Limit Theorems for Infinite Variance Sequences

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ABSTRACT This article discusses limit theorems for some stationary sequences not having finite variances.

1 Independent sequences

Although the concern of this volume is “long-range dependence,” we start this article by explaining the case of independent random variables for the sake of completeness.

Throughout this article, all random variables and stochastic processes are real-valued. In the following, \( \mathcal{L}(X) \) stands for the law of a random variable \( X \), and \( \hat{\mu}(\theta), \theta \in \mathbb{R} \), is the characteristic function of a probability distribution \( \mu \) on \( \mathbb{R} \).

Let \( \{X_j\} \) be a sequence of independent and identically distributed (iid) random variables and suppose that there exist \( a_n > 0, \uparrow \infty, \{k_n\} \subset \mathbb{N}, k_n \uparrow \infty, \) and \( c_n \in \mathbb{R} \) such that

\[
\mathcal{L} \left( \frac{1}{a_n} \sum_{j=1}^{k_n} X_j + c_n \right) \to \mu
\]  

(1.1)

for some probability distribution \( \mu \), where \( \to \) denotes weak convergence of probability distributions. Then \( \mu \) is infinitely divisible, where \( \mu \) is said to be infinitely divisible if for any \( n \geq 1 \), there exists a probability distribution \( \mu_n \) such that \( \hat{\mu}(\theta) = \{\hat{\mu}_n(\theta)\}^n \). Any infinitely divisible probability distribution occurs as a limiting distribution as in (1.1), (see [4]). Three infinitely divisible distributions which are concerned with in this article are the following.

(i) Gaussian, if \( \hat{\mu}(\theta) = \exp\{i\theta m - \frac{1}{2} \theta^2 \sigma^2\}, m \in \mathbb{R}, \sigma^2 > 0. \)

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(ii) Non-Gaussian $\alpha$-stable, $0 < \alpha < 2$, if it is not a delta measure, $\hat{\mu}(\theta)$ does not vanish and for any $a > 0,$

$$\hat{\mu}(\theta)^{\alpha} = \hat{\mu}(a^{1/\alpha} \theta)e^{i\theta c}$$

for some $c \in \mathbb{R}$.

(iii) Non-Gaussian $\alpha$-semi-stable, $0 < \alpha < 2$, if it is not a delta measure, $\hat{\mu}(\theta)$ does not vanish and for some $r \in (0, 1)$ and some $c \in \mathbb{R},$

$$\hat{\mu}(\theta)^{r} = \hat{\mu}(r^{1/\alpha} \theta)e^{i\theta c}.$$  \hspace{1cm} (1.2)

When we want to emphasize $r$ in (1.2), we refer to the distribution as $(r, \alpha)$-semi-stable. If $k_n$ in (1.1) satisfies that $k_n/k_{n+1} \to 1$ as $n \to \infty$, then $\mu$ in (1.1) is Gaussian or non-Gaussian stable, and if $k_n$ in (1.1) satisfies that $k_n/k_{n+1} \to r$ as $n \to \infty$ for some $r \in (0, 1)$, then $\mu$ in (1.1) is non-Gaussian $(r, \alpha)$-semi-stable, (see [11]). It follows from the definitions that $\mu$ is $\alpha$-stable if and only if it is $(r, \alpha)$-semi-stable for any $r \in (0, 1)$. The “if” part can be weakened to that if $\mu$ is $(r, \alpha)$-semi-stable for $r = r_1, r_2 \in (0, 1)$ such that $\log r_1/\log r_2$ is irrational, then $\mu$ is $\alpha$-stable ([11]). Non-Gaussian stability and semi-stability are also characterized by the Lévy-Khintchine representation of $\hat{\mu}$ as follows ([8]):

$$\hat{\mu}(\theta) = \exp \left\{ i\theta c + \int_{0}^{\infty} (e^{i\theta s} - 1 - i\theta sI[|s| \leq 1]) d \left( -\frac{H_+(s)}{s^\alpha} \right) \right\}$$

$$+ \int_{-\infty}^{0} (e^{i\theta s} - 1 - i\theta sI[|s| \leq 1]) d \left( \frac{H_-(|s|)}{|s|^\alpha} \right),$$  \hspace{1cm} (1.3)

where $H_\pm(s)$ are nonnegative functions on $(0, \infty)$ such that (i) $H_\pm(s)/s^\alpha$ are nonincreasing in $s > 0$ and (ii) $H_\pm(r^{1/\alpha} s) = H_\pm(s), s > 0$, with $r$ in (1.2).

$\mu$ in (1.3) is $(r, \alpha)$-semi-stable in general, and if $H_\pm(s) = C_\pm$ (constants), then it is $\alpha$-stable.

Non-Gaussian semi-stable distributions do not have finite variances. Actually, if $X$ is non-Gaussian $\alpha$-semi-stable, then $E[|X|^{\gamma}] < \infty$ for any $\gamma < \alpha$ but $E[|X|^\alpha] = \infty$, (and thus $E[|X|^2] = \infty$, namely its variance is infinite). Also if the limiting distribution $\mu$ in (1.1) is non-Gaussian semi-stable, then iid random variables $\{X_j\}$ cannot have finite variances. (If $\{X_j\}$ are strongly dependent, then partial sums of bounded random variables may produce an infinite variance limit law. See Section 5.) On the other hand, if $\mu$ is Gaussian, it, of course, has finite variance, (and moreover all moments). However, this does not necessarily mean that, to get a Gaussian limit in (1.1), we need $\{X_j\}$ with finite variance. Namely, some infinite variance sequences produce Gaussian limits. We first explain it. In the following, $\nu$ is the law of $X_j$ in (1.1).

**Theorem 1.1.** Suppose $k_n = n$ in (1.1). Then (1.1) holds for a Gaussian $\mu$ if and only if

$$h(z) = \int_{|x| \leq z} x^2 \nu(dx)$$
is slowly varying.

For the proof, see [4] Section 2.6. If $h(z)$ is a slowly varying function with $h(z) \to \infty$, then, of course, $E[|X|^p] = \infty$, but we get a Gaussian limit $\mu$.

In the following, whenever we say “semi-stable” without any further comments, we include “stable” case as its special case.

A stochastic process $\{X(t), t \geq 0\}$ is said to be a Lévy process if it has independent and stationary increments, it is stochastically continuous, its sample paths are right continuous and have left limits, and $X(0) = 0$ almost surely. If $\{X(t), t \geq 0\}$ is a Lévy process and $\mathcal{L}(X(1))$ is $\alpha$-semi-stable, then it is called an $\alpha$-semi-stable Lévy process and denoted by $\{Z_\alpha(t), t \geq 0\}$.

“Process” version of the limit theorem (1.1) is the following. See [15].

**Theorem 1.2.** Suppose (1.1) holds and $\mu = \mathcal{L}(Z_\alpha(1))$. Then

$$\frac{1}{a_n} \sum_{j=1}^{[k_n \xi]} X_j + c_n \xrightarrow{w} Z_\alpha(t),$$

where $\xrightarrow{w}$ means the convergence in the space $D([0, \infty))$.

For later use, we extend the definition of $\{Z_\alpha(t), t \geq 0\}$ to the case for $t < 0$ in the following way. Let $\{Z_\alpha^-\}(t), t \geq 0\}$ be an independent copy of $\{Z_\alpha(t), t \geq 0\}$ and define for $t < 0$, $Z_\alpha(t) = -Z_\alpha^-(t)$.

### 2 Semi-stable integrals

Let $f : \mathbb{R} \to \mathbb{R}$ be a nonrandom function. We consider integrals of $f$ with respect to $\{Z_\alpha(t), t \in \mathbb{R}\}$. From now on, for simplicity, we assume that they are symmetric in the sense that $\mathcal{L}(Z_\alpha(t)) = \mathcal{L}(-Z_\alpha(t))$ for every $t$. Denote the characteristic function of $\mathcal{L}(Z_\alpha(1))$ by $\varphi(\theta), \theta \in \mathbb{R}$. In the rest of this article, we always assume that $\alpha \neq 2$, because we are only interested in infinite variance cases.

**Theorem 2.1.** If $f \in L^\alpha(\mathbb{R})$, then

$$X_\alpha = \int_{-\infty}^{\infty} f(u) dZ_\alpha(u)$$

can be defined in the sense of convergence in probability, and $X_\alpha$ is also symmetric $\alpha$-semi-stable such that

$$E[e^{i\theta X_\alpha}] = \exp \left\{ \int_{-\infty}^{\infty} \log \varphi(f(u)\theta)\,du \right\}. \quad (2.1)$$

For the stable case, see, e.g. [14], and for the semi-stable case, see [13]. We call $X_\alpha$ a semi-stable integral.
We define semi-stable integral processes by

\[ X_\alpha(t) := \int_{-\infty}^{\infty} f_t(u) dZ_\alpha(u), \quad t \geq 0, \]

where \( f_t : \mathbb{R} \to \mathbb{R} \) and \( f_t \in L^\alpha(\mathbb{R}) \) for each \( t \geq 0 \).

3 Limit theorems converging to semi-stable integral processes

Let \( H > 0 \). A stochastic process \( \{X(t)\} \) is said to be \( H \)-semi-selfsimilar if for some \( a \in (0, 1) \)

\[ \{X(at)\} \overset{d}{=} \{a^H X(t)\}, \quad (3.1) \]

and to be \( H \)-selfsimilar if (3.1) holds for any \( a > 0 \), where \( \overset{d}{=} \) denotes equality of all finite dimensional distributions. For more about semi-selfsimilarity, see [10].

Here we consider two semi-stable integral processes of moving average type with infinite variances represented by

\[ X_1(t) := \int_{-\infty}^{\infty} \left( |t-u|^{H-1/\alpha} - |u|^{H-1/\alpha} \right) dZ_\alpha(u), \quad t \geq 0, 0 < H < 1, H \neq \frac{1}{\alpha}, \quad (3.2) \]

and

\[ X_2(t) := \int_{-\infty}^{\infty} \log \left| \frac{t-u}{u} \right| dZ_\alpha(u), \quad t \geq 0, 1 < \alpha < 2. \quad (3.3) \]

Both integrals are well defined because the integrands are \( L^\alpha \)-integrable in the respective cases. \( \{X_1(t), t \geq 0\} \) is \( H \)-semi-selfsimilar ([10]), and is \( H \)-selfsimilar if \( \{Z_\alpha(t)\} \) is stable ([9]). \( \{X_2(t)\} \) is \((1/\alpha)\)-semi-selfsimilar, and is \((1/\alpha)\)-selfsimilar if \( \{Z_\alpha(t)\} \) is \( \alpha \)-stable ([5]). Both processes have \( \alpha \)-semi-stable joint distributions (and thus do not have finite variances). The dependence structure of the increments of \( \{X_1(t)\} \) and \( \{X_2(t)\} \) in the stable case is studied in [2].

Here we give limit theorems converging to \( \{X_k(t)\}, k = 1, 2 \). In the following, \( \overset{d}{\to} \) denotes convergence in law. Suppose \( \{X_j\} \) are iid symmetric random variables satisfying

\[ n^{-1/\alpha} \sum_{j=1}^{n} X_j \overset{d}{\to} Z_\alpha(1) \quad (3.4) \]

when \( \{Z_\alpha(t)\} \) is \( \alpha \)-stable, or

\[ r^{n/\alpha} \sum_{j=1}^{[r^{-\gamma}]} X_j \overset{d}{\to} Z_\alpha(1) \]
when \( \{Z_\alpha(t)\} \) is \((r, \alpha)\)-semi-stable. Take \( \delta \) such that \( \frac{1}{\alpha} - 1 < \delta < \frac{1}{\alpha} \), and define a stationary sequence

\[
Y_k = \sum_{j \in \mathbb{Z}} c_j X_{k-j}, \quad k = 1, 2, \ldots,
\]

where

\[
c_j = \begin{cases} 
0, & \text{if } j = 0, \\
 j^{-\delta - 1}, & \text{if } j > 0, \\
 -|j|^{-\delta - 1}, & \text{if } j < 0.
\end{cases} \tag{3.5}
\]

We can easily see that the infinite series \( Y_k \) is well defined for each \( k \) and \( Y_k \) does not have finite variance. Define further for \( H = \frac{1}{\alpha} - \delta \),

\[
W_n(t) = n^{-H} \sum_{k=1}^{\lfloor nf \rfloor} Y_k \tag{3.6}
\]

and

\[
V_n(t) = r^{nH} \sum_{k=1}^{\lfloor r^nt \rfloor} Y_k.
\]

**Theorem 3.1.** Let \( \{X_1(t)\} \) and \( \{X_2(t)\} \) be the processes defined in (3.2) and (3.3), respectively.

(i) Suppose that \( \{Z_\alpha(t)\} \) is \( \alpha \)-stable. Then

\[
W_n(t) \overset{d}{\rightarrow} \begin{cases} 
|\delta|^{-1} X_1(t) & \text{when } \delta \neq 0, \\
 X_2(t) & \text{when } \delta = 0,
\end{cases}
\]

where \( \overset{d}{\rightarrow} \) denotes convergence of all finite dimensional distributions.

(ii) Suppose that \( \{Z_\alpha(t)\} \) is \((r, \alpha)\)-semi-stable. Then

\[
V_n(t) \overset{d}{\rightarrow} \begin{cases} 
|\delta|^{-1} X_1(t) & \text{when } \delta \neq 0, \\
 X_2(t) & \text{when } \delta = 0.
\end{cases}
\]

As to the proofs, for the case of stable \( \{X_1(t)\} \), see [9], and for the case of stable \( \{X_2(t)\} \), see [5]. For the semi-stable case, the proof can be carried out in almost the same way as for the stable case.

**Remark 3.1.** If \( \delta < 0 \) (necessarily \( \alpha > 1 \)), then \( H = 1/\alpha - \delta > 1/\alpha \). Thus the normalization \( n^H \) in (3.6) grows much faster than \( n^{1/\alpha} \) in (3.4), the case of partial sums of independent random variables. This explains that \( \{Y_k\} \) exhibits long range dependence.
4 Random walks in random scenery

Consider two independent Lévy processes \( \{Z_\alpha(t)\} \) and \( \{Z_\beta(t)\} \). Suppose that \( \{Z_\alpha(t)\} \) is \( \alpha \)-semi-stable with \( 0 < \alpha < 2 \) and \( \{Z_\beta(t)\} \) is \( \beta \)-stable with \( 1 < \beta \leq 2 \). Then the local time of \( \{Z_\beta(t)\} \), \( L_t(x) \), exists (see [3]) and we can define

\[
\Delta(t) = \int_{-\infty}^{\infty} L_t(x) dZ_\alpha(x), \quad t \geq 0,
\]

since \( \{Z_\alpha(t)\} \) is a semimartingale (see [12]). \( \{\Delta(t)\} \) is \( H \)-selfsimilar if \( \{Z_\alpha(t)\} \) is stable ( [6] ), and is \( H \)-semi-selfsimilar if \( \{Z_\alpha(t)\} \) is semi-stable ([1]), where \( H = 1 - 1/\beta + 1/(\alpha\beta) \) ( \( > 1/2 \) ). Although \( \Delta(t) \) itself is not distributed as to be semi-stable, the variance of \( \Delta(t) \) is infinite, because of the infinite variance property of \( \{Z_\alpha\} \).

Let \( \{S_n, n \geq 0\} \) be an integer-valued random walk with mean zero and \( \{\xi(j), j \in \mathbb{Z}\} \) be a sequence of independent and identically distributed symmetric random variables, independent of \( \{S_n\} \). Further assume that

\[
n^{-1/\beta} S_n \xrightarrow{d} Z_\beta(1),
\]

and

\[
n^{-1/\alpha} \sum_{j=1}^{n} \xi(j) \xrightarrow{d} Z_\alpha(1) \quad \text{when } \{Z_\alpha(t)\} \text{ is stable}
\]

and

\[
r^{r/\alpha} \sum_{j=1}^{[r-n]} \xi(j) \xrightarrow{d} Z_\alpha(1) \quad \text{when } \{Z_\alpha(t)\} \text{ is semi-stable}.
\]

Consider a stationary infinite variance sequence \( \{\xi(S_k), k \geq 0\} \). This is the problem of random walks in random scenery.

**Theorem 4.1.** Let \( H = 1 - 1/\beta + 1/(\alpha\beta) \). Under the above assumptions, we have

\[
n^{-H} \sum_{k=1}^{[nt]} \xi(S_k) \xrightarrow{d} \Delta(t) \quad \text{when } \{Z_\alpha(t)\} \text{ is stable} \quad (4.1)
\]

and

\[
r^{\beta nH} \sum_{k=1}^{[r-n\beta t]} \xi(S_k) \xrightarrow{d} \Delta(t) \quad \text{when } \{Z_\alpha(t)\} \text{ is semi-stable}. \quad (4.2)
\]


**Remark 4.1.** If \( \alpha > 1 \), then \( H = 1 - 1/\beta + 1/(\alpha\beta) > 1/\alpha \). By the same reason as in Remark 3.1, \( \{\xi(S_k), k \geq 0\} \) exhibits long range dependence.
An infinite variance limit law may arise from sums of bounded random variables

Recently, Koul and Surgailis [7] gave an example that shows the title of this section above.

Consider a stationary sequence \( \{Y_k\} \) in Section 3, and assume that (3.4) holds for \( \alpha \)-stable \( Z_\alpha(1) \) with \( 1 < \alpha < 2 \) and \( \delta \) in (3.6) is negative. They proved the following.

**Theorem 5.1.** Let \( h \) be a real-valued measurable function of bounded variation such that \( E[h(Y_0)] = 0 \). Then

\[
  n^{\alpha-1/\alpha} \sum_{k=1}^{n} h(Y_k) \overset{d}{\to} C \tilde{h} Z_\alpha(1),
\]

where \( C > 0, \tilde{h} = -\int_{\mathbb{R}} f(x) dh(x) \) and \( f \) is the density function of \( L(Y_0) \).

Their comment on this theorem is “This theorem is surprising in the sense it shows that an \( \alpha \)-stable \( (1 < \alpha < 2) \) limit law may arise from sums of bounded random variables \( \{h(Y_k)\} \). This is unlike the case of independent and identically distributed or weakly dependent summands.”

**References**


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