Limit Theorems for Infinite Variance Sequences

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ABSTRACT This article discusses limit theorems for some stationary sequences not having finite variances.

1 Independent sequences

Although the concern of this volume is "long-range dependence," we start this article by explaining the case of independent random variables for the sake of completeness.

Throughout this article, all random variables and stochastic processes are real-valued. In the following, $\mathcal{L}(X)$ stands for the law of a random variable X, and $\widehat{\mu}(\theta), \theta \in \mathbb{R}$, is the characteristic function of a probability distribution μ on $\mathbb{R}$.

Let $\{X_j\}$ be a sequence of independent and identically distributed (iid) random variables and suppose that there exist $a_n > 0, \uparrow \infty, \{k_n\} \subset \mathbb{N}, k_n \uparrow \infty$, and $c_n \in \mathbb{R}$ such that

$$\mathcal{L}\left(\frac{1}{a_n}\sum_{j=1}^{k_n} X_j + c_n\right) \to \mu \tag{1.1}$$

for some probability distribution μ, where $\to$ denotes weak convergence of probability distributions. Then μ is infinitely divisible, where μ is said to be infinitely divisible if for any $n \geq 1$, there exists a probability distribution μ_n such that $\widehat{\mu}(\theta) = \{\widehat{\mu}_n(\theta)\}^n$. Any infinitely dibisible probability distribution occurs as a limiting distribution as in (1.1), (see [4]). Three infinitely divisible distributions which are concerned with in this article are the following.

(i) Gaussian, if $\widehat{\mu}(\theta) = \exp\{i\theta m - \frac{1}{2}\theta^2\sigma^2\}, m \in \mathbb{R}, \sigma^2 > 0$.

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(ii) Non-Gaussian α-stable, $0 < \alpha < 2$, if it is not a delta measure, $\widehat{\mu}(\theta)$ does not varnish and for any $a > 0$,

$$\widehat{\mu}(\theta)^a = \widehat{\mu}(a^{1/\alpha}\theta)e^{i\theta c} \quad \text{for some } c \in \mathbb{R}.$$

(iii) Non-Gaussian α-semi-stable, $0 < \alpha < 2$, if it is not a delta measure, $\widehat{\mu}(\theta)$ does not varnish and for some $r \in (0,1)$ and some $c \in \mathbb{R}$,

$$\widehat{\mu}(\theta)^r = \widehat{\mu}(r^{1/\alpha}\theta)e^{i\theta c}. \tag{1.2}$$

When we want to emphasize r in (1.2), we refer to the distribution as (r,α)-semi-stable. If k_n in (1.1) satisfies that $k_n/k_{n+1} \to 1$ as $n \to \infty$, then μ in (1.1) is Gaussian or non-Gaussian stable, and if k_n in (1.1) satisfies that $k_n/k_{n+1} \to r$ as $n \to \infty$ for some $r \in (0,1)$, then μ in (1.1) is non-Gaussian (r,α)-semi-stable, (see [11]). It follows from the defintions that μ is α-stable if and only if it is (r,α)-semi-stable for any $r \in (0,1)$. The "if" part can be weakened to that if μ is (r,α)-semi-stable for $r = r_1, r_2 \in (0,1)$ such that $\log r_1/\log r_2$ is irrational, then μ is α-stable ([11]). Non-Gaussian stability and semi-stability are also characterized by the Lévy-Khintchine representation of $\widehat{\mu}$ as follows ([8]):

$$\widehat{\mu}(\theta) = \exp\left\{ i\theta c + \int_0^\infty \left(e^{i\theta s} - 1 - i\theta s I[\{|s| \le 1\}]\right) d\left(-\frac{H_+(s)}{s^\alpha}\right) \right. \tag{1.3}$$
$$\left. + \int_{-\infty}^0 \left(e^{i\theta s} - 1 - i\theta s I[\{|s| \le 1\}]\right) d\left(\frac{H_-(|s|)}{|s|^\alpha}\right) \right\},$$

where $H_\pm(s)$ are nonnegative functions on $(0,\infty)$ such that (i) $H_\pm(s)/s^\alpha$ are nonincreasing in $s > 0$ and (ii) $H_\pm(r^{1/\alpha}s) = H_\pm(s)$, $s > 0$, with r in (1.2). μ in (1.3) is (r,α)-semi-stable in general, and if $H_\pm(s) = C_\pm$ (constants), then it is α-stable.

Non-Gaussian semi-stable distributions do not have finite variances. Actually, if X is non-Gaussian α-semi-stable, then $E[|X|^\gamma] < \infty$ for any $\gamma < \alpha$ but $E[|X|^\alpha] = \infty$, (and thus $E[|X|^2] = \infty$, namely its variance is infinite). Also if the limiting distribution μ in (1.1) is non-Gaussian semi-stable, then iid random variables $\{X_j\}$ cannot have finite variances. (If $\{X_j\}$ are strongly dependent, then partial sums of bounded random variables may produce an infinite variance limit law. See Section 5.) On the other hand, if μ is Gaussian, it, of course, has finite variance, (and moreover all moments). However, this does not necessarily mean that, to get a Gaussian limit in (1.1), we need $\{X_j\}$ with finite variance. Namely, some infinite variance sequences produce Gaussian limits. We first explain it. In the following, ν is the law of X_j in (1.1).

Theorem 1.1. *Suppose $k_n = n$ in (1.1). Then (1.1) holds for a Gaussian μ if and only if*

$$h(z) = \int_{|x| \le z} x^2 \nu(dx)$$

is slowly varying.

For the proof, see [4] Section 2.6. If $h(z)$ is a slowly varying function with $h(z) \to \infty$, then, of course, $E[|X_j|^2] = \infty$, but we get a Gaussian limit μ.

In the following, whenever we say "semi-stable" without any further comments, we include "stable" case as its special case.

A stochastic process $\{X(t), t \geq 0\}$ is said to be a Lévy process if it has independent and stationary increments, it is stochastically continuous, its sample paths are right continuous and have left limits, and $X(0) = 0$ almost surely. If $\{X(t), t \geq 0\}$ is a Lévy process and $\mathcal{L}(X(1))$ is α-semi-stable, then it is called an α-semi-stable Lévy process and denoted by $\{Z_\alpha(t), t \geq 0\}$. "Process" version of the limit theorem (1.1) is the following. See [15].

Theorem 1.2. *Suppose* (1.1) *holds and* $\mu = \mathcal{L}(Z_\alpha(1))$. *Then*

$$\frac{1}{a_n} \sum_{j=1}^{[k_n t]} X_j + c_n \overset{w}{\to} Z_\alpha(t),$$

where $\overset{w}{\to}$ *means the convergence in the space* $D([0,\infty))$.

For later use, we extend the definition of $\{Z_\alpha(t), t \geq 0\}$ to the case for $t < 0$ in the following way. Let $\{Z_\alpha^{(-)}(t), t \geq 0\}$ be an independent copy of $\{Z_\alpha(t), t \geq 0\}$ and define for $t < 0$, $Z_\alpha(t) = -Z_\alpha^{(-)}(-t)$.

2 Semi-stable integrals

Let $f : \mathbb{R} \to \mathbb{R}$ be a nonrandom function. We consider integrals of f with respect to $\{Z_\alpha(t), t \in \mathbb{R}\}$. From now on, for simplicity, we assume that they are symmetric in the sense that $\mathcal{L}(Z_\alpha(t)) = \mathcal{L}(-Z_\alpha(t))$ for every t. Denote the characteristic function of $\mathcal{L}(Z_\alpha(1))$ by $\varphi(\theta), \theta \in \mathbb{R}$. In the rest of this article, we always assume that $\alpha \neq 2$, because we are only interested in infinite variance cases.

Theorem 2.1. *If* $f \in L^\alpha(\mathbb{R})$, *then*

$$X_\alpha = \int_{-\infty}^{\infty} f(u) dZ_\alpha(u)$$

can be defined in the sense of convergence in probability, and X_α *is also symmetric* α*-semi-stable such that*

$$E\left[e^{i\theta X_\alpha}\right] = \exp\left\{\int_{-\infty}^{\infty} \log \varphi(f(u)\theta) du\right\}. \tag{2.1}$$

For the stable case, see, e.g. [14], and for the semi-stable case, see [13]. We call X_α a semi-stable integral.

We define semi-stable integral processes by

$$X_\alpha(t) := \int_{-\infty}^{\infty} f_t(u) dZ_\alpha(u), \quad t \geq 0,$$

where $f_t : \mathbb{R} \to \mathbb{R}$, and $f_t \in L^\alpha(\mathbb{R})$ for each $t \geq 0$.

3 Limit theorems converging to semi-stable integral processes

Let $H > 0$. A stochastic process $\{X(t)\}$ is said to be H-semi-selfsimilar if for some $a \in (0,1)$

$$\{X(at)\} \stackrel{d}{=} \{a^H X(t)\}, \tag{3.1}$$

and to be H-selfsimilar if (3.1) holds for any $a > 0$, where $\stackrel{d}{=}$ denotes equality of all finite dimensional distributions. For more about semi-selfsimilarlity, see [10].

Here we consider two semi-stable integral processes of moving average type with infinite variances represented by

$$X_1(t) := \int_{-\infty}^{\infty} \left(|t-u|^{H-1/\alpha} - |u|^{H-1/\alpha}\right) dZ_\alpha(u), \quad t \geq 0, 0 < H < 1, H \neq \frac{1}{\alpha}, \tag{3.2}$$

and

$$X_2(t) := \int_{-\infty}^{\infty} \log\left|\frac{t-u}{u}\right| dZ_\alpha(u), \quad t \geq 0, 1 < \alpha < 2. \tag{3.3}$$

Both integrals are well defined because the integrands are L^α-integrable in the respective cases. $\{X_1(t), t \geq 0\}$ is H-semi-selfsimilar ([10]), and is H-selfsimilar if $\{Z_\alpha(t)\}$ is stable ([9]). $\{X_2(t)\}$ is $(1/\alpha)$-semi-selfsimilar, and is $(1/\alpha)$-selfsimilar if $\{Z_\alpha(t)\}$ is α-stable ([5]). Both processes have α-semi-stable joint distributions (and thus do not have finite variances). The dependence structure of the increments of $\{X_1(t)\}$ and $\{X_2(t)\}$ in the stable case is studied in [2].

Here we give limit theorems converging to $\{X_k(t)\}, k = 1, 2$. In the following, $\stackrel{d}{\to}$ denotes convergence in law. Suppose $\{X_j\}$ are iid symmetric random variables satisfying

$$n^{-1/\alpha} \sum_{j=1}^{n} X_j \stackrel{d}{\to} Z_\alpha(1) \tag{3.4}$$

when $\{Z_\alpha(t)\}$ is α-stable, or

$$r^{n/\alpha} \sum_{j=1}^{[r^{-n}]} X_j \stackrel{d}{\to} Z_\alpha(1)$$

when $\{Z_\alpha(t)\}$ is (r,α)-semi-stable. Take δ such that $\frac{1}{\alpha}-1<\delta<\frac{1}{\alpha}$, and define a stationary sequence

$$Y_k=\sum_{j\in\mathbb{Z}} c_j X_{k-j}, \quad k=1,2,...,$$

where

$$c_j=\begin{cases}0, & \text{if } j=0,\\ j^{-\delta-1}, & \text{if } j>0,\\ -|j|^{-\delta-1}, & \text{if } j<0.\end{cases} \tag{3.5}$$

We can easily see that the infinite series Y_k is well defined for each k and Y_k does not have finite variance. Define further for $H=\frac{1}{\alpha}-\delta$,

$$W_n(t)=n^{-H}\sum_{k=1}^{[nt]} Y_k \tag{3.6}$$

and

$$V_n(t)=r^{nH}\sum_{k=1}^{[r^{-n}t]} Y_k.$$

Theorem 3.1. *Let $\{X_1(t)\}$ and $\{X_2(t)\}$ be the processes defined in* (3.2) *and* (3.3)*, respectively.*

(i) Suppose that $\{Z_\alpha(t)\}$ is α-stable. Then

$$W_n(t)\overset{d}{\Rightarrow}\begin{cases}|\delta|^{-1}X_1(t) & \text{when } \delta\neq 0\\ X_2(t) & \text{when } \delta=0,\end{cases}$$

where $\overset{d}{\Rightarrow}$ denotes convergence of all finite dimensional distributions.

(ii) Suppose that $\{Z_\alpha(t)\}$ is (r,α)-semi-stable. Then

$$V_n(t)\overset{d}{\Rightarrow}\begin{cases}|\delta|^{-1}X_1(t) & \text{when } \delta\neq 0\\ X_2(t) & \text{when } \delta=0.\end{cases}$$

As to the proofs, for the case of stable $\{X_1(t)\}$, see [9], and for the case of stable $\{X_2(t)\}$, see [5]. For the semi-stable case, the proof can be carried out in almost the same way as for the stable case.

Remark 3.1. If $\delta<0$ (necessarily $\alpha>1$), then $H=1/\alpha-\delta>1/\alpha$. Thus the normalization n^H in (3.6) grows much faster than $n^{1/\alpha}$ in (3.4), the case of partial sums of independent random variables. This explains that $\{Y_k\}$ exhibits long range dependence.

4 Random walks in random scenery

Consider two independent Lévy processes $\{Z_\alpha(t)\}$ and $\{Z_\beta(t)\}$. Suppose that $\{Z_\alpha(t)\}$ is α-semi-stable with $0 < \alpha < 2$ and $\{Z_\beta(t)\}$ is β-stable with $1 < \beta \le 2$. Then the local time of $\{Z_\beta(t)\}$, $L_t(x)$, exists (see [3]) and we can define

$$\Delta(t) = \int_{-\infty}^{\infty} L_t(x) dZ_\alpha(u), \quad t \ge 0,$$

since $\{Z_\alpha(t)\}$ is a semimartingale (see [12]). $\{\Delta(t)\}$ is H-selfsimilar if $\{Z_\alpha(t)\}$ is stable ([6]), and is H-semi-selfsimilar if $\{Z_\alpha(t)\}$ is semi-stable ([1]), where $H = 1 - 1/\beta + 1/(\alpha\beta)$ $(> 1/2)$. Although $\Delta(t)$ itself is not distributed as to be semi-stable, the variance of $\Delta(t)$ is infinite, because of the infinite variance property of $\{Z_\alpha\}$.

Let $\{S_n, n \ge 0\}$ be an integer-valued random walk with mean zero and $\{\xi(j), j \in \mathbf{Z}\}$ be a sequence of independent and identically distributed symmetric random variables, independent of $\{S_n\}$. Further assume that

$$n^{-1/\beta} S_n \xrightarrow{d} Z_\beta(1),$$

and

$$n^{-1/\alpha} \sum_{j=1}^{n} \xi(j) \xrightarrow{d} Z_\alpha(1) \quad \text{when } \{Z_\alpha(t)\} \text{ is stable}$$

and

$$r^{r/\alpha} \sum_{j=1}^{[r^{-n}]} \xi(j) \xrightarrow{d} Z_\alpha(1) \quad \text{when } \{Z_\alpha(t)\} \text{ is semi-stable.}$$

Consider a stationary infinite variance sequence $\{\xi(S_k), k \ge 0\}$. This is the problem of random walks in random scenery.

Theorem 4.1. *Let $H = 1 - 1/\beta + 1/(\alpha\beta)$. Under the above assumptions, we have*

$$n^{-H} \sum_{k=1}^{[nt]} \xi(S_k) \stackrel{d}{\Rightarrow} \Delta(t) \quad \textit{when } \{Z_\alpha(t)\} \textit{ is stable} \tag{4.1}$$

and

$$r^{\beta n H} \sum_{k=1}^{[r^{-\beta n} t]} \xi(S_k) \stackrel{d}{\Rightarrow} \Delta(t) \quad \textit{when } \{Z_\alpha(t)\} \textit{ is semi-stable.} \tag{4.2}$$

Kesten-Spitzer [6] proved (4.1) and Arai [1] extended (4.1) to the semi-stable case (4.2).

Remark 4.1. If $\alpha > 1$, then $H = 1 - 1/\beta + 1/(\alpha\beta) > 1/\alpha$. By the same reason as in Remark 3.1, $\{\xi(S_k), k \ge 0\}$ exhibits long range dependence.

5 An infinite variance limit law may arise from sums of bounded random variables

Recently, Koul and Surgailis [7] gave an example that shows the title of this section above.

Consider a stationary sequence $\{Y_k\}$ in Section 3, and assume that (3.4) holds for α-stable $Z_\alpha(1)$ with $1 < \alpha < 2$ and δ in (3.6) is negative. They proved the following.

Theorem 5.1. *Let h be a real-valued measurable function of bounded variation such that $E[h(Y_0)] = 0$. Then*

$$n^{\delta-1/\alpha} \sum_{k=1}^{n} h(Y_k) \xrightarrow{d} C\widetilde{h} Z_\alpha(1),$$

where $C > 0$, $\widetilde{h} = -\int_{\mathbb{R}} f(x)dh(x)$ and f is the density function of $\mathcal{L}(Y_0)$.

Their comment on this theorem is "This theorem is surprising in the sense it shows that an α-stable $(1 < \alpha < 2)$ limit law may arise from sums of bounded random variables $\{h(Y_k)\}$. This is unlike the case of independent and identically distributed or weakly dependent summands."

References

[1] T. Arai (2001) : A class of semi-selfsimilar processes related to random walks in random scenery. To appear in *Tokyo J. Math.*

[2] A. Astraukas, J.R. Levy and M.S. Taqqu (1991) : The asymptotic dependence structure of the linear fractional Lévy motion. *Lithuanian Math. J.* **31** (1), 1–28.

[3] E. Boylan (1964) : Local times for a class of Markoff processes. *Illinois J. Math.* **8**, 19–39.

[4] B.V. Gnedenko and A.N. Kolmogorov (1968) : *Limit Distributions for Sums of Independent Random Variables.* Addison-Wesley.

[5] Y. Kasahara, M. Maejima and W. Vervaat (1988) : Log-fractional stable processes. Stoch. Proc. Appl. **30**, 329–229.

[6] H. Kesten and F. Spitzer (1979) : A limit theorem related to a new class of self similar processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **50**, 5–25.

[7] H.L. Koul and D. Surgailis (2001) : Asymptotics of weighted emprical processes of long memory moving averages with infinite variance. To appear in *Stoch. Proc. Appl.*

[8] P. Lévy (1937) : *Théorie de l'Addition des Variables Aléatoires.* Genthier-Villars.

[9] M. Maejima (1983) : On a class of self-similar processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **62**, 235–245.

[10] M. Maejima and K. Sato (1999) : Semi-selfsimilar processes. *J. Theoret. Probab.* **12**, 347–383.

[11] D. Mejzler (1973) : On a certain class of infinitely divisible distributions. *Israel J. Math.* **16**, 1–19.

[12] P.A. Meyer (1976) : Un cours sur les integrales stochastiques. Séminaire de Probabilités X. Univ. de Strasbourg. *Lecture Notes in Math.* **511**, Springer.

[13] B. Rajput and K. Rama-Murthy (1987) : Spectral representation of semistable preocesses, and semistable laws on Banach spaces. *J. Multivariate Anal.* **21**, 139–157.

[14] G. Samorodnitsky and M.S. Taqqu (1994) : *Stable Non-Gaussian Random Processes.* Chapman and Hall.

[15] A.V. Skorohod (1957) : Limit theorems for stochastic processes with independent increments. *Theory Probab. Appl.* **2**, 138–171.

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