies $\lim_{r\to\infty} r^\alpha \int_{|x|>r} \nu(dx) = 0$.

(II) (i) *Let $\alpha < 0$. If $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$, then there exists, uniquely in law, an α-quasi-selfsimilar additive process $\{Y_t, t \geq 0\}$ on $\mathbb{R}^d$ satisfying $\mathcal{L}(Y_1) = \mu$.*

(ii) *Let $0 < \alpha < 1$. If $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$, then there exist an $\mathbb{R}^d$-valued strictly α-stable random variable S_α and an α-quasi-selfsimilar additive process $\{Y_t, t \geq 0\}$ on $\mathbb{R}^d$ independent of S_α satisfying $\mathcal{L}(S_\alpha + Y_1) = \mu$. These S_α and $\{Y_t\}$ are determined uniquely in law.*

(iii) *Let $\alpha = 1$. If $\mu \in L^{\langle 1\rangle}(\mathbb{R}^d)$, then there exist an $\mathbb{R}^d$-valued 1-stable random variable S_1 and a broad-sense 1-quasi-selfsimilar natural additive process $\{Y_t, t \geq 0\}$ on $\mathbb{R}^d$ independent of S_1 satisfying $\mathcal{L}(S_1 + Y_1) = \mu$. If $S_1^{(j)}$ and $\{Y_t^{(j)}\}$ in place of S_1 and $\{Y_t\}$ respectively fulfill the statement above for $j = 1, 2$, then there exists a continuous function $f\colon [0, \infty) \to \mathbb{R}^d$ which is locally of bounded variation and satisfies that $f(0) = 0$, $\mathcal{L}\left(S_1^{(1)}\right) = \mathcal{L}\left(S_1^{(2)} - f(1)\right)$ and $\left\{Y_t^{(1)}\right\} \stackrel{\mathrm{d}}{=} \left\{Y_t^{(2)} + f(t)\right\}$.*

(iv) *Let $1 < \alpha < 2$. If $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$, then there exist an $\mathbb{R}^d$-valued α-stable random variable S_α and an α-quasi-selfsimilar additive process $\{Y_t, t \geq 0\}$ on $\mathbb{R}^d$ independent of S_α satisfying $E[|Y_1|] < \infty$, $E[Y_1] = 0$ and $\mathcal{L}(S_\alpha + Y_1) = \mu$. These S_α and $\{Y_t\}$ are determined uniquely in law.*

3. Preliminaries for proofs

In this section, we give some preliminaries for proofs of the theorems in Section 2.

Throughout this paper, we use the Lévy-Khintchine representation of the characteristic function of $\mu \in I(\mathbb{R}^d)$ in the following form:

(3.1)
$$\widehat{\mu}(z) = \exp\left\{ -\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z, x\rangle}{1+|x|^2} \right) \nu(dx) \right\}, \quad z \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^d$, A is a nonnegative-definite symmetric $d \times d$ matrix, $\gamma \in \mathbb{R}^d$, and ν is a nonnegative measure, called the Lévy measure, satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$. We call (A, ν, γ) the Lévy-Khintchine triplet of μ and we write $\mu = \mu_{(A,\nu,\gamma)}$ when we want to emphasize the Lévy-Khintchine triplet. $C_\mu(z), z \in \mathbb{R}^d$, denotes the cumulant function of $\mu \in I(\mathbb{R}^d)$, that is, $C_\mu(z)$ is the unique continuous function satisfying $\widehat{\mu}(z) = e^{C_\mu(z)}$ and $C_\mu(0) = 0$. For $t \geq 0$ and $\mu \in I(\mathbb{R}^d)$, we write μ^t for the distribution with characteristic function $\widehat{\mu}(z)^t := e^{tC_\mu(z)}$. For a random variable X with distribution $\mu \in I(\mathbb{R}^d)$, we write $\widehat{\mathcal{L}}(X)(z)$ and $C_X(z)$ for $\widehat{\mu}(z)$ and $C_\mu(z)$, respectively.

We need the notion of stochastic integrals. Let J be an interval in $\mathbb{R}$ and let $\mathcal{B}_J^0$ denote the totality of $B \in \mathcal{B}(J)$ whose closure in the relative topology on J is compact. An $\mathbb{R}^d$-valued i.s.r.m. X over J is said to be homogeneous if $\mathcal{L}(X(B)) = \mathcal{L}(X(B+a))$ for all $B \in \mathcal{B}_J^0$ and $a \in \mathbb{R}$ satisfying $B + a \in \mathcal{B}_J^0$. See [8, 13, 14, 17], for the definition and the deep study of stochastic integrals of nonrandom measurable functions $f \colon J \to \mathbb{R}$ with respect to $\mathbb{R}^d$-valued i.s.r.m.'s X over J, denoted by $\int_B f(s)X(ds), B \in \mathcal{B}_J^0$. For $s, t \in J$, we use the symbol

$$\int_s^t f(u)X(du) = \begin{cases} \int_{(s,t]} f(u)X(du), & \text{for } s < t, \\ 0, & \text{for } t = s, \\ -\int_{(t,s]} f(u)X(du), & \text{for } t < s, \end{cases}$$

which is understood to be a càdlàg process in $s \in J$ for each fixed $t \in J$ and a càdlàg process in $t \in J$ for each fixed $s \in J$, since such a modification always exists. Indeed, by Remark 3.16 of [8], for a fixed $t_0 \in J$, $Y_t := \int_{t_0}^t f(s)X(ds), t \in J$ has a càdlàg modification $\{\widetilde{Y}_t, t \in J\}$, and $\{\widetilde{Y}_t - \widetilde{Y}_s, s, t \in J\}$ is a desired modification of $\left\{\int_s^t f(u)X(du), s, t \in J\right\}$. If J is infinite to the right, then the improper stochastic integral $\int_t^\infty f(s)X(ds), t \in J$, is defined as the limit in probability of $\int_{(t,u]} f(s)X(ds)$ as $u \to \infty$ whenever the limit exists. Then we understand $\left\{\int_t^\infty f(s)X(ds), t \in J\right\}$ to be a càdlàg process, since such a modification always exists. If J is infinite to the left, then $\int_{-\infty}^t f(s)X(ds)$, $t \in J$, is defined in a similar way and $\left\{\int_{-\infty}^t f(s)X(ds), t \in J\right\}$ is regarded as a càdlàg process. Similarly, for a nonrandom continuous function

$q\colon J \to \mathbb{R}^d$, we regard $\left\{\text{p-lim}_{s\downarrow -\infty}\left(\int_s^t f(u)X(du) - q(s)\right), t \in J\right\}$ as a càdlàg process, if J is infinite to the left and the limit in probability exists, (see [9]). Using the notion of i.s.r.m., we can define stochastic integrals with respect to natural additive processes in law. The concept of natural additive processes in law was defined in [13] as a characterization of semimartingale additive processes in terms of their Lévy-Khintchine triplets. Namely, an additive process in law $\{X_t\}$ with Lévy-Khintchine triplets (A_t, ν_t, γ_t) is natural if and only if γ_t is locally of bounded variation in t. Note that for any natural additive processes in law $\{X_t, t \geq 0\}$ on $\mathbb{R}^d$, there exists a unique $\mathbb{R}^d$-valued i.s.r.m. X over $[0,\infty)$ satisfying $X_t = X([0,t])$ a.s. for each $t \geq 0$. Then, stochastic integrals $\int_B f(s)dX_s$ of nonrandom measurable functions $f\colon [0,\infty) \to \mathbb{R}$ with respect to natural additive processes in law $\{X_t, t \geq 0\}$ are defined by $\int_B f(s)X(ds)$ for $B \in \mathcal{B}^0_{[0,\infty)}$.

Let $f\colon [0,\infty) \to \mathbb{R}$ be a nonrandom measurable function and $\{X_t\}$ a natural additive process on $\mathbb{R}^d$. When there exists a nonrandom function $q\colon [0,\infty) \to \mathbb{R}^d$ such that $\int_0^t f(s)dX_s - q(t)$ converges in probability as $t \to \infty$, we call the limit the essential improper stochastic integral. For details on essential improper stochastic integrals, see [14, 15, 16].

Using stochastic integrals with respect to Lévy processes, we can define a mapping

$$\Phi_f(\mu) = \mathcal{L}\left(\int_0^\infty f(t)dX_t^{(\mu)}\right), \quad \mu \in \mathfrak{D}(\Phi_f) \subset I(\mathbb{R}^d),$$

for a nonrandom measurable function $f\colon [0,\infty) \to \mathbb{R}$, where $\{X_t^{(\mu)}, t \geq 0\}$ is a Lévy process on $\mathbb{R}^d$ satisfying $\mathcal{L}(X_1^{(\mu)}) = \mu$ and $\mathfrak{D}(\Phi_f)$ is the domain of the mapping Φ_f that is the class of $\mu \in I(\mathbb{R}^d)$ for which $\int_0^\infty f(t)dX_t^{(\mu)}$ is definable in the sense above. See also [15, 16]. The range of Φ_f denoted by $\mathfrak{R}(\Phi_f)$ is defined as $\Phi_f(\mathfrak{D}(\Phi_f))$. As to essential improper stochastic integrals, we use the following symbols. $\mathfrak{D}(\Phi_{f,\mathrm{es}})$ denotes the totality of $\mu \in I(\mathbb{R}^d)$ such that the essential improper integral of f with respect to $\{X_t^{(\mu)}\}$ is definable. Also, for $\mu \in \mathfrak{D}(\Phi_{f,\mathrm{es}})$,

$$\Phi_{f,\mathrm{es}}(\mu) := \Bigg\{\mathcal{L}\left(\operatorname*{p-lim}_{t\to\infty}\left(\int_0^t f(s)dX_s^{(\mu)} - q(t)\right)\right) : q \text{ is an } \mathbb{R}^d\text{-valued nonrandom function such that } \operatorname*{p-lim}_{t\to\infty}\left(\int_0^t f(s)dX_s^{(\mu)} - q(t)\right) \text{ exists}\Bigg\}.$$

We also need to define mappings Φ_α, which characterize $L^{\langle\alpha\rangle}(\mathbb{R}^d)$. For $\alpha \in (-\infty,0)\cup(0,2)$, let

$$\Phi_\alpha(\mu) = \begin{cases} \mathcal{L}\left(\int_0^{-1/\alpha}(1+\alpha t)^{-1/\alpha}dX_t^{(\mu)}\right), & \text{when } \alpha<0, \\ \mathcal{L}\left(\int_0^{\infty}(1+\alpha t)^{-1/\alpha}dX_t^{(\mu)}\right), & \text{when } 0<\alpha<2. \end{cases} \tag{3.2}$$

We also write $\Phi_{1,\mathrm{es}}$ for $\Phi_{f,\mathrm{es}}$ with $f(t)=(1+t)^{-1}$. Due to Theorems 2.4 and 2.8 of [16], the domains $\mathfrak{D}(\Phi_\alpha)$ are as follows, (see also p. 49 of [16]).

$$\mathfrak{D}(\Phi_\alpha) = \begin{cases} I(\mathbb{R}^d), & \text{when } \alpha<0, \\ I_\alpha(\mathbb{R}^d), & \text{when } 0<\alpha<1, \\ I_1^*(\mathbb{R}^d), & \text{when } \alpha=1, \\ I_\alpha^0(\mathbb{R}^d), & \text{when } 1<\alpha<2, \end{cases}$$

$$\mathfrak{D}(\Phi_{1,\mathrm{es}}) = I_1(\mathbb{R}^d).$$

As to the ranges $\mathfrak{R}(\Phi_\alpha)$, Theorem 4.6 of [7] says the following.

$$\mathfrak{R}(\Phi_\alpha) = \begin{cases} L^{\langle\alpha\rangle}(\mathbb{R}^d), & \text{when } \alpha<0, \\ L^{\langle\alpha\rangle}(\mathbb{R}^d)\cap\mathcal{C}_\alpha(\mathbb{R}^d), & \text{when } 0<\alpha<1, \\ L^{\langle 1\rangle}(\mathbb{R}^d)\cap\mathcal{C}_1^*(\mathbb{R}^d), & \text{when } \alpha=1, \\ L^{\langle\alpha\rangle}(\mathbb{R}^d)\cap\mathcal{C}_\alpha^0(\mathbb{R}^d), & \text{when } 1<\alpha<2, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} \mathcal{C}_1^*(\mathbb{R}^d) = \Bigg\{ & \mu_{(A,\nu,\gamma)} \in L^{\langle 1\rangle}(\mathbb{R}^d)\cap\mathcal{C}_1(\mathbb{R}^d)\colon \nu(B) = \int_S \lambda(d\xi)\int_0^\infty 1\!\!1_B(r\xi)r^{-2}k_\xi(r)dr, \\ & \text{where } k_\xi(r) \text{ is nonnegative valued, measurable in } \xi, \\ & \text{and nonincreasing and right-continuous in } r, \\ & \text{and satisfies that } \lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 tdt\int_S \xi\lambda(d\xi)\int_0^\infty \frac{r^2}{1+t^2r^2}dk_\xi(r) \text{ exists in } \mathbb{R}^d \text{ and equals } \gamma \Bigg\}, \end{aligned}$$

$$\mathcal{C}_\alpha^0(\mathbb{R}^d) = \mathcal{C}_\alpha(\mathbb{R}^d)\cap I_1^0(\mathbb{R}^d), \quad \text{for } 1<\alpha<2.$$

We also know the following, which will be used later.

Lemma 3.1 (Theorems 5.1 and 5.2 of [9])**.** (i) *When $\alpha<0$, $\widetilde{\mu}\in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ if and only if $\widetilde{\mu}=\Phi_\alpha(\mu)$ for some $\mu\in I(\mathbb{R}^d)$. This μ is uniquely determined by $\widetilde{\mu}$.*
(ii) *When $0<\alpha<1$, $\widetilde{\mu}\in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ if and only if*

$$\widetilde{\mu} = \sigma_\alpha * \Phi_\alpha(\mu), \tag{3.4}$$

where $\mu\in I_\alpha(\mathbb{R}^d)$ and σ_α is a strictly α-stable distribution. These σ_α and μ are uniquely determined by $\widetilde{\mu}$.

(iii) *When* $\alpha = 1$, $\widetilde{\mu} \in L^{\langle 1 \rangle}(\mathbb{R}^d)$ *if and only if*

$$\widetilde{\mu} = \sigma_1 * \widetilde{\rho},$$

where $\widetilde{\rho} \in \Phi_{1,\mathrm{es}}(\rho)$ *for some* $\rho \in I_1(\mathbb{R}^d)$ *and* σ_1 *is a* 1*-stable distribution. If* $\rho_j \in I_1(\mathbb{R}^d)$, $\widetilde{\rho}_j \in \Phi_{1,\mathrm{es}}(\rho_j)$, $\sigma_{1,j}$ *is a* 1*-stable distribution for* $j = 1, 2$, *and* $\sigma_{1,1} * \widetilde{\rho}_1 = \sigma_{1,2} * \widetilde{\rho}_2$, *then* $\sigma_{1,1} = \sigma_{1,2} * \delta_{-\widetilde{c}}$, $\widetilde{\rho}_1 = \widetilde{\rho}_2 * \delta_{\widetilde{c}}$ *and* $\rho_1 = \rho_2 * \delta_c$ *for some* $\widetilde{c}, c \in \mathbb{R}^d$.

(iv) *When* $1 < \alpha < 2$, $\widetilde{\mu} \in L^{\langle \alpha \rangle}(\mathbb{R}^d)$ *if and only if* (3.4) *holds for some* $\mu \in I_\alpha^0(\mathbb{R}^d)$ *and some* α*-stable distribution* σ_α. *These* σ_α *and* μ *are uniquely determined by* $\widetilde{\mu}$.

4. Proofs of the main results

In this section, we prove the theorems stated in Section 2.

Theorem 2.1 is a consequence of Lemma 3.1 above and Theorem 8.3 of [9].

We next prove Theorem 2.2 (I).

Proof of Theorem 2.2 (I). (i) $\{Y_t\}$ in (2.1) satisfies that

$$Y_t = \int_{1/\alpha}^{(1-t^{-\alpha})/\alpha} (1-\alpha u)^{-1/\alpha} X(du), \quad t > 0,$$

where X is the background driving homogeneous i.s.r.m. of $\{Z_t\}$. Thus $\{Y_t\}$ is an additive process. The α-quasi-selfsimilarity of $\{Y_t\}$ follows from the calculation that for all $a > 0$,

$$C_{Y_{at}}(z) = \int_{1/\alpha}^{\{1-(at)^{-\alpha}\}/\alpha} C_{X((0,1])}\left((1-\alpha u)^{-1/\alpha} z\right) du$$
$$= a^{-\alpha} \int_{1/\alpha}^{(1-t^{-\alpha})/\alpha} C_{X((0,1])}\left((1-\alpha v)^{-1/\alpha} a z\right) dv = a^{-\alpha} C_{aY_t}(z). \tag{4.1}$$

(ii) $\{Y_t\}$ in (2.2) satisfies that

$$Y_t = \int_{-\infty}^{(1-t^{-\alpha})/\alpha} (1-\alpha u)^{-1/\alpha} X(du), \quad t > 0,$$

where X is the background driving homogeneous i.s.r.m. of $\{Z_t\}$, and hence $\{Y_t\}$ is an additive process. The α-quasi-selfsimilarity of $\{Y_t\}$ follows from a similar calculation to (4.1).

(iii) Let $\mathcal{L}(X((0,1])) = \mu = \mu_{(A,\nu,\gamma)}$. Then $\{Y_t\}$ in (2.3) satisfies that

$$Y_t = \operatorname*{p-lim}_{s \downarrow -\infty} \left(\int_s^{1-t^{-1}} (1-u)^{-1} X(du) - q_\mu(s) \right) + q_\mu(1-t^{-1}), \quad t > 0,$$

and thus it is a natural additive process since the Lévy-Khintchine triplet $(A_{Y_t}, \nu_{Y_t}, \gamma_{Y_t})$ of $\mathcal{L}(Y_t)$ fulfills that

$$\gamma_{Y_t} = \begin{cases} 0, & 0 \le t \le 1, \\ \displaystyle\int_1^t v^{-1} dv \left(\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+v^2|x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right), & t > 1. \end{cases}$$

We also have that for all $a > 0$,

(4.2)

$$\begin{aligned} C_{Y_{at}}(z) &= \lim_{s \downarrow -\infty} \left\{ \int_s^{1-(at)^{-1}} C_\mu \left((1-u)^{-1} z \right) du + i \left\langle -q_\mu(s) + q_\mu(1-(at)^{-1}), z \right\rangle \right\} \\ &= \lim_{s \downarrow -\infty} \Bigg\{ a^{-1} \Bigg(\int_{1-a(1-s)}^{1-t^{-1}} C_\mu \left((1-v)^{-1} az \right) dv \\ &\qquad\qquad + i \left\langle -q_\mu(1-a(1-s)) + q_\mu(1-t^{-1}), az \right\rangle \Bigg) \\ &\qquad + i \left\langle q_\mu(1-a(1-s)) - q_\mu(s) + q_\mu(1-(at)^{-1}) - q_\mu(1-t^{-1}), z \right\rangle \Bigg\} \\ &= a^{-1} C_{aY_t}(z) \\ &\qquad + \lim_{s \downarrow -\infty} i \left\langle q_\mu(1-a(1-s)) - q_\mu(s) + q_\mu(1-(at)^{-1}) - q_\mu(1-t^{-1}), z \right\rangle \\ &=: a^{-1} \left(C_{aY_t}(z) + i \langle c_a(t), z \rangle \right), \end{aligned}$$

where the convergence of $q_\mu(1-a(1-s)) - q_\mu(s) + q_\mu(1-(at)^{-1}) - q_\mu(1-t^{-1})$ as $s \downarrow -\infty$ is assured by that of $\int_{1-a(1-s)}^{1-t^{-1}} C_\mu \left((1-v)^{-1} az \right) dv + i \left\langle -q_\mu(1-a(1-s)) + q_\mu(1-t^{-1}), az \right\rangle$. The calculation (4.2) yields the broad-sense 1-quasi-selfsimilarity of $\{Y_t\}$.

(iv) Due to the same argument as that in (ii), $\{Y_t\}$ is an α-quasi-selfsimilar additive process. Since

$$Y_1 = \int_{-\infty}^0 (1-\alpha u)^{-1/\alpha} X(du),$$

we have $\mathcal{L}(Y_1) = \Phi_\alpha \left(\mathcal{L}(X((0,1])) \right) \in L^{\langle \alpha \rangle}(\mathbb{R}^d) \cap \mathcal{C}_\alpha^0(\mathbb{R}^d)$, which yields $E[|Y_1|] < \infty$ and $E[Y_1] = 0$. □

For the proof of Theorem 2.2 (II), we need the content of Theorem 2.3. Therefore, we first prove Theorem 2.3.

Proof of Theorem 2.3. (I) If $\{Y_t\}$ is a broad-sense α-quasi-selfsimilar additive process on $\mathbb{R}^d$, then for any $a > 0$, there exists a function $c_a(t)$ satisfying (1.7) for each $t \ge 0$.

For $0 \leq s < t$, let μ_s and $\mu_{s,t}$ denote $\mathcal{L}(Y_s)$ and $\mathcal{L}(Y_t - Y_s)$, respectively. By letting $a = s/t$, it follows from (1.7) that

$$\widehat{\mu}_s(z) = \widehat{\mu}_t\left((s/t)z\right)^{(s/t)^{-\alpha}} e^{i\left\langle (s/t)^{-\alpha} c_{s/t}(t), z\right\rangle}.$$

Thus we have

$$\widehat{\mu}_t(z) = \widehat{\mu}_s(z)\widehat{\mu}_{s,t}(z) = \widehat{\mu}_t\left((s/t)z\right)^{(s/t)^{-\alpha}} \left\{e^{i\left\langle (s/t)^{-\alpha} c_{s/t}(t), z\right\rangle}\widehat{\mu}_{s,t}(z)\right\}.$$

For any $b > 1$, by putting $s = b^{-1}t$, it follows that

$$\widehat{\mu}_t(z) = \widehat{\mu}_t\left(b^{-1}z\right)^{b^\alpha} \left\{e^{i\left\langle b^\alpha c_{b^{-1}}(t), z\right\rangle}\widehat{\mu}_{b^{-1}t,t}(z)\right\},$$

which yields $\mu_t \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$. Furthermore, if $\alpha \in (0, 2)$, then $\mathcal{L}(Y_t) \in \mathcal{C}_\alpha(\mathbb{R}^d)$, since

$$\mathcal{L}\left(r^{-1}Y_t + c_{r^{-1}}(t)\right)^{r^\alpha} = \mathcal{L}(Y_{r^{-1}t}) \to \delta_0 \quad \text{as } r \to \infty.$$

(II) (i) If $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$, then there exists an α-quasi-selfsimilar additive process $\{Y_t\}$ on $\mathbb{R}^d$ satisfying $\mathcal{L}(Y_1) = \mu$ due to Theorems 2.1 and 2.2 (I). It remains to show the uniqueness in law of $\{Y_t\}$. Let $\{Y_t^{(1)}\}$ and $\{Y_t^{(2)}\}$ be both α-quasi-selfsimilar additive processes on $\mathbb{R}^d$ having μ as their distribution at time 1. Then, we have

$$\widehat{\mathcal{L}}\left(Y_t^{(1)}\right)(z) = \left\{\widehat{\mathcal{L}}\left(tY_1^{(1)}\right)(z)\right\}^{t^{-\alpha}} = \widehat{\mu}(tz)^{t^{-\alpha}} = \left\{\widehat{\mathcal{L}}\left(tY_1^{(2)}\right)(z)\right\}^{t^{-\alpha}} = \widehat{\mathcal{L}}\left(Y_t^{(2)}\right)(z),$$

for all $t > 0$. Then Theorem 9.7 (iii) of [12] yields that $\{Y_t^{(1)}\} \overset{\mathrm{d}}{=} \{Y_t^{(2)}\}$.

(ii) If $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$, then there exist a strictly α-stable random variable S_α and an α-quasi-selfsimilar additive process $\{Y_t\}$ on $\mathbb{R}^d$ independent of S_α satisfying $\mathcal{L}(S_\alpha + Y_1) = \mu$ due to Theorems 2.1 and 2.2 (I). It remains to show the uniqueness in law of S_α and $\{Y_t\}$. Due to (I), we have $\mathcal{L}(Y_1) \in L^{\langle\alpha\rangle}(\mathbb{R}^d) \cap \mathcal{C}_\alpha(\mathbb{R}^d) = \mathfrak{R}(\Phi_\alpha)$. Then, Lemma 3.1 entails the uniqueness in law of S_α. Let $\{Y_t^{(j)}\}$ is an α-quasi-selfsimilar additive processes on $\mathbb{R}^d$ independent of S_α such that $\mathcal{L}(S_\alpha + Y_1^{(j)}) = \mu$ for $j = 1, 2$. Noting that $\widehat{\mathcal{L}}(S_\alpha)(z)$ does not vanish, we have

$$\widehat{\mathcal{L}}\left(Y_t^{(1)}\right)(z) = \widehat{\mu}(tz)^{t^{-\alpha}} \left\{\widehat{\mathcal{L}}\left(S_\alpha\right)(tz)\right\}^{-t^{-\alpha}} = \widehat{\mathcal{L}}\left(Y_t^{(2)}\right)(z),$$

for all $t > 0$. Then Theorem 9.7 (iii) of [12] yields that $\{Y_t^{(1)}\} \overset{\mathrm{d}}{=} \{Y_t^{(2)}\}$.

(iii) The existence of S_1 and $\{Y_t\}$ satisfying the statement is showed in the same way as that in (ii). It remains to prove the uniqueness in law of S_α and $\{Y_t\}$ up to addition of a continuous function which is locally of bounded variation. Let $S_1^{(j)}$ be a 1-stable random variable and $\{Y_t^{(j)}\}$ a broad-sense 1-quasi-selfsimilar natural additive process independent of $S_1^{(j)}$ such that $\mathcal{L}(S_1^{(j)} + Y_1^{(j)}) = \mu$ for $j = 1, 2$. Then $\mathcal{L}(Y_1^{(j)}) \in L^{\langle 1\rangle}(\mathbb{R}^d) \cap \mathcal{C}_1(\mathbb{R}^d)$ by (I). Theorem 4.1 and Lemma 4.3 of [9] and Lemma

5.1 of [7] yields that $\mathcal{L}(Y_1^{(j)}) \in \Phi_{1,\mathrm{es}}(\rho_j)$ for some $\rho_j \in I_1(\mathbb{R}^d)$. Therefore Lemma 3.1 entails that $\mathcal{L}(S_1^{(1)}) = \mathcal{L}(S_1^{(2)} + c)$ and $\mathcal{L}(Y_1^{(1)}) = \mathcal{L}(Y_1^{(2)} - c)$ for some $c \in \mathbb{R}^d$. Since

$$\begin{aligned}\mathcal{L}\left(Y_t^{(1)}\right) &= \mathcal{L}\left(tY_1^{(1)} + c_t^{(1)}(1)\right)^{t^{-1}} = \mathcal{L}\left(tY_1^{(2)} + c_t^{(1)}(1) - tc\right)^{t^{-1}} \\ &= \mathcal{L}\left(Y_t^{(2)} + t^{-1}\left(c_t^{(1)}(1) - c_t^{(2)}(1)\right) - c\right), \quad t > 0,\end{aligned}$$

for some functions $c_a^{(1)}(t)$ and $c_a^{(2)}(t)$, we have $\mathcal{L}(Y_t^{(1)}) = \mathcal{L}(Y_t^{(2)} + f(t))$, where

$$f(t) = \begin{cases} t^{-1}\left(c_t^{(1)}(1) - c_t^{(2)}(1)\right) - c, & \text{for } t > 0, \\ 0, & \text{for } t = 0. \end{cases}$$

f is continuous and locally of bounded variation since $\mathcal{L}(Y_t^{(1)}) = \mathcal{L}(Y_t^{(2)} + f(t))$. Moreover, Theorem 9.7 (iii) of [12] yields that $\{Y_t^{(1)}\} \stackrel{\mathrm{d}}{=} \{Y_t^{(2)} + f(t)\}$. Since $\mathcal{L}(S_1^{(1)} + Y_1^{(1)}) = \mathcal{L}(S_1^{(2)} + Y_1^{(2)})$, we have $\mathcal{L}(S_1^{(1)}) = \mathcal{L}(S_1^{(2)} - f(1))$.

(iv) It is proved in a similar way to (ii). □

Finally, we prove Theorem 2.2 (II). For that, it is sufficient to show the following.

Theorem 4.1. (i) *Let $\alpha < 0$. If $\{Y_t, t \geq 0\}$ is an α-quasi-selfsimilar additive process on $\mathbb{R}^d$, then*

$$Y_t = \int_{1/\alpha}^{(1-t^{-\alpha})/\alpha} (1 - \alpha u)^{-1/\alpha} X(du), \quad t > 0,$$

for some $\mathbb{R}^d$-valued homogeneous i.s.r.m. X over $\mathbb{R}$.

(ii) *Let $0 < \alpha < 1$. If $\{Y_t, t \geq 0\}$ is an α-quasi-selfsimilar additive process on $\mathbb{R}^d$, then*

$$Y_t = \int_{-\infty}^{(1-t^{-\alpha})/\alpha} (1 - \alpha u)^{-1/\alpha} X(du), \quad t > 0, \tag{4.3}$$

for some $\mathbb{R}^d$-valued homogeneous i.s.r.m. X over $\mathbb{R}$ with $\mathcal{L}(X((0,1])) \in I_\alpha(\mathbb{R}^d)$.

(iii) *Let $\alpha = 1$. If $\{Y_t, t \geq 0\}$ is a broad-sense 1-quasi-selfsimilar natural additive process on $\mathbb{R}^d$, then*

$$Y_t = \operatorname*{p\text{-}lim}_{s \downarrow -\infty}\left(\int_s^{1-t^{-1}} (1-u)^{-1} X(du) - q_{X((0,1])}(s)\right) + q_{X((0,1])}(1 - t^{-1}) + f(t), \ t > 0,$$

for some $\mathbb{R}^d$-valued homogeneous i.s.r.m. X over $\mathbb{R}$ with $\mathcal{L}(X((0,1])) \in I_1(\mathbb{R}^d)$ and some continuous function $f\colon [0,\infty) \to \mathbb{R}^d$ which is locally of bounded variation and satisfies $f(0) = 0$.

(iv) *Let $1 < \alpha < 2$. If $\{Y_t, t \geq 0\}$ is an α-quasi-selfsimilar additive process on $\mathbb{R}^d$ satisfying $E[|Y_1|] < \infty$ and $E[Y_1] = 0$, then (4.3) holds for some $\mathbb{R}^d$-valued homogeneous i.s.r.m. X over $\mathbb{R}$ with $\mathcal{L}(X((0,1])) \in I_\alpha^0(\mathbb{R}^d)$.*

Remark 4.2. Recall Theorem 2.2 (I). If we combine, for instance, (1.4) and (2.1), then we can say that when $\alpha < 0$, for a given $\mathbb{R}^d$-valued homogeneous i.s.r.m. X over $\mathbb{R}$,

$$Y_t = \begin{cases} tZ_{(1-t^{-\alpha})/\alpha} = \displaystyle\int_{1/\alpha}^{(1-t^{-\alpha})/\alpha} (1-\alpha u)^{-1/\alpha} X(du), & t > 0, \\ 0, & t = 0, \end{cases}$$

is an α-quasi-selfsimilar additive process on $\mathbb{R}^d$. Similar statements also hold for the cases $0 < \alpha < 1$, $\alpha = 1$ and $1 < \alpha < 2$. Theorem 4.1 assures their converses. The following fact is known, (see Theorem 1 of [2] and Theorem 6.1 of [8]). Let $\{Y_t, t \geq 0\}$ be a H-selfsimilar additive process on $\mathbb{R}^d$. Then there exists an $\mathbb{R}^d$-valued homogeneous i.s.r.m. X over $\mathbb{R}$ with $\mathcal{L}(X((0,1])) \in I_{\log}(\mathbb{R}^d)$ such that $Y_t = \int_{-\infty}^{\log t} e^{Hu} X(du), t > 0$. Hence, Theorem 4.1 corresponds to this fact.

To prove Theorem 4.1, we need the following proposition, which is about naturalness of quasi-selfsimilar additive processes and corresponds to Theorem 2.14 of [13].

Proposition 4.3. (i) *Let $\alpha < 1$ and let $\{Y_t, t \geq 0\}$ be an α-quasi-selfsimilar additive process on $\mathbb{R}^d$. Then, $\{Y_t\}$ is natural.*
(ii) *Let $\alpha \geq 1$ and let $\{Y_t, t \geq 0\}$ be an α-quasi-selfsimilar additive process on $\mathbb{R}^d$ satisfying $E[|Y_1|] < \infty$ and $E[Y_1] = 0$. Then, $\{Y_t\}$ is natural.*

Proof. By (1.7) with $c_a(t) \equiv 0$, a quasi-selfsimilar additive process $\{Y_t, t \geq 0\}$ with Lévy-Khintchine triplets (A_t, ν_t, γ_t) satisfies that for any $a > 0$ and each $t \geq 0$,

$$\gamma_{at} = a^{1-\alpha}\left(\gamma_t + \int_{\mathbb{R}^d} x r_a(x) \nu_t(dx)\right), \quad \nu_{at}(dx) = a^{-\alpha}\nu_t(a^{-1}dx), \tag{4.4}$$

with $r_a(x) = (1 + |ax|^2)^{-1} - (1 + |x|^2)^{-1}$.

(i) It follows from (4.4) that

$$\gamma_t = t^{1-\alpha}\left(\gamma_1 + \int_{\mathbb{R}^d} x r_t(x) \nu_1(dx)\right),$$

which has continuous derivative in t on any bounded closed interval $[u, v] \subset (0, \infty)$. Therefore γ_t is of bounded variation in t on any bounded closed interval $[u, v] \subset (0, \infty)$. Fix $a > 1$. By (4.4), we have $\gamma_t = a^{\alpha-1}\gamma_{at} - \int_{\mathbb{R}^d} x r_a(x) \nu_t(dx)$, which yields that for all $n \in \mathbb{Z}_+$,

$$|\gamma|_{[a^{-n-1}, a^{-n}]} \leq a^{\alpha-1}|\gamma|_{[a^{-n}, a^{-n+1}]} + \int_{\mathbb{R}^d} |x||r_a(x)|(\nu_{a^{-n}} - \nu_{a^{-n-1}})(dx),$$

where $|\gamma|_{[u,v]}$ denotes the variation of $t \mapsto \gamma_t$ on $[u,v]$. Then it follows that

$$(1-a^{\alpha-1})|\gamma|_{[a^{-n-1},a^{-n}]} \\ \leq a^{\alpha-1}\left(|\gamma|_{[a^{-n},a^{-n+1}]} - |\gamma|_{[a^{-n-1},a^{-n}]}\right) + \int_{\mathbb{R}^d} |x||r_a(x)|(\nu_{a^{-n}} - \nu_{a^{-n-1}})(dx),$$

which entails that

$$(1-a^{\alpha-1})|\gamma|_{[0,1]} \leq a^{\alpha-1}|\gamma|_{[1,a]} + \int_{\mathbb{R}^d} |x||r_a(x)|\nu_1(dx) < \infty.$$

Thus γ_t is of bounded variation in t on any bounded closed interval $[u,v] \subset [0,\infty)$.

(ii) Since $E[|Y_1|] < \infty$ and $E[Y_1] = 0$, we have $\gamma_1 = -\int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2}\nu_1(dx)$. Then, it follows from (4.4) that for $t > 0$,

$$\gamma_t = t^{1-\alpha}\left(\gamma_1 + \int_{\mathbb{R}^d} x r_t(x)\nu_1(dx)\right) = -t^{-\alpha}\int_{\mathbb{R}^d} \frac{tx|tx|^2\nu_1(dx)}{1+|tx|^2} = -\int_{\mathbb{R}^d} \frac{x|x|^2\nu_t(dx)}{1+|x|^2}.$$

Hence we have

$$|\gamma|_{[0,t]} \leq \int_{\mathbb{R}^d} \frac{|x|^3}{1+|x|^2}\nu_t(dx) < \infty.$$

Therefore γ_t is locally of bounded variation in t on $[0,\infty)$. □

We now prove Theorem 4.1 by using Proposition 4.3.

Proof of Theorem 4.1. (i) If $\{Y_t, t \geq 0\}$ is an α-quasi-selfsimilar additive process on $\mathbb{R}^d$, then $\mathcal{L}(Y_1) \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ due to Theorem 2.3 (I). Then by Lemma 3.1 there exists an $\mathbb{R}^d$-valued homogeneous i.s.r.m. $\widetilde{X}$ over $\mathbb{R}$ such that

$$\mathcal{L}\left(\int_{1/\alpha}^{0} (1-\alpha u)^{-1/\alpha}\widetilde{X}(du)\right) = \Phi_\alpha\left(\mathcal{L}(\widetilde{X}((0,1]))\right) = \mathcal{L}(Y_1).$$

If we put

$$\widetilde{Y}_t := \int_{1/\alpha}^{(1-t^{-\alpha})/\alpha} (1-\alpha u)^{-1/\alpha}\widetilde{X}(du), \quad t > 0,$$

and $\widetilde{Y}_0 := 0$, then due to the same calculation as (4.1), $\{\widetilde{Y}_t, t \geq 0\}$ is an α-quasi-selfsimilar additive process on $\mathbb{R}^d$ satisfying $\mathcal{L}(\widetilde{Y}_1) = \mathcal{L}(Y_1)$. Theorem 2.3 (II) (i) yields that $\{\widetilde{Y}_t\} \stackrel{\mathrm{d}}{=} \{Y_t\}$. Note that $\{\widetilde{Y}_t\}$ and $\{Y_t\}$ are natural by virtue of Proposition 4.3 (i). Define an $\mathbb{R}^d$-valued i.s.r.m. X over $(1/\alpha,\infty)$ by $X(B) := \int_{(1-\alpha B)^{-1/\alpha}} u^{-1}dY_u, B \in \mathcal{B}^0_{(1/\alpha,\infty)}$, where $(1-\alpha B)^{-1/\alpha} = \{(1-\alpha t)^{-1/\alpha} \colon t \in B\}$. This X is definable due to Proposition 3.12 of [8]. Since for each $B \in \mathcal{B}^0_{(1/\alpha,\infty)}$, $X(B)$ is equal in law to

$$\int_{(1-\alpha B)^{-1/\alpha}} u^{-1}d\widetilde{Y}_u = \int_B (1-\alpha v)^{1/\alpha}d\widetilde{Y}_{(1-\alpha v)^{-1/\alpha}} \\ = \int_B (1-\alpha v)^{1/\alpha}(1-\alpha v)^{-1/\alpha}\widetilde{X}(dv) = \widetilde{X}(B),$$

it follows from the homogeneity of $\widetilde{X}$ that X is homogeneous. Then $X = \{X(B), B \in \mathcal{B}^0_{[1/\alpha,\infty)}\}$ can be extended to an $\mathbb{R}^d$-valued homogeneous i.s.r.m. $X = \{X(B), B \in \mathcal{B}^0_{\mathbb{R}}\}$ with $\mathcal{L}(X((0,1])) = \mathcal{L}(\widetilde{X}((0,1]))$. Letting

$$X^\sharp(B) := X((1-B^{-\alpha})/\alpha) = \int_B u^{-1} dY_u, \quad B \in \mathcal{B}^0_{(0,\infty)},$$

where $(1-B^{-\alpha})/\alpha = \{(1-t^{-\alpha})/\alpha \colon t \in B\}$, we have that for each $t > 0$,

$$\int_{1/\alpha}^{(1-t^{-\alpha})/\alpha} (1-\alpha u)^{-1/\alpha} X(du) = \operatorname*{p\text{-}lim}_{s \downarrow 0} \int_s^t v X^\sharp(dv) = \operatorname*{p\text{-}lim}_{s \downarrow 0} \int_s^t v v^{-1} dY_v = Y_t \quad \text{a.s.}$$

Since $\left\{\int_{1/\alpha}^{(1-t^{-\alpha})/\alpha} (1-\alpha u)^{-1/\alpha} X(du)\right\}$ and $\{Y_t\}$ are càdlàg processes, it holds almost surely that

$$Y_t = \int_{1/\alpha}^{(1-t^{-\alpha})/\alpha} (1-\alpha u)^{-1/\alpha} X(du), \quad \text{for all } t > 0.$$

(ii) It is proved in a similar way to (i).

(iii) If $\{Y_t, t \geq 0\}$ is a broad-sense 1-quasi-selfsimilar natural additive process on $\mathbb{R}^d$, then $\mathcal{L}(Y_1) \in L^{\langle 1 \rangle}(\mathbb{R}^d) \cap \mathcal{C}_1(\mathbb{R}^d)$ due to Theorem 2.3 (I). By Theorem 4.1 and Lemma 4.3 of [9] and Lemma 5.1 of [7], there exists an $\mathbb{R}^d$-valued homogeneous i.s.r.m. $\widetilde{X}$ over $\mathbb{R}$ with $\mathcal{L}(\widetilde{X}((0,1])) \in I_1(\mathbb{R}^d)$ such that $\mathcal{L}(Y_1) \in \Phi_{1,\mathrm{es}}\left(\mathcal{L}(\widetilde{X}((0,1]))\right)$. Then there exists $c \in \mathbb{R}^d$ such that

$$\mathcal{L}\left(\operatorname*{p\text{-}lim}_{s \downarrow -\infty}\left(\int_s^0 (1-u)^{-1} \widetilde{X}(du) - q_{\widetilde{X}((0,1])}(s)\right) + c\right) = \mathcal{L}(Y_1).$$

If we put

$$\widetilde{Y}_t := \operatorname*{p\text{-}lim}_{s \downarrow -\infty}\left(\int_s^{1-t^{-1}} (1-u)^{-1} \widetilde{X}(du) - q_{\widetilde{X}((0,1])}(s)\right) + q_{\widetilde{X}((0,1])}(1-t^{-1}) + ct, \quad t > 0,$$

and $\widetilde{Y}_0 := 0$, then due to a similar calculation to (4.2), $\{\widetilde{Y}_t, t \geq 0\}$ is a broad-sense 1-quasi-selfsimilar natural additive process on $\mathbb{R}^d$ satisfying $\mathcal{L}(\widetilde{Y}_1) = \mathcal{L}(Y_1)$. Theorem 2.3 (II) (iii) yields that $\{\widetilde{Y}_t\} \stackrel{\mathrm{d}}{=} \{Y_t + f(t)\}$ for some continuous function $f \colon [0,\infty) \to \mathbb{R}^d$ which is locally of bounded variation and satisfies $f(0) = 0$. Define an $\mathbb{R}^d$-valued i.s.r.m. X over $(-\infty, 1)$ by

$$X(B) := \int_{(1-B)^{-1}} u^{-1} d\left\{Y_u + f(u)\right\} - \int_B (1-u) d\left\{q_{\widetilde{X}((0,1])}(u) + c(1-u)^{-1}\right\}$$

for $B \in \mathcal{B}^0_{(-\infty,1)}$. This X is definable due to Proposition 3.12 of [8]. Since for each $B \in \mathcal{B}^0_{(-\infty,1)}$, $X(B)$ is equal in law to

$$\int_{(1-B)^{-1}} u^{-1} d\widetilde{Y}_u - \int_B (1-u) d\left\{q_{\widetilde{X}((0,1])}(u) + c(1-u)^{-1}\right\}$$

$$= \int_B (1-v) d\widetilde{Y}_{(1-v)^{-1}} - \int_B (1-u) d\left\{ q_{\widetilde{X}((0,1])}(u) + c(1-u)^{-1} \right\}$$
$$= \int_B (1-v)(1-v)^{-1} \widetilde{X}(du) = \widetilde{X}(B),$$

it follows from the homogeneity of $\widetilde{X}$ that X is homogeneous. Then $X = \{X(B), B \in \mathcal{B}^0_{(-\infty,1)}\}$ can be extended to an $\mathbb{R}^d$-valued homogeneous i.s.r.m. $X = \{X(B), B \in \mathcal{B}^0_{\mathbb{R}}\}$ with $\mathcal{L}(X((0,1])) = \mathcal{L}(\widetilde{X}((0,1])) \in I_1(\mathbb{R}^d)$. Then, it holds almost surely that

$$Y_t = \operatorname*{p\text{-}lim}_{s \downarrow -\infty} \left(\int_s^{1-t^{-1}} (1-u)^{-1} X(du) - q_{X((0,1])}(s) \right) + q_{X((0,1])}(1-t^{-1}) + ct - f(t), \quad t > 0,$$

and $ct - f(t)$ is continuous and locally of bounded variation and satisfies $c \cdot 0 - f(0) = 0$.

(iv) It is proved in a similar way to (i). □

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