Moments and Projections of Semistable Probability Measures on p-adic Vector Spaces

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Abstract

In this paper, two topics on semistable probability measures on p-adic vector spaces are studied. One is the existence of absolute moments of operator-semistable probability measures and another is an answer to the question whether one can get semistability of a probability measure from that of all its projections. All results obtained here are extensions of known results for real vector spaces to p-adic vector spaces.

Keywords : semistable probability measure, p-adic vector space, moment of probability measure

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Semistable Probability Measures on p-adic Vector Spaces

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1 Introduction

Let V be a finite dimensional p-adic vector space, where p is a prime number. Let $\mathcal{P}(V)$ denote the topological semigroup of probability measures on V, with weak topology and convolution '$*$' as the semigroup operation defined with respect to the additive group structure on V. Let $GL(V)$ denote the group of all invertible linear operators on V, namely the set of all bi-continuous automorphisms on V. In the following, $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ and $\mathbb{N}$ stand for the set of real numbers, rational numbers, integers and natural numbers, respectively.

A probability measure μ on V is said to be *operator-semistable* if there exist $\tau \in GL(V)$, $c \in]0,1[$ and $\{x_t\}_{t\geq 0} \subset V$ such that μ is embeddable in a continuous real one-parameter semigroup $\{\mu_t\}_{t\geq 0} \subset \mathcal{P}(V)$ as $\mu = \mu_1$ satisfying $\tau(\mu_t) = \mu_{ct} * \delta_{x_t}$ for all $t \geq 0$, where δ_x denotes the Dirac measure supported on $x \in V$. However, if H is the compact subgroup such that $\mu_0 = \omega_H$, the normalised Haar measure of H, then the image of $\{x_t\}_{t\geq 0}$ on V/H is a continuous real one-parameter semigroup and hence it is trivial. Thus, $\tau(\mu_t) = \mu_{ct}$ for all $t \geq 0$. We call μ and $\{\mu_t\}_{t\geq 0}$ as above (τ, c)-*semistable*; when τ is a scaler automorphism, we say that μ (or $\{\mu_t\}_{t\geq 0}$) is *semistable*.

Let $\|\cdot\|$ be a p-adic vector space norm on V. For $r \in (0, \infty)$, we say that $\mu \in \mathcal{P}(V)$ has an *absolute moment of order* r if $\int_V \|x\|^r \, \mathrm{d}\mu(x) < \infty$. Note that this definition is independent of norm on V, since any two vector space norms on V are equivalent (see, e.g. Cassels [1], Chapter 7, Lemma 2.1).

Operator-semistable probability measures on real vector spaces have been studied extensively. For a complete survey of results on (operator-)semistable probability measures, see Hazod and Siebert [3] and Sato [10]. For some related results for p-adic vector spaces, see Dani and Shah [2], Shah [11], Telöken [17] and Yasuda [19].

In this paper, we discuss the existence of absolute moments of order r of an operator-semistable probability measure μ on a finite dimensional p-adic vector space V as an extension of the corresponding result on real vector spaces (Theorem 1). Using this result, we investigate the relation between semistability of a probability measure μ on V and that of all its one-dimensional projections under certain conditions (Theorem 2).

2 Moments of operator-semistable probability measures on p-adic vector spaces

In this section, we discuss the existence of absolute moments of operator-semistable probability measures on a finite dimensional p-adic vector space V. There exist a lot of operator-semistable probability measures on a p-adic vector space (see Dani and Shah [2], Theorem 4.2).

For a prime number p, let $\mathbb{Q}_p$ denote the topological field of p-adic numbers with the p-adic norm $|\cdot|_p$. Namely, for any rational number $x \in \mathbb{Q}$, if $x = (h/k)p^n$ for some integers $h, k, n \in \mathbb{Z}$, where p does not divide h or k, then $|x|_p = p^{-n}$ and $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to this norm. Let $d = \dim V$. Then V is isomorphic to $\mathbb{Q}_p^d$. Let $M(V)$ denote the space of all linear operators on V. Then $M(V)$ (resp. $GL(V)$) is isomorphic to $M_d(\mathbb{Q}_p)$, the vector space of $d \times d$ matrices (resp. $GL_d(\mathbb{Q}_p)$, the group of nonsingular $d \times d$ matrices) with entries in $\mathbb{Q}_p$, having the usual topology as a subset of $\mathbb{Q}_p^{d^2}$. Here, $M(V)$ is a d^2-dimensional p-adic vector space. Given a vector space norm $\|\cdot\|$ on V, we define a vector space norm $\|\cdot\|$ on $M(V)$ as follows: $\|\tau\| = \sup\{\|\tau(x)\| : x \in V, \|x\| = 1\}$ for $\tau \in M(V)$. Here, $\|\tau\tau'\| \leq \|\tau\|\,\|\tau'\|$ and $\|\tau(x)\| \leq \|\tau\|\,\|x\|$ for $\tau, \tau' \in M(V)$ and $x \in V$. We define the spectral radius $s(\tau) = \lim_{n\to\infty} \|\tau^n\|^{1/n}$ for $\tau \in M(V)$; it is easy to see that the limit exists. Note that $s(\tau) \leq \|\tau^n\|^{1/n}$ for all n. Clearly, $s(\tau)$ is independent of the norm defined as above, since any two vector space norms on $M(V)$ are equivalent (see, e.g. Cassels [1], Chapter 7, Lemma 2.1).

We now define a vector space norm $\|\cdot\|_p$ on V (resp. on $M(V)$) as follows: We fix a basis $\{e_1, \ldots, e_d\}$ on V. For $x = (x_1, \ldots, x_d) = \sum_{i=1}^d x_i e_i \in V$, let $\|x\|_p = \max_i |x_i|_p$. Using this norm on V, we can define $\|\cdot\|_p$ on $M(V)$ as follows; for any $\tau \in M(V)$, $\|\tau\|_p = \sup\{\|\tau(x)\|_p : \|x\|_p = 1\}$. Note that if $\tau = (a_{ij}) \in M_d(\mathbb{Q}_p)$ with respect to the basis mentioned above, $\|\tau\|_p = \max_{i,j} |a_{ij}|_p$. Here, for $x = \sum_{i=1}^d x_i e_i$ and $y = \sum_{i=1}^d y_i e_i$ in V, $\|x + y\|_p = \max_i |x_i + y_i|_p = |x_j + y_j|_p \leq \max\{|x_j|_p, |y_j|_p\}$ for some j. Then $\|x + y\|_p \leq \max\{\|x\|_p, \|y\|_p\}$. Therefore, for any $r > 0$, $\|x+y\|_p^r \leq \max\{\|x\|_p^r, \|y\|_p^r\}$, and hence $x \mapsto \|x\|_p^r$ is a continuous subadditive function on V (see its definition below).

A probability measure μ on V is said to be *full* if the support of μ, denoted by $\operatorname{supp}\mu$, is not contained in a proper subspace of V. For $\tau \in GL(V)$, let $C(\tau) = \{x \in V : \tau^n(x) \to 0\}$. Clearly $C(\tau)$ is a τ-invariant vector subspace of V. We say that τ is *contracting* on V if $C(\tau) = V$. Note that, since $C(\tau)$ is closed in V, for any $\mu \in \mathcal{P}(V)$, $\tau^n(\mu) \to \delta_0$ if and only if $\operatorname{supp}\mu \subset C(\tau)$. A measure $\mu \in \mathcal{P}(V)$ is said to be an *idempotent* if $\mu^2 = \mu$, equivalently, if $\mu = \omega_H$, the normalised Haar measure of some compact subgroup H. Clearly, any idempotent has an absolute moment of any order.

The following is a generalization of a result on existence of absolute moments of operator-semistable probability measures on real vector spaces to p-adic vector spaces, (for the result on real vector spaces, see Luzak [5], [6]).

Theorem 1 *Let V be a finite dimensional p-adic vector space. Let $\mu \in \mathcal{P}(V)$ be full, non-idempotent and (τ, c)-semistable for some $\tau \in GL(V)$ and some $c \in]0.1[$. Then μ has an absolute moment of order r if and only if $s(\tau^{-1})^r c < 1$.*

Remark: The fullness condition is not needed for the "if" part of the above theorem. Suppose μ is a non-idempotent probability measure embeddable in a (τ, c)-semistable $\{\mu_t\}_{t\geq 0}$ as $\mu = \mu_1$. Let V_μ be the subspace generated by $\operatorname{supp}\mu$. Since V is totally disconnected, $\operatorname{supp}\mu_t \subset V_\mu$ for all $t \geq 0$, and hence V_μ is τ-invariant. Let τ_μ be the restriction of τ to V_μ. Since μ is full on V_μ, μ has an absolute moment of order r if and only if $s(\tau_\mu^{-1})^r c < 1$, by Theorem 1. Thus, the "if" part of the assertion without the fullness condition follows since $s(\tau_\mu^{-1}) \leq s(\tau^{-1})$.

Before proving the above theorem, let us state a result on subadditive functions on a locally compact (Hausdorff) group G. A function $\phi : G \to [0, \infty[$ (resp. $\phi : G \to]0, \infty[$) is said to be *subadditive* (resp. *submultiplicative*) if $\phi(xy) \leq \phi(x) + \phi(y)$ (resp. $\phi(xy) \leq \phi(x)\phi(y)$) for all $x, y \in G$ and if there exists a positive real number $r = r(\phi)$ such that $U_r = \{x \in G : \phi(x) \leq r\}$ is a neighbourhood of the identity e in G. Note that if a function ϕ is subadditive, then $1 + \phi$ is submultiplicative. The following result (which is perhaps well known) follows from the same result about submultiplicative functions on G, (see Siebert [14], Theorem 1, and Siebert [15], Theorem 5).

Proposition 1 *Let G be a locally compact group with identity e and let $\mathcal{P}(G)$ be the convolution semigroup of probability measures on G. Let $\{\mu_t\}_{t\geq 0}$, $\mu_0 = \delta_e$, be a continuous one-parameter semigroup in $\mathcal{P}(G)$ with the Lévy measure η. Let $\mathcal{U}(e)$ denote the set of all neighbourhoods of the identity e in G. Let ϕ be a subadditive function and let $r = r(\phi) > 0$ and $U_r \in \mathcal{U}(e)$ be as above. Then the following are equivalent:*

(i) $\int_G \phi \, \mathrm{d}\mu_t < \infty$ *for some* $t > 0$.

(ii) $\sup_{0 \le s \le t} \int_G \phi \, \mathrm{d}\mu_s < \infty$ *for all* $t > 0$.

(iii) $\int_{G \setminus U_r} \phi \, \mathrm{d}\eta < \infty$.

Moreover, if ϕ *is continuous, then the above are equivalent to each of the following statements:*

(iv) $\int_{G \setminus U} \phi \, \mathrm{d}\eta < \infty$ *for all* $U \in \mathcal{U}(e)$.

(v) $\int_{G \setminus U} \phi \, \mathrm{d}\eta < \infty$ *for some compact neighbourhood* $U \in \mathcal{U}(e)$.

Proof of Theorem 1. As in the hypothesis, since μ is (τ, c)-semistable, there exists a (τ, c)-semistable one-parameter semigroup $\{\mu_t\}_{t \ge 0} \subset \mathcal{P}(V)$ with $\mu_1 = \mu$. Now recall that $C(\tau) = \{x \in V : \tau^n(x) \to 0\}$; it is a vector subspace of V. It is shown in Dani and Shah [2] that for all $t \ge 0$, $\mu_t = \mu_t^{(0)} * \omega_H$, $\mu_0 = \omega_H$, for some compact subgroup H such that $\tau(H) = H$, $\{\mu_t^{(0)}\}$ is a (τ, c)-semistable one-parameter semigroup supported on $C(\tau)$, $\mu_0^{(0)} = \delta_0$ and $H \cap C(\tau) = \{0\}$. Since any norm on V satisfies the triangular inequality, it is easy to see that μ has an absolute moment of order r if and only if $\mu_1^{(0)}$ does. Also, since μ, and hence $\mu_1^{(0)}$, is not an idempotent, $\mu_1^{(0)} \ne \delta_0$. Hence $C(\tau) \ne \{0\}$. Let τ_1 be the restriction of τ to $C(\tau)$. Since τ_1 is contracting on $C(\tau)$, it follows that $s(\tau_1^{-1}) > 1$. Now we are going to show that $\mu_1^{(0)}$ is full on $C(\tau)$ and $s(\tau^{-1}) = s(\tau_1^{-1})$.

Let V_H be the subspace of V generated by H. Since $\tau(H) = H$, $\tau(V_H) = V_H$. Moreover, since H is compact, if τ_2 is the restriction of τ to V_H, then it can easily be shown that for some $M > 0$, $\|\tau_2^n\| < M$ for all $n \in \mathbb{Z}$, and hence $s(\tau_2^{-1}) = 1 = s(\tau_2)$. Also, $V_H \cap C(\tau) = \{0\}$. Since $\mu = \mu_1^{(0)} * \omega_H$ is full on V, $V = C(\tau) \oplus V_H$, a direct product, and $\mu_1^{(0)}$ is full on $C(\tau)$. Now, we have $s(\tau^{-1}) = \max\{s(\tau_1^{-1}), s(\tau_2^{-1})\} = s(\tau_1^{-1})$. In particular, $s(\tau^{-1}) > 1$.

Now without loss of generality, we may assume that $\mu = \mu_1^{(0)}$, $V = C(\tau)$ and $\tau = \tau_1$, i.e. μ is embeddable in a (τ, c)-semistable $\{\mu_t\}_{t \ge 0}$ with $\mu_0 = \delta_0$ and τ is contracting on V. Let η be the Lévy measure of $\{\mu_t\}_{t \ge 0}$ on $V \setminus \{0\}$. Then we know that η is finite on $V \setminus U$ for any neighbourhood U of 0 in V and $\tau(\eta) = c\eta$. Since $\mu \ne \delta_0$ and V is totally disconnected, it is easy to show that η is not a zero measure (see also Heyer [4], Theorems 6.2.10 and 6.2.3).

Here, V is isomorphic to $\mathbb{Q}_p^d$, where $d = \dim V$. Since any two norms on V are equivalent, it is enough to consider the norm $\|\cdot\|_p$ defined above.

Since τ is contracting on V, by Lemma 3.3 in Siebert [16] there exist distinct open compact subgroups G_n, $n \in \mathbb{Z}$, $V = \cup_n G_n$, $\cap_n G_n = \{0\}$, and for all n, $G_n \subset G_{n+1}$ and $\tau(G_n) = G_{n-1}$.

We first prove the "if" part. Let $r > 0$ be fixed such that $s(\tau^{-1})^r c < 1$. Since the map $x \mapsto \|x\|_p^r$ is subadditive, by Proposition 1, it is enough to show that $\int_{V\setminus G_0} \|x\|_p^r \, \mathrm{d}\eta(x) < \infty$. Using the relation $\tau(\eta) = c\eta$, we have (for $\tau^0 = I$),

$$
\begin{aligned}
\int_{V\setminus G_0} \|x\|_p^r \, \mathrm{d}\eta(x) &= \sum_{n=0}^{\infty} \int_{G_{n+1}\setminus G_n} \|x\|_p^r \, \mathrm{d}\eta(x) \\
&= \sum_{n=0}^{\infty} \int_{G_1\setminus G_0} \|\tau^{-n}(x)\|_p^r c^n \, \mathrm{d}\eta(x) \\
&\leq \sum_{n=0}^{\infty} \int_{G_1\setminus G_0} \|\tau^{-n}\|_p^r \, \|x\|_p^r c^n \, \mathrm{d}\eta(x) \\
&\leq M \left(1 + \sum_{n=1}^{\infty} \{ (\|\tau^{-n}\|_p^{1/n})^r c \}^n \right),
\end{aligned}
$$

where $M = \sup\{\|x\|_p^r : x \in G_1\} \eta(G_1 \setminus G_0)$, which is finite. Let $a_n = (\|\tau^{-n}\|_p^{1/n})^r c$, $n \in \mathbb{N}$. Since $\|\tau^{-n}\|_p^{1/n} \to s(\tau^{-1})$, we get that $a_n \to s(\tau^{-1})^r c < 1$, and hence $\sum a_n^n$ converges. Therefore

$$
\int_{V\setminus G_0} \|x\|_p^r \, \mathrm{d}\eta(x) < \infty.
$$

Hence, μ has an absolute moment of order r if $s(\tau^{-1})^r c < 1$.

For the "only if" part, we need to show that $\int_V \|x\|_p^r \, \mathrm{d}\mu(x) = \infty$ if $s(\tau^{-1})^r c \geq 1$. If possible, suppose this integral is finite for some $r > 0$ satisfying $s(\tau^{-1})^r c \geq 1$. Now from above,

$$
\sum_{n=0}^{\infty} \int_{G_1\setminus G_0} \|\tau^{-n}(x)\|_p^r c^n \, \mathrm{d}\eta(x) = \int_{V\setminus G_0} \|x\|_p^r \, \mathrm{d}\eta(x) < \infty.
$$

Hence

$$
\int_{G_1\setminus G_0} \sum_{n=0}^{\infty} \|\tau^{-n}(x)\|_p^r c^n \, \mathrm{d}\eta(x) < \infty.
$$

Let η_0 be the restriction of η to $G_1 \setminus G_0$. Then the above implies that for η_0-almost all x,

$$\sum_{n=0}^{\infty} \|\tau^{-n}(x)\|_p^r c^n < \infty \text{ and hence } \|\tau^{-n}(x)\|_p^r c^n \to 0.$$

Let $t_n = \|\tau^{-n}\|_p$. By definition, $t_n \in \mathbb{Q} \setminus \{0\}$. Let $\psi_n = t_n \tau^{-n}$ for all n. Then $\psi_n \in GL(V)$ and $\|\psi_n\|_p = |t_n|_p \|\tau^{-n}\|_p = t_n^{-1} t_n = 1$ for all n. Also, since $s(\tau^{-1})^r c \geq 1$, $t_n^{-r} \leq c^n$. Now we get for η_0-almost all x,

$$\|\psi_n(x)\|_p^r = t_n^{-r} \|\tau^{-n}(x)\|_p^r \leq c^n \|\tau^{-n}(x)\|_p^r \to 0,$$

This implies that, for η_0-almost all x, $\|\psi_n(x)\|_p \to 0$ and hence $\psi_n(x) \to 0$. Let

$$V' = \{x \in V : \psi_n(x) \to 0\}.$$

Then V' is a vector subspace, hence it is closed and $\operatorname{supp} \eta_0 \subset V'$. Since $\tau \circ \psi_n = \psi_n \circ \tau$ for all n, for any $x \in V'$,

$$\psi_n(\tau(x)) = \tau(\psi_n(x)) \to 0.$$

Therefore, $\tau(V') \subset V'$ and hence $\tau(V') = V'$ as $\tau \in GL(V)$. Since $\eta = \sum_{n \in \mathbb{Z}} c^{-n} \tau^n(\eta_0)$, $\operatorname{supp} \eta \subset V' \setminus \{0\}$. Now we show that $V' = V$. If possible, suppose V' is proper. Let $\pi : V \to V/V'$ be the natural projection. Then $\{\pi(\mu_t)\}_{t \geq 0}$ is a continuous one-parameter semigroup with the Lévy measure $\pi(\eta)$ defined on $(V/V') \setminus \{\pi(0)\}$. From above, $\pi(\eta)$ is a zero measure. Hence, since V/V' is totally disconnected, it is easy to show that $\pi(\mu_t) = \delta_{\pi(0)}$, for all t, (see also Heyer [4], Theorems 6.2.10 and 6.2.3), and hence $\operatorname{supp}\mu_1 \subset V'$. But since $\mu_1 = \mu$ is full on V, we get a contradiction. Hence $V' = V$. That is, $\psi_n(x) \to 0$ for all $x \in V$. But $\|\psi_n\|_p = 1$ for all n, which is a contradiction. Thus, $\int_V \|x\|_p^r \, \mathrm{d}\mu(x)$ must be infinite if $s(\tau^{-1})^r c \geq 1$. This completes the proof. □

We now state two simple lemmas about operator-semistable probability measures which will be used in the next section.

Lemma 1 *Let V be a finite dimensional p-adic vector space. Let $\tau \in GL(V)$ and $c \in \,]0,1[$.*

(i) If $\mu \in \mathcal{P}(V)$ is (τ, c)-semistable, then for $k_n = [c^{-n}]$, $\tau^n(\mu)^{k_n} \to \mu$.

(ii) *If* $\tau^n(\nu)^{l_n} \to \mu \in \mathcal{P}(V)$ *for some* $\nu \in \mathcal{P}(V)$ *and* $\{l_n\} \subset \mathbb{N}$ *such that* $\tau^n(\nu) \to \delta_0$ *and* $l_n/l_{n+1} \to c$*, then* μ *is* (τ, c)*-semistable.*

Proof. (i) Let $\mu \in \mathcal{P}(V)$ be (τ, c)-semistable. Let $\{\mu_t\}_{t\geq 0}$ be a continuous one-parameter semigroup with $\mu_1 = \mu$ and $\tau(\mu_t) = \mu_{ct}$, for all t. Then for $k_n = [c^{-n}]$, $\tau^n(\mu)^{k_n} = \mu_{c^n k_n} \to \mu_1 = \mu$.

(ii) Let ν, $\{l_n\}$, τ, c be as above. Then the set $T = \{\tau^n(\nu)^m : m \leq l_n,\, n \in \mathbb{N}\}$ is relatively compact (see Shah [13], Theorem 2.1 and Remark following it). Also, since $l_n/l_{n+1} \to c$, we can prove the assertion along the proof of Theorem 4.6 of Telöken [17] using Theorem 2.3 of Telöken [17]. In Telöken [17], the fullness of μ is not assumed. However, it is not necessary here to assume that μ is full, since, for $\tau_n = \tau^n$, we have $\tau_{n+1}\tau_n^{-1} = \tau$ for all n. □

Lemma 2 *Let* V *be a finite dimensional p-adic vector space. Let* $\tau \in GL(V)$ *and* $c \in]0, 1[$ *. Let* $\{\psi_m\} \subset K$*, a compact subgroup of* $GL(V)$*, be such that* ψ_m *commutes with* τ *for each* m*. Let* $\{\nu_m\} \subset \mathcal{P}(V)$ *be such that* $\nu_m \to \nu \in \mathcal{P}(V)$*. Suppose each* ν_m *is embeddable in a* $(\psi_m\tau, c)$*-semistable one-parameter semigroup* $\{\mu_t^{(m)}\}_{t\geq 0}$ *as* $\mu_1^{(m)} = \nu_m$*, such that* $\mu_0^{(m)} = \delta_0$*. Then for* $k_n = [c^{-n}]$*,* $\psi^n\tau^n(\nu)^{k_n} * y_n \to \nu$*, for some* $\psi \in K$ *and some sequence* $\{y_n\}$ *in* V*.*

Proof. We may assume, without loss of generality, that $\{\psi_m\}$ itself converges. Let ψ be the limit point of it, then $\psi \in K$ and ψ commutes with τ. Recall that $C(\tau) = \{x \in V : \tau^n(x) \to 0\}$, which is a vector subspace. Moreover, if $\rho \in K$ commutes with τ, then ρ keeps $C(\tau)$ invariant and $C(\rho\tau) = C(\tau)$ (cf. Wang [18], Proposition 2.1). For any $m \in \mathbb{N}$, since $(\psi_m\tau)^n(\nu_m) = \mu_{c^n}^{(m)} \to \delta_0$, we have that each ν_m, and hence ν, is supported on $C(\tau)$. Therefore, without loss of generality, we may assume that $V = C(\tau)$, that is, τ is contracting on V.

Let $n \in \mathbb{N}$ be fixed. Let $s_n = 1 - k_n c^n$. Here, for all m,

$$\nu_m = \mu_1^{(m)} = \mu_{k_n c^n}^{(m)} * \mu_{s_n}^{(m)} = \psi_m^n \tau^n(\nu_m)^{k_n} * \mu_{s_n}^{(m)}.$$

Also, $\mu_{k_n c^n}^{(m)} = \psi_m^n\tau^n(\nu_m)^{k_n} \to \psi^n\tau^n(\nu)^{k_n}$. As $\nu_m \to \nu$, the above implies that $\{\mu_{s_n}^{(m)}\}_{m\in\mathbb{N}}$ is relatively compact (cf. Parthasarathy [9], Chapter III, Theorem 2.1). Let α_n be a limit point of it. Since $s_n < c^n$, $\mu_{s_n}^{(m)}$ is a factor of $\mu_{c^n}^{(m)}$, and since $\mu_{c^n}^{(m)} = \psi_m^n\tau^n(\nu_m) \to \psi^n\tau^n(\nu)$, α_n is a factor of $\psi^n\tau^n(\nu)$.

Here, since $\psi \in K$ and it commutes with τ, $C(\psi\tau) = C(\tau) = V$, and hence $\psi^n\tau^n(\nu) \to \delta_0$. Therefore, it follows that there exists a sequence $\{x_n\} \subset V$ such that $\alpha_n * \delta_{x_n} \to \delta_0$ (cf. Shah [13], Lemma 2.3). Now from the above equation, we have that $\nu = \psi^n\tau^n(\nu)^{k_n} * \alpha_n$, for all n, and hence for $y_n = x_n^{-1}$, $\psi^n\tau^n(\nu)^{k_n} * \delta_{y_n} \to \nu$. This completes the proof. □

3 Semistable probability measures on p-adic vector spaces and their projections

In this section, we compare semistability of a probability measure on a p-adic vector space V with that of all its one-dimensional projections. Clearly, if a probability measure μ on V is semistable then all its projections are also semistable. Conversely, we are interested in finding out conditions under which semistabilty of μ is implied by that of all its one-dimensional projections.

For V isomorphic to $\mathbb{Q}_p^d$ and for $x, y \in V$, $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$, let $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$. It is a continuous bi-linear map from V^2 to $\mathbb{Q}_p$. Any one-dimensional projection of V is of the form $y \mapsto \langle x, y\rangle$ for some $x \in V$. For $x \in V \setminus \{0\}$, let $V_x = \mathrm{Ker}(y \mapsto \langle x, y\rangle)$; it is a subspace of co-dimension 1 in V.

For $\mu \in \mathcal{P}(V)$ and $x \in V$, let $\mu_x = (x, \mu)$ denote the image of μ under the map $y \mapsto \langle x, y\rangle$. We can see by injectivity of Fourier transform that $\mu = \nu$ if and only if $(x, \mu) = (x, \nu)$ for all $x \in V$. Moreover, it follows from Lévy's continuity theorem that $\mu_n \to \mu$ in $\mathcal{P}(V)$ if and only if $(x, \mu_n) \to (x, \mu)$, for all $x \in V$.

In the following, we consider semistable probability measures on V. Here, we identify $a \in \mathbb{Q}_p$ with the map $x \mapsto ax$. Recall that a probability measure μ on V is semistable if it is (a, c)-semistable for some $a \in \mathbb{Q}_p \setminus \{0\}$, $|a|_p < 1$, and some $c \in]0, 1[$, that is, μ is embeddable in a continuous one-parameter semigroup $\{\mu_t\}_{t\geq 0} \subset \mathcal{P}(V)$ as $\mu = \mu_1$ and $a(\mu_t) = \mu_{ct}$, for all $t \geq 0$. This automatically implies that $C(a) = V$ and $\mu_0 = \delta_0$. Now for a probability measure μ on V, we denote $\Gamma(\mu) = \{c \in]0, \infty[: \mu$ is embeddable in $\{\mu_t\}_{t\geq 0} \subset \mathcal{P}(V)$ such that $a(\mu_t) = \mu_{ct}, t \geq 0$, for some $a \in \mathbb{Q}_p \setminus \{0\}\}$. Clearly, $\Gamma(\mu)$ is always nonempty as $1 \in \Gamma(\mu)$ and if μ is (a, c)-semistable for some $a \in \mathbb{Q}_p \setminus \{0\}$, then $c, c^{-1} \in \Gamma(\mu)$. Let $\mathbb{Z}_p$ denote the ring of p-adic integers, namely,

$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$. Let $\mathbb{Z}_p(1) = \{x \in \mathbb{Z}_p : |x|_p = 1\}$. It is the maximal compact subgroup of the multiplicative group $\mathbb{Q}_p \setminus \{0\}$. For $\mu \in \mathcal{P}(V)$, let $\mathrm{Inv}(\mu) = \{\tau \in GL(V) : \tau(\mu) = \mu\}$. It is a closed subgroup of $GL(V)$.

The following theorem is a generalization (to p-adic vector spaces) of Theorem 1 in Maejima and Samorodnitsky [8] which is for real vector spaces. For semistable probability measures on a real vector space $\mathbb{R}^d$, see Maejima [7] and references cited therein.

Theorem 2 *Let V be a finite dimensional p-adic vector space and let $\mu \in \mathcal{P}(V)$. Suppose that for all one-dimensional projections π of V, $\pi(\mu)$ is semistable and $\Gamma = \cap_\pi \Gamma(\pi(\mu)) \neq \{1\}$. Then there exist unbounded sequences $\{a_n\}$ and $\{d_n\}$ in $\mathbb{N}$ such that $p^{a_n}(\mu)^{d_n} \to \mu$; in particular μ is embeddable. Moreover, if $\mathrm{Inv}(\mu) \cap \mathbb{Z}_p(1) = \{z \in \mathbb{Z}_p(1) : z(\mu) = \mu\}$ is an open subgroup of $\mathbb{Z}_p(1)$, then μ is semistable.*

Remark: 1. Given any subgroup K of $\mathbb{Z}_p(1)$, there exist a lot of semistable probability measures on V which are K-invariant. In the proof of Theorem 4.2 in Dani and Shah [2], for any contracting $\tau \in GL(V)$, (we may choose $\tau = aI$, $a \in \mathbb{Q}_p$, $0 < |a|_p < 1$), and any $c \in]0, 1[$, one can construct a Lévy measure λ such that $\tau(\lambda) = c\lambda$ and λ is K-invariant; for this one has to take a K-invariant subgroup H_0 and a K-invariant measure ρ_0 on $H_0 \setminus H_1$, which will imply that the corresponding one-parameter semigroup $\{\mu_t\}_{t \geq 0}$ is such that it is (τ, c)-semistable and each μ_t is K-invariant (see also Yasuda [19]).

2. Any closed infinite subgroup of $\mathbb{Z}_p(1)$ is open, so the additional condition in the above theorem leaves out only those probability measures which have finite invariance subgroups in $\mathbb{Z}_p(1)$.

Proof of Theorem 2. Step 1. We may assume that $\mu \neq \delta_0$. Let W be the subspace generated by $\mathrm{supp}\,\mu$. If $\dim W = 1$, then the statement is trivial. So we assume that $\dim W \geq 2$. Since scaler automorphisms keep any subspace invariant, for all one-dimensional projections π of W, $\pi(\mu)$ is semistable and $\cap_\pi \Gamma(\pi(\mu)) \neq \{1\}$. In view of the above arguments, we may assume that $V = W$. That is, it is enough to prove the theorem for a full probability measure μ on V.

From the hypothesis, $\mu_x = (x, \mu)$ is semistable for all $x \in V$ and $\Gamma = \cap_x \Gamma(\mu_x) \neq \{1\}$. Choose $c \in \Gamma$ such that $c \in]0, 1[$. Then for each $x \in V \setminus \{0\}$, μ_x is (a_x, c)-semistable for some $a_x \in \mathbb{Q}_p \setminus \{0\}$, $|a_x|_p < 1$.

Step 2. Suppose for any $x \in V \setminus \{0\}$, μ_x is also (d_x, c)-semistable for some $d_x \in \mathbb{Q}_p \setminus \{0\}$. We show that $|a_x|_p = |d_x|_p$. This can be shown by using Theorem 1, but we give a direct proof here. Let $a_x = b_x p^{m(x)}$, and $d_x = s_x p^{k(x)}$, where $b_x, s_x \in \mathbb{Z}_p(1)$ and $m(x), k(x) \in \mathbb{N} \setminus \{1\}$. Let $k_n = [c^{-n}]$, $n \in \mathbb{N}$. Then by Lemma 1 (i), $a_x^n (\mu_x)^{k_n} \to \mu_x$ and $d_x^n (\mu_x)^{k_n} \to \mu_x$. If possible, suppose $m(x) \neq k(x)$. Without loss of generality, we may assume that $m(x) > k(x)$. Then for $i(x) = m(x) - k(x) \in \mathbb{N}$,

$$\mu_x = \lim_n a_x^n (\mu_x)^{k_n} = \lim_n (a_x d_x^{-1})^n d_x^n (\mu_x)^{k_n} = \lim_n b_x^n s_x^{-n} p^{i(x)n} d_x^n (\mu_x)^{k_n}.$$

Since $\{b_x^n s_x^{-n}\} \subset \mathbb{Z}_p(1)$, which is compact, $p^{i(x)n} \to 0$ in $M(V)$ and since $d_x^n (\mu_x)^{k_n} \to \mu_x$, we have $\mu_x = \delta_0$, and hence $\operatorname{supp} \mu \subset V_x$, which is proper as $x \neq 0$. This is a contradiction as μ is full on V. Therefore $m(x) = k(x)$ and hence $|a_x|_p = p^{-m(x)} = |d_x|_p$.

Step 3. Let $V(1) = \{x \in V : \|x\|_p = 1\}$. Then $V(1)$ is compact. We now define a map $m : V(1) \mapsto \mathbb{N}$ as follows: If μ_x is (a_x, c)-semistable, then $m(x) = -\log_p(|a_x|_p)$, i.e. $a_x = b_x p^{m(x)}$, where $|b_x|_p = 1$. From the above arguments, the map m is well-defined.

For each x, we fix a_x and an (a_x, c)-semistable one-parameter semigroup $\{(x, \mu)_t\}_{t \geq 0}$ with $(x, \mu)_1 = (x, \mu)$.

We now show that the image of m, $F = \{m(x) : x \in V(1)\}$, is a finite subset in $\mathbb{N}$. If possible, suppose that for some sequence $\{x_l\} \subset V(1)$, $m(x_l) \to \infty$. Then $a_{x_l} \to 0$ in $\mathbb{Q}_p$. This implies that $a_{x_l}(x_l, \mu) = (x_l, \mu)_c \to \delta_0$. But (x_l, μ) is a factor of $[(x_l, \mu)_c]^{n_0}$ for some fixed $n_0 \in \mathbb{N}$ with $n_0 c > 1$. From above, $[(x_l, \mu)_c]^{n_0} \to \delta_0$. But since $x_l \in V(1)$, $\{x_l\}$ is relatively compact, and for any limit point x of it, $x \in V(1) \subset V \setminus \{0\}$ and $\mu_x = (x, \mu)$ is a factor of δ_0. Therefore, $\mu_x = \delta_g$ for some $g \in \mathbb{Q}_p$ and hence $\mu_x = \delta_0$ as μ_x is semistable. Now $\operatorname{supp} \mu \subset V_x$, a proper subspace, hence μ is not full, which is a contradiction. This implies that F is finite.

Step 4. We next show that the map m from $V(1)$ to $F \subset \mathbb{N}$ is continuous. Let $\{x_l\} \subset V(1)$, $x_l \to x$ in $V(1)$. Since F is finite, we may assume that $m(x_l) = i_0$ for all l. We have to show that $m(x) = i_0$. Here, (x_l, μ) is (a_{x_l}, c)-semistable. Then by Lemma 2, there exist $b \in \mathbb{Z}_p(1)$ and a sequence $\{y_n\} \subset \mathbb{Q}_p$ such that $b^n p^{i_0 n} (x, \mu)^{k_n} * \delta_{y_n} \to (x, \mu)$. Since (x, μ) is (a_x, c)-semistable, we have that $a_x^n (x, \mu)^{k_n} \to (x, \mu)$. Let $a_x = b_x p^{m(x)}$ as above. If possible, suppose $m(x) < i_0$. Then

$$\lim_n (x,\mu) * \delta_{y_n^{-1}} = \lim_n b^n p^{i_0 n}(x,\mu)^{k_n} = \lim_n (bb_x^{-1})^n p^{(i_0-m(x))n} a_x^n (x,\mu)^{k_n} = \delta_0,$$

since $\{(bb_x^{-1})^n\} \subset \mathbb{Z}_p(1)$, which is compact, $p^{(i_0-m(x))n} \to 0$ in $M(V)$ and since $a_x^n(x,\mu)^{k_n} \to (x,\mu)$. This implies that $\{y_n^{-1}\}$ is relatively compact and $(x,\mu) = \delta_y$ for some limit point y of $\{y_n\}$ in $\mathbb{Q}_p$. Hence $(x,\mu) = \delta_0$ as it is semistable, Therefore, we have $\operatorname{supp}\mu \subset V_x$, a proper subspace, and this is a contradiction. Therefore, $m(x) \geq i_0$.

If possible suppose $m(x) > i_0$. Then for $z_n = (b_x b^{-1})^n p^{(m(x)-i_0)n}(y_n^{-1})$,

$$(x,\mu) = \lim_n a_x^n (x,\mu)^{k_n} = \lim_n (b_x b^{-1})^n p^{(m(x)-i_0)n}[b^n p^{i_0 n}(x,\mu)^{k_n} * \delta_{y_n}] * \delta_{z_n}.$$

Therefore, arguing as earlier, we get that $(x,\mu) * \delta_{z_n^{-1}} \to \delta_0$. Now we get a contradiction as above. Hence $m(x) = i_0$. That is, the map m from $V(1)$ to F is continuous.

Step 5. Now we show that F consists of only one natural number. If possible, suppose there exist $g_1, g_2 \in V(1)$ such that $m(g_1) \neq m(g_2)$. We may also assume that $m(g_1) > m(g_2)$. Here, $p^{m(g_i)} b_{g_i}(g_i,\mu) = (g_i,\mu)_c$, for $i = 1,2$. Let $\{\rho_n\} \subset \mathbb{Q}_p \setminus \{0\}$ be such that $|\rho_n|_p \to 0$. Let $h_n = \rho_n g_1 + g_2, n \in \mathbb{N}$. Here, $h_n \to g_2$ and $\|g_2\|_p = 1$, we get that, for all large n, $\|h_n\|_p = 1$. Without loss of generality, we may assume that $\|h_n\|_p = 1$ for all n. We know that since μ is full on V, for any $x \in V$, (x,μ) is full on the image space $\{\langle x, y\rangle : y \in V\}$; moreover, since (x,μ) is semistable, it is not an idempotent. Now for $i = 1,2$, let $r_i = -\log c / m(g_i) \log p$. By Theorem 1,

$$\int_{\mathbb{Q}_p} |x|_p^s \,\mathrm{d}(g_i,\mu)(x) = \int_V |\langle g_i, y\rangle|_p^s \,\mathrm{d}\mu(y) < \infty \text{ if and only if } s < r_i.$$

Now for a fixed $s \in [r_1, r_2[$ and a fixed n, using the subadditivity of the function $x \mapsto |x|_p^s$, we get that

$$\begin{aligned}
\int_{\mathbb{Q}_p} |x|_p^s \,\mathrm{d}(h_n,\mu)(x) &= \int_V |\langle h_n, y\rangle|_p^s \,\mathrm{d}\mu(y) \\
&\geq |\rho_n|_p^s \int_V |\langle g_1, y\rangle|_p^s \,\mathrm{d}\mu(y) - \int_V |\langle g_2, y\rangle|_p^s \,\mathrm{d}\mu(y) \\
&= |\rho_n|_p^s \int_{\mathbb{Q}_p} |x|_p^s \,\mathrm{d}(g_1,\mu)(x) - \int_{\mathbb{Q}_p} |x|_p^s \,\mathrm{d}(g_2,\mu)(x) \\
&= \infty,
\end{aligned}$$

as the first integral above with respect to the measure (g_1, μ) is infinite and the second one with respect to (g_2, μ) is finite for $s \in [r_1, r_2[$. Since $h_n \to g_2$ and $h_n \in V(1)$, by continuity of the map m, we get that, for all large n, $m(h_n) = m(g_2)$, i.e. (h_n, μ) is $(b_n p^{m(g_2)}, c)$-semistable and $|b_n|_p = 1$. Hence, from Theorem 1 and above equations, we get that $s \geq -\log c / m(g_2) \log p = r_2$. But $s \in [r_1, r_2[$, which is a contradiction. Therefore, our assumption that $m(g_1) \neq m(g_2)$ is wrong. Hence there exists a unique $u_0 \in \mathbb{N}$ such that $F = \{u_0\}$ and μ_x is (a_x, c)-semistable, where $a_x = b_x p^{u_0}$, $b_x \in \mathbb{Z}_p(1)$, for all $x \in V(1)$. That is, $b_x p^{u_0}(x, \mu)_t = (x, \mu)_{ct}$, for all $x \in V(1)$ and all $t > 0$.

Step 6. Let $m_1 = p - 1$ and let $m_n = m_{n-1}^{p-1}$ for $n \geq 2$. Since $b_x \in \mathbb{Z}_p(1)$, we know that $b_x^{m_n} \to 1$ in $\mathbb{Z}_p(1)$. Let $a_n = u_0 m_n$ and let $d_n = [c^{-m_n}]$. Clearly, $a_n \to \infty$, $d_n \to \infty$ and $c^{m_n} d_n \to 1$. Then for all $x \in V(1)$,

$$\begin{aligned}
\lim_{n\to\infty} (x, p^{a_n}(\mu)^{d_n}) &= \lim_{n\to\infty} p^{a_n}(x, \mu^{d_n}) \\
&= \lim_{n\to\infty} b_x^{m_n} p^{u_0 m_n}(x, \mu^{d_n}) \\
&= \lim_{n\to\infty} (x, \mu)_{c^{m_n} d_n} \\
&= (x, \mu).
\end{aligned}$$

Since for $x \in V \setminus \{0\}, (x, \mu) = \|x\|_p^{-1}(x', \mu), x' \in V(1)$, the above equation also holds for all $x \in V \setminus \{0\}$. Hence $p^{a_n}(\mu)^{d_n} \to \mu$. By Theorem 1.5 of Shah [12], μ is embeddable.

Step 7. We suppose $K = \text{Inv}(\mu) \cap \mathbb{Z}_p(1)$ is open in $\mathbb{Z}_p(1)$. Then $\mathbb{Z}_p(1)/K$ is a finite group and $K \subset \text{Inv}(\mu_x)$ for all x. Let $q \in \mathbb{N}$ be the order of $\mathbb{Z}_p(1)/K$. Then $b_x^q \in K$ for all $x \in V(1)$. This implies that $p^{u_0 q}(x, \mu) = (x, \mu)_{c^q}$. That is, for $j_0 = u_0 q$ and $r = c^q$, we get that $p^{j_0}(x, \mu) = (x, \mu)_r$, $x \in V(1)$. Now for each n, let $l_n = [r^{-n}]$. Since $0 < r < 1$, $r^n \to 0$, $l_n \to \infty$ and $l_n r^n \to 1$. Then we have that for $x \in V(1)$,

$$\lim_{n\to\infty} (x, p^{j_0 n}(\mu)^{l_n}) = \lim_{n\to\infty} p^{j_0 n}(x, \mu)^{l_n} = \lim_{n\to\infty} (x, \mu)_{l_n r^n} = (x, \mu).$$

Again, since for $x \in V \setminus \{0\}, (x, \mu) = \|x\|_p^{-1}(x', \mu), x' \in V(1)$, the above equation also holds for all $x \in V \setminus \{0\}$. Hence $p^{j_0 n}(\mu)^{l_n} \to \mu$. Since $p^{j_0 n}(\mu) \to \delta_0$, $p^{j_0(n+1)}/p^{j_0 n} = p^{j_0}$ is contracting on V and $l_n/l_{n+1} \to r$, by Lemma 1 (ii), μ is (p^{j_0}, r)-semistable, i.e. μ is semistable. This completes the proof. $\square$

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