# Examples of $\alpha$-selfdecomposable distributions

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**Abstract**

We show that the logarithms of gamma random variables are $\alpha$-selfdecomposable, and apply the result to show that logarithms of several positive random variables are 1-selfdecomposable.

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## 1. Introduction

Let $I(\mathbb{R}^d)$ be the class of all infinitely divisible distributions on $\mathbb{R}^d$ and $\widehat{\mu}(z), z \in \mathbb{R}^d$, the characteristic function of $\mu \in I(\mathbb{R}^d)$. The Lévy-Khintchine representation of $\widehat{\mu}$ we use in this paper is

$$\widehat{\mu}(z) = \exp\left\{-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z, x\rangle}{1+|x|^2}\right)\nu(dx)\right\},$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$, $\langle\cdot,\cdot\rangle$ is the Euclidean inner product on $\mathbb{R}^d$, $A$ is a nonnegative-definite symmetric $d \times d$ matrix, $\gamma \in \mathbb{R}^d$ and $\nu$ is the Lévy measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$.

Let $S = \{x \in \mathbb{R}^d : |x| = 1\}$. The polar decomposition of the Lévy measure $\nu$ of $\mu \in I(\mathbb{R}^d)$, with $0 < \nu(\mathbb{R}^d) \le \infty$, is the following: There exist a measure $\lambda$ on $S$ with $0 < \lambda(S) \le \infty$ and a family $\{\nu_\xi, \xi \in S\}$ of measures on $(0, \infty)$

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such that $\nu_\xi(B)$ is measurable in $\xi$ for each $B \in \mathcal{B}((0,\infty))$, $0 < \nu_\xi((0,\infty)) \leq \infty$ for each $\xi \in S$ and

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty \mathbb{1}_B(r\xi)\nu_\xi(dr), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).$$

Here $\lambda$ and $\{\nu_\xi\}$ are uniquely determined by $\nu$ up to multiplication of measurable functions $c(\xi)$ and $c(\xi)^{-1}$ with $0 < c(\xi) < \infty$. We say that $\nu$ has the polar decomposition $(\lambda, \nu_\xi)$ and $\nu_\xi$ is called the radial component of $\nu$. (See, e.g., Barndorff-Nielsen et al. (2006), Lemma 2.1.)

The class of selfdecomposable distributions is one of the most important subclasses of $I(\mathbb{R}^d)$, and the following are extensions of selfdecomposability.

**Definition 1.1.** Let $\alpha \in \mathbb{R}$. We say that $\mu \in I(\mathbb{R}^d)$ is *$\alpha$-selfdecomposable* if for any $b > 1$, there exists $\rho_b \in I(\mathbb{R}^d)$ satisfying

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)^{b^\alpha}\widehat{\rho}_b(z), \quad z \in \mathbb{R}^d.$$

We denote the totality of $\alpha$-selfdecomposable distributions on $\mathbb{R}^d$ by $L^{\langle\alpha\rangle}(\mathbb{R}^d)$.

The class of selfdecomposable distributions is $L^{\langle 0\rangle}(\mathbb{R}^d)$. The classes $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ have already been studied by Alf and O'Connor (1977), O'Connor (1979a,b, 1981), Jurek (1988, 1989, 1992), Jurek and Schreiber (1992). Several properties of the classes $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ are summarized in Maejima and Ueda (2010a).

**Proposition 1.2.** (i) *If $\beta < \alpha$, then $L^{\langle\beta\rangle}(\mathbb{R}^d) \supset L^{\langle\alpha\rangle}(\mathbb{R}^d)$.*

(ii) *For $\alpha > 2$, $L^{\langle\alpha\rangle}(\mathbb{R}^d) = \{\delta_\gamma : \gamma \in \mathbb{R}^d\}$.*

(iii) *$L^{\langle 2\rangle}(\mathbb{R}^d) = \{$all Gaussian distributions$\}$.*

(iv) *$L^{\langle\alpha\rangle}(\mathbb{R}^d)$ is left-continuous in $\alpha \in \mathbb{R}$, namely,*

$$\bigcap_{\beta<\alpha} L^{\langle\beta\rangle}(\mathbb{R}^d) = L^{\langle\alpha\rangle}(\mathbb{R}^d) \quad \text{for all } \alpha \in \mathbb{R}.$$

(v) *Let $\alpha \in (-\infty, 2)$. Let $\mu \in I(\mathbb{R}^d)$ and let $\nu_\xi$ be the radial component of its Lévy measure. Then $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ if and only if*

$$\nu_\xi(dr) = r^{-\alpha-1}\ell_\xi(r)dr, \quad r > 0,$$

*for some $\ell_\xi(r)$, where $\ell_\xi(r)$ is a nonnegative-valued function which is measurable in $\xi$, and nonincreasing and right-continuous in $r$.*

(i) follows directly from Definition 1.1. (ii) is Corollary 1.1(f) of Jurek (1988) and (iii) is Proposition 1.1 of Jurek (1989). (iv) is Proposition 3.4 of Maejima and Ueda (2010a). (v) for $\alpha \in [-1, 2)$ and $d = 1$ is found in Alf and O'Connor (1977) and O'Connor (1979a,b, 1981).

As to concrete examples of $\alpha$-selfdecomposable distributions, except the case of $\alpha = 0$ (selfdecomposable distributions), only few are known. In Proposition 3.3 of Maejima and Ueda (2010a), we showed the following.

**Example 1.3.** Let $\alpha \in (0, 2]$ and let $S_\alpha$ be a non-constant $\alpha$-stable random variable. Then

$$\mathcal{L}(S_\alpha) \begin{cases} \in L^{\langle\alpha\rangle}(\mathbb{R}^d), \\ \notin L^{\langle\beta\rangle}(\mathbb{R}^d), \quad \beta > \alpha. \end{cases}$$

This shows that, in comparison to Proposition 1.2 (iv), $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ is not right-continuous at $\alpha \in (0, 2]$, namely,

$$L^{\langle\alpha\rangle}(\mathbb{R}^d) \supsetneqq \bigcup_{\beta>\alpha} L^{\langle\beta\rangle}(\mathbb{R}^d).$$

One more example in $\mathbb{R}^2$ is found in Maejima and Ueda (2010b).

**Example 1.4.** Let $(Z_1, Z_2)$ be a bivariate Gaussian random variable, where $Z_1$ and $Z_2$ are standard Gaussian random variables. Define a bivariate gamma random variable by $X := (Z_1^2, Z_2^2)$. Suppose that the correlation of $Z_1$ and $Z_2$ is in $(-1, 0) \cup (0, 1)$. Then

$$\mathcal{L}(X) \begin{cases} \in L^{\langle-2\rangle}(\mathbb{R}^2), \\ \notin L^{\langle\alpha\rangle}(\mathbb{R}^2), \quad \alpha > -2. \end{cases}$$

This is an example showing that $L^{\langle\alpha\rangle}(\mathbb{R}^2)$ is not right-continuous at $\alpha = -2$, namely,

$$L^{\langle-2\rangle}(\mathbb{R}^2) \supsetneqq \bigcup_{\beta>-2} L^{\langle\beta\rangle}(\mathbb{R}^2).$$

In this paper, we give new examples of $\alpha$-selfdecomposable random variables for $\alpha \in (0, 1]$ when $d = 1$. They are logarithms of: gamma random variable, absolute value of standard Gaussian random variable, absolute value of Student's $t$-random variable, $F$-random variable, positive strictly stable random variable and absolute value of non-zero symmetric stable random variable.

## 2. New Examples

Let $\Gamma_{c,\lambda}$ be a gamma random variable with parameters $c > 0, \lambda > 0$. Its density function $f(x)$ is

$$f(x) = \frac{\lambda^c}{\Gamma(\lambda)} x^{c-1} e^{-\lambda x}, \quad x > 0.$$

(When $c = 1$, is it an exponential random variable.) Note that in the case $d = 1$, $S = \{-1, 1\}$. The radial component of the Lévy measure of $\Gamma_{c,\lambda}$ is $\nu_{-1} = 0$ and

$$\nu_1(dr) = \frac{e^{-\lambda r}}{r} dr, \quad r > 0.$$

Comparing this with Proposition 1.2 (v), we have the following.

**Remark 2.1.**

$$\mathcal{L}(\Gamma_{c,\lambda}) \begin{cases} \in L^{\langle 0 \rangle}(\mathbb{R}), \\ \notin L^{\langle \alpha \rangle}(\mathbb{R}), \quad \alpha > 0. \end{cases}$$

Thus, $L^{\langle \alpha \rangle}(\mathbb{R})$ is also not right-continuous at $\alpha = 0$.

In this paper, we consider $\log \Gamma_{c,\lambda}$. It is known (Theorem 2 of Shanbhag and Sreehari (1977)) that $\mathcal{L}(\log \Gamma_{c,\lambda}) \in L^{\langle 0 \rangle}(\mathbb{R})$. To state our main result, we need the following. Let

$$h(\alpha, r) := \frac{\alpha}{r} - \frac{e^{-r}}{1 - e^{-r}}, \quad r > 0,$$

$$k(r) := \frac{r^2 e^{-r}}{(1 - e^{-r})^2}, \quad r > 0.$$

As we will show in the next section, this function $k$ is a strictly deceasing function on $(0, \infty)$ with $k(0+) = 1$ and $k(\infty) = 0$. Hence $0 < k(r) < 1$ for $r > 0$. Thus for $0 < \alpha < 1$, there is a unique $r_\alpha \in (0, \infty)$ satisfying $k(r_\alpha) = \alpha$. Let

$$A_1 = (-\infty, 0] \times (0, \infty),$$

$$A_2 = \{(\alpha, c) \in (0, 1) \times (0, \infty) \colon c \geq h(\alpha, r_\alpha)\}$$

and

$$A_3 = \{1\} \times [1/2, \infty).$$

Also as we will show in the next section, the behavior of $(0,1) \ni \alpha \mapsto h(\alpha, r_\alpha)$ is the following: $(0,1) \ni \alpha \mapsto h(\alpha, r_\alpha)$ is nondecreasing and continuous. As $\alpha \downarrow 0$, $h(\alpha, r_\alpha) \to 0$ and as $\alpha \uparrow 1$, $h(\alpha, r_\alpha) \to 1/2$.

Our main theorem in this paper is the following.

**Theorem 2.2.**

$$\mathcal{L}(\log \Gamma_{c,\lambda}) \begin{cases} \in L^{\langle \alpha \rangle}(\mathbb{R}), & \textit{if } (\alpha, c) \in A_1 \cup A_2 \cup A_3, \\ \notin L^{\langle \alpha \rangle}(\mathbb{R}) & \textit{if } (\alpha, c) \notin A_1 \cup A_2 \cup A_3. \end{cases}$$

This also shows that $L^{\langle \alpha \rangle}(\mathbb{R})$ is not right-continuous at $\alpha \in (0,1]$.

**Corollary 2.3.** (i) *Let $E$ be an exponential random variable.*

(ii) *Let $Z$ be a one-dimensional standard Gaussian random variable.*

(iii) *Let $T$ be a Student's t-random variable.*

(iv) *Let $F$ be an $F$-random variable.*

(v) *Let $S^+_{1/2}$ be a positive strictly $(1/2)$-stable random variable.*

*Then, for $X = E, |Z|, |T|, F, S^+_{1/2}$, we have*

$$\mathcal{L}(\log X) \begin{cases} \in L^{\langle 1 \rangle}(\mathbb{R}), \\ \notin L^{\langle \alpha \rangle}(\mathbb{R}), & \alpha > 1. \end{cases}$$

Corollary 2.3 (v) can be generalized as follows.

**Proposition 2.4.** (i) *Let $\beta \in (0,1)$ and let $S^+_\beta$ be a positive strictly $\beta$-stable random variable.*

(ii) *Let $\beta \in (0,2)$ and let $S^{\mathrm{sym}}_\beta$ be a one-dimensional non-zero symmetric $\beta$-stable random variable.*

*Then, for $X = S^+_\beta, |S^{\mathrm{sym}}_\beta|$, we have*

$$\mathcal{L}(\log X) \begin{cases} \in L^{\langle 1 \rangle}(\mathbb{R}), \\ \notin L^{\langle \alpha \rangle}(\mathbb{R}), & \alpha > 1. \end{cases}$$

The proofs of the theorem, the corollary and the proposition will be given in the next section.

## 3. Proofs

We first prove the main theorem.

*Proof of Theorem 2.2.* Recall that $\mathcal{L}(\log \Gamma_{c,\lambda}) \in L^{\langle 0 \rangle}(\mathbb{R})$. Thus when $\alpha \le 0$, $\mathcal{L}(\log \Gamma_{c,\lambda}) \in L^{\langle \alpha \rangle}(\mathbb{R})$ for any $c > 0$ by Proposition 1.2 (i).

Let $\alpha > 0$. It is known that $\nu_\xi$ of $\log \Gamma_{c,\lambda}$ is $\nu_1 = 0$ and

$$\nu_{-1}(dr) = \frac{e^{-cr}}{r(1-e^{-r})} dr,$$

(see, e.g., Linnik and Ostrovskii (1977), Eq. (2.6.13)). This does not depend on the parameter $\lambda > 0$. Then

$$\nu_{-1}(dr) = \frac{1}{r^{\alpha+1}} \cdot \frac{r^\alpha e^{-cr}}{1-e^{-r}} dr =: \frac{1}{r^{\alpha+1}} \ell_{\alpha,c}(r) dr \tag{3.1}$$

and it is enough to check the monotonicity or non-monotonicity of $\ell_{\alpha,c}(r), r > 0$, depending on $(\alpha, c)$.

When $\alpha > 1$, since $\ell_{\alpha,c}(r) = r^{\alpha-1} e^{-cr}/(1 + o(r)) \to 0$ as $r \downarrow 0$ and $\ell_{\alpha,c}(1) > 0$, $\ell_{\alpha,c}$ is not nonincreasing. Thus $\mathcal{L}(\log \Gamma_{c,\lambda}) \notin L^{\langle \alpha \rangle}(\mathbb{R})$ for any $c > 0$.

When $0 < \alpha \le 1$, we need some calculation. We have

$$\frac{d\ell_{\alpha,c}}{dr}(r) = \frac{-r^\alpha e^{-cr}}{1-e^{-r}} (c - h(\alpha, r)). \tag{3.2}$$

Furthermore

$$\frac{\partial h}{\partial r}(\alpha, r) = r^{-2}(k(r) - \alpha).$$

We show here that $k(r) = \frac{r^2 e^{-r}}{(1-e^{-r})^2}$ is strictly deceasing on $(0, \infty)$ with $k(0+) = 1$ and $k(\infty) = 0$. We have

$$\frac{d}{dr}(k(r)^{1/2}) = \frac{e^{-r/2}}{(1-e^{-r})^2} a(r),$$

where $a(r) = 1 - e^{-r} - r/2 - re^{-r}/2$. Since $a'(r) = \{e^{-r}(1 + r) - 1\}/2$, $a''(r) = -re^{-r}/2 < 0$ for $r > 0$ and $a'(0) = 0$, we have $a'(r) < 0$. Noting that $a(0) = 0$, we have $a(r) < 0$, implying that $\frac{d}{dr}(k(r)^{1/2}) < 0$ for $r > 0$. Thus $k$ is strictly deceasing on $(0, \infty)$. We also have $k(r) = e^{-r}(1 + o(r))^{-2} \to 1$ as $r \downarrow 0$ and $k(\infty) = 0$.

Suppose $\alpha = 1$. Since $0 < k(r) < 1$,

$$\frac{\partial h}{\partial r}(1, r) = r^{-2}(k(r) - 1) < 0, \quad r > 0.$$

Also, $h(1, 0+) = 1/2$ and thus $\sup_{r>0} h(1, r) = 1/2$. Therefore, if $c \geq 1/2$ in (3.2), then $\frac{d\ell_{\alpha,c}}{dr}(r) \leq 0$ for $r > 0$ and thus $\ell_{\alpha,c}$ is nonincreasing. If $c < 1/2$, then $\frac{d\ell_{\alpha,c}}{dr}(r) > 0$ for some $r$ and $\ell_{\alpha,c}$ is not nonincreasing.

Next suppose $0 < \alpha < 1$. Since $k$ is strictly decreasing, we have $k(r) > \alpha$ for $r < r_\alpha$ and $k(r) < \alpha$ for $r > r_\alpha$, namely, $\frac{\partial h}{\partial r}(\alpha, r) > 0$ for $r < r_\alpha$ and $\frac{\partial h}{\partial r}(\alpha, r) < 0$ for $r > r_\alpha$. Therefore $\sup_{r>0} h(\alpha, r) = h(\alpha, r_\alpha)$. Note that $h(\alpha, r) = \{(\alpha - 1) + \alpha r/2 + o(r^2)\}/(r + o(r)) \to -\infty$ as $r \downarrow 0$ and $h(\alpha, \infty) = 0$. Since $\frac{\partial h}{\partial r}(\alpha, r) < 0$ for $r > r_\alpha$, we have $h(\alpha, r_\alpha) > 0$. If $c \geq h(\alpha, r_\alpha)$ in (3.2), then $\frac{d\ell_{\alpha,c}}{dr}(r) \leq 0$ for $r > 0$ and $\ell_{\alpha,c}$ is nonincreasing. If $c < h(\alpha, r_\alpha)$, then $\frac{d\ell_{\alpha,c}}{dr}(r) > 0$ for some $r$ and $\ell_{\alpha,c}$ is not nonincreasing.

We finally show the behavior of the function $\alpha \mapsto h(\alpha, r_\alpha)$. For $0 \leq \alpha \leq \beta \leq 1$,

$$h(\alpha, r_\alpha) = \sup_{r>0} h(\alpha, r) \leq \sup_{r>0} h(\beta, r) = h(\beta, r_\beta).$$

Since $k$ is continuous, $(0, 1) \ni \alpha \mapsto k^{-1}(\alpha) = u_\alpha \in (0, \infty)$ is continuous. Hence $(0, 1) \ni \alpha \mapsto h(\alpha, r_\alpha)$ is continuous. As $\alpha \downarrow 0$, $r_\alpha \to \infty$ and $h(\alpha, r_\alpha) \to 0$. Also, as $\alpha \uparrow 1$, $r_\alpha \to 0$ and

$$\begin{aligned} h(\alpha, r_\alpha) &= \frac{\alpha}{r_\alpha} - \frac{e^{-r_\alpha}}{1 - e^{-r_\alpha}} = \frac{k(r_\alpha)}{r_\alpha} - \frac{e^{-r_\alpha}}{1 - e^{-r_\alpha}} \\ &= \frac{e^{-r_\alpha}(r_\alpha - 1 + e^{-r_\alpha})}{(1 - e^{-r_\alpha})^2} = \frac{e^{-r_\alpha}(r_\alpha^2/2 + o(r_\alpha^2))}{(r_\alpha + o(u_\alpha))^2} \to 1/2. \end{aligned}$$

□

In the following, we prove Corollary 2.3.

*Proof of Corollary 2.3.* (i) This is obvious from Theorem 2.2.

(ii) This follows from the fact that $Z^2 \stackrel{\mathrm{d}}{=} \Gamma_{1/2,1/2}$ and Theorem 2.2.

(iii) Note that $T \stackrel{\mathrm{d}}{=} Z/\sqrt{\Gamma_{n/2,1/2}/n}$, where $n \in \mathbb{N}$ and $Z$ is a standard Gaussian random variable independent of $\Gamma_{n/2,1/2}$. We have $\log|T| \stackrel{\mathrm{d}}{=} \log|Z| - 2^{-1}\log\Gamma_{n/2,1/2} + 2^{-1}\log n$. By (2) and Theorem 2.2, $\mathcal{L}(\log|T|) \in L^{\langle 1\rangle}(\mathbb{R})$. The radial component $\nu_\xi$ of the Lévy measure of $\log|T|$ is $\nu_1(dr) =$

$2^{-\alpha}r^{-\alpha-1}\ell_{\alpha,n/2}(2r)dr$ and $\nu_{-1}(dr) = r^{-\alpha-1}\ell_{\alpha,1/2}(r)dr$. If $\alpha > 1$, then $\ell_{\alpha,n/2}(0+) = \ell_{\alpha,1/2}(0+) = 0$ and hence $\mathcal{L}(\log|T|) \notin L^{\langle\alpha\rangle}(\mathbb{R})$.

(iv) Note that $F \stackrel{\mathrm{d}}{=} (\Gamma_{m/2,1/2}/m)/(\Gamma_{n/2,1/2}/n)$, where $m, n \in \mathbb{N}$ and $\Gamma_{m/2,1/2}$ and $\Gamma_{n/2,1/2}$ are independent. Then $\log F \stackrel{\mathrm{d}}{=} -\log m + \log n + \log\Gamma_{m/2,1/2} - \log\Gamma_{n/2,1/2}$. By Theorem 2.2, $\mathcal{L}(\log F) \in L^{\langle 1\rangle}(\mathbb{R})$. The radial component $\nu_\xi$ of the Lévy measure of $\log F$ is $\nu_1(dr) = r^{-\alpha-1}\ell_{\alpha,n/2}(2r)dr$ and $\nu_{-1}(dr) = r^{-\alpha-1}\ell_{\alpha,m/2}(r)dr$. If $\alpha > 1$, then $\ell_{\alpha,m/2}(0+) = \ell_{\alpha,n/2}(0+) = 0$ and hence $\mathcal{L}(\log F) \notin L^{\langle\alpha\rangle}(\mathbb{R})$.

(v) This follows from (ii) and the fact that $S^+_{1/2} \stackrel{\mathrm{d}}{=} aZ^{-2}$ with some constant $a > 0$ and a standard Gaussian random variable $Z$. □

We finally prove Proposition 2.4.

*Proof of Proposition 2.4.* (i) Without loss of generality, we may assume that the Laplace transform of $S^+_\beta$ is $E[e^{-uS^+_\beta}] = \exp(-u^\beta)$. Shanbhag and Sreehari (1977) proved that $(\Gamma_{1,1}/S^+_\beta)^\beta \stackrel{\mathrm{d}}{=} \Gamma_{1,1}$, where $\Gamma_{1,1}$ is independent of $S^+_\beta$. Hence $\log\Gamma_{1,1} \stackrel{\mathrm{d}}{=} \beta\log\Gamma_{1,1} - \beta\log S^+_\beta$, which is another proof of the selfdecomposability of $\log\Gamma_{1,1}$ and shows the infinite divisibility of $\log S^+_\beta$. By (3.1), the radial component $\nu_\xi$ of the Lévy measure of $\log S^+_\beta$ is $\nu_{-1} = 0$ and

$$\nu_1(dr) = \left\{\beta(\beta r)^{-\alpha-1}\ell_{\alpha,1}(\beta r) - r^{-\alpha-1}\ell_{\alpha,1}(r)\right\} dr =: r^{-\alpha-1}k_{\alpha,\beta}(r)dr.$$

We have

$$\frac{dk_{\alpha,\beta}}{dr}(r) = \beta^{1-\alpha}\ell'_{\alpha,1}(\beta r) - \ell'_{\alpha,1}(r). \tag{3.3}$$

$\mathcal{L}(\log S^+_\beta) \in L^{\langle\alpha\rangle}(\mathbb{R})$ if and only if $\frac{dk_{\alpha,\beta}}{dr}(r) \le 0$ for $r > 0$. If $\alpha > 1$, then $k_{\alpha,\beta}(0+) = \beta^{-\alpha}\ell_{\alpha,1}(0+) - \ell_{\alpha,1}(0+) = 0$ and $k_{\alpha,\beta}(1) > 0$ for every $\beta \in (0,1)$ and hence $\mathcal{L}(\log S^+_\beta) \notin L^{\langle\alpha\rangle}(\mathbb{R})$. Let $\alpha = 1$. We have $\ell'_{1,1}(r) = (e^r - 1 - re^r)/(e^r - 1)^2$ and

$$\ell''_{1,1}(r) = \frac{e^r}{(e^r-1)^3}(re^r + r - 2e^r + 2) =: \frac{e^r}{(e^r-1)^3}f(r).$$

It follows that $f'(r) = 1 - e^r + re^r$ and that $f''(r) = re^r > 0$, implying that $f'(r) \ge f'(0) = 0$ and that $f(r) \ge f(0) = 0$. Therefore $\ell''_{1,1}(r) \ge 0$ for $r > 0$ and $\ell'_{1,1}$ is nondecreasing. Hence for any $\beta \in (0,1)$ and any $r > 0$, $\ell'_{1,1}(\beta r) \le \ell'_{1,1}(r)$, which yields that $\frac{dk_{1,\beta}}{dr}(r) \le 0$ for $r > 0$. Thus $\mathcal{L}(\log S^+_\beta) \in L^{\langle 1\rangle}(\mathbb{R})$.

(ii) A one-dimensional non-zero symmetric $\beta$-stable random variable $S_\beta^{\mathrm{sym}}$ is expressible as $S_\beta^{\mathrm{sym}} \stackrel{\mathrm{d}}{=} \sqrt{S_{\beta/2}^+}Z$, where $S_{\beta/2}^+$ is a positive strictly $(\beta/2)$-stable random variable independent of a standard Gaussian random variable $Z$. Hence $\log|S_\beta^{\mathrm{sym}}| \stackrel{\mathrm{d}}{=} 2^{-1}\log S_{\beta/2}^+ + \log|Z|$. By (i) and Corollary 2.3 (ii), $\mathcal{L}(\log|S_\beta^{\mathrm{sym}}|) \in L^{\langle 1\rangle}(\mathbb{R})$. The radial component $\nu_\xi$ of the Lévy measure of $\log|S_\beta^{\mathrm{sym}}|$ is $\nu_1(dr) = 2^{-\alpha}r^{-\alpha-1}k_{\alpha,\beta/2}(2r)dr$ and $\nu_{-1}(dr) = 2^{-\alpha}r^{-\alpha-1}\ell_{\alpha,1/2}(2r)dr$. If $\alpha > 1$, then $k_{\alpha,\beta/2}(0+) = \ell_{\alpha,1/2}(0+) = 0$ and hence $\mathcal{L}(\log|S_\beta^{\mathrm{sym}}|) \notin L^{\langle\alpha\rangle}(\mathbb{R})$. □

## 4. A concluding remark

Let $X$ be a random variable on $\mathbb{R}^d$ satisfying $\mathcal{L}(X) \in \bigcup_{\alpha\in\mathbb{R}} L^{\langle\alpha\rangle}(\mathbb{R}^d) \setminus \{\delta_\gamma\colon \gamma\in\mathbb{R}^d\}$. Then there exists the unique $\alpha \le 2$ such that

$$\mathcal{L}(X) \in L^{\langle\alpha\rangle}(\mathbb{R}^d) \text{ and } \mathcal{L}(X) \notin L^{\langle\beta\rangle}(\mathbb{R}^d) \text{ for any } \beta > \alpha. \tag{4.1}$$

We call this $\alpha$ the index of $X$ and denote it by $\mathrm{ind}(X)$. Note that the index of a non-constant stable random variable $S_\alpha$ is the same as the index of stability $\alpha$ by Example 1.3.

Let $d = 1$. In this paper, we have treated the logarithms of absolute values of several random variables. A natural question may be the following.

**Question.** Let $\mathcal{L}(X), \mathcal{L}(\log|X|) \in \bigcup_{\alpha\in\mathbb{R}} L^{\langle\alpha\rangle}(\mathbb{R}) \setminus \{\delta_\gamma\colon \gamma\in\mathbb{R}\}$. Which is greater, $\mathrm{ind}(X)$ or $\mathrm{ind}(\log|X|)$?

From what we have shown in this paper, we have the following.

**Example 4.1.** (The case $\mathrm{ind}(X) < \mathrm{ind}(\log|X|)$.)

(i) $X = \Gamma_{c,\lambda}$ (gamma random variable with parameters $c > 0$ and $\lambda > 0$): $\mathrm{ind}(\Gamma_{c,\lambda}) = 0$ and $\mathrm{ind}(\log\Gamma_{c,\lambda}) > 0$.

(ii) $X = S_\alpha^+, \alpha \in (0,1)$ (positive strictly $\alpha$-stable random variable with $\alpha\in(0,1)$): $\mathrm{ind}(S_\alpha^+) = \alpha < 1$ and $\mathrm{ind}(\log S_\alpha^+) = 1$.

(iii) $X = e^Z$ (log-normal random variable): $\mathrm{ind}(\log e^Z) = 2$ and $\mathrm{ind}(e^Z) < 2$. (Proof: $\mathrm{ind}(\log e^Z) = \mathrm{ind}(Z) = 2$. Since $e^Z$ is a generalized gamma convolution (see Thorin (1977)), $\mathcal{L}(e^Z) \in L^{\langle 0\rangle}(\mathbb{R})$ and hence $\mathrm{ind}(e^Z)$ exists. Because $e^Z$ is not Gaussian, $\mathrm{ind}(e^Z) < 2$.)

(iv) $X = S_\alpha^{\mathrm{sym}}, \alpha \in (0,1)$ (one-dimensional non-zero symmetric $\alpha$-stable random variable with $\alpha \in (0,1)$): $\mathrm{ind}(S_\alpha^{\mathrm{sym}}) = \alpha < 1$ and $\mathrm{ind}(\log|S_\alpha^{\mathrm{sym}}|) = 1$.

**Example 4.2.** (The case $\mathrm{ind}(X) > \mathrm{ind}(\log|X|)$.)

(i) $X = Z$ (standard Gaussian random variable): $\mathrm{ind}(Z) = 2$ and $\mathrm{ind}(\log|Z|) = 1$.

(ii) $X = S_\alpha^{\mathrm{sym}}, \alpha \in (1,2)$ (one-dimensional non-zero symmetric $\alpha$-stable random variable with $\alpha \in (1,2)$): $\mathrm{ind}(S_\alpha^{\mathrm{sym}}) = \alpha > 1$ and $\mathrm{ind}(\log|S_\alpha^{\mathrm{sym}}|) = 1$.

**Example 4.3.** (The case $\mathrm{ind}(X) = \mathrm{ind}(\log|X|)$.)
$X = S_1^{\mathrm{sym}}$ (one-dimensional non-zero symmetric 1-stable random variable): $\mathrm{ind}(S_1^{\mathrm{sym}}) = 1$ and $\mathrm{ind}(\log|S_\alpha^{\mathrm{sym}}|) = 1$.

In the examples above, it seems to hold that $\mathrm{ind}(X) < \mathrm{ind}(\log X)$ if $X$ is a positive random variable. Is this generally true? This is an open question. If $X$ is a symmetric random variable, then each of the relations $\mathrm{ind}(X) < \mathrm{ind}(\log|X|)$, $\mathrm{ind}(X) > \mathrm{ind}(\log|X|)$ and $\mathrm{ind}(X) = \mathrm{ind}(\log|X|)$ can hold as in Examples 4.1(iv), 4.2(ii) and 4.3.

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