The dichotomy of recurrence and transience of semi-Lévy processes

(*Running title:* Recurrence and transience of semi-Lévy processes)

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Semi-Lévy process is an additive process with periodically stationary increments. In particular, it is a generalization of Lévy process. The dichotomy of recurrence and transience of Lévy processes is well known, but this is not necessarily true for general additive processes. In this paper, we prove the recurrence and transience dichotomy of semi-Lévy processes. For the proof, we introduce a concept of semi-random walk and discuss its recurrence and transience properties. An example of semi-Lévy process constructed from two independent Lévy processes is investigated. Finally, we prove the laws of large numbers for semi-Lévy processes.

keywords: Lévy process; semi-Lévy process; recurrence; transience; semi-random walk; law of large numbers

1. Introduction

Recurrence and transience problem has been studied for many stochastic processes. Among those, the dichotomy of recurrence and transience of Lévy processes is well known (see, e.g., Section 35 of Sato [5]), which is not necessarily true for general additive processes (see Remark 3.4 of Sato and Yamamuro [7]). In this paper, we show the dichotomy for semi-Lévy processes.

A stochastic process $\{X_t\}_{t\ge 0}$ on $\mathbb{R}^d$ is called an additive process if $X_0 = 0$ a.s., it is stochastically continuous, it has independent increments and its sample paths are right-continuous in $t \ge 0$ and have left-limits in $t > 0$. Further, if $\{X_t\}_{t\ge 0}$ has stationary increments, it is a Lévy process. As an extension of Lévy process, we define a subclass of additive processes with the property that for some $p > 0$,

$$X_{t+p} - X_{s+p} \stackrel{\mathrm{d}}{=} X_t - X_s \quad \text{for any } s, t \ge 0.$$

Process with this property is called a semi-Lévy process with period p. Semi-Lévy processes have already been considered in the literature by Maejima and Sato [4] and in Lemma 3.3 of Becker-Kern [1].

We give several comments regarding semi-Lévy processes. An infinitely divisible distribution μ on $\mathbb{R}^d$ is called selfdecomposable if for any $b > 1$ there exists a distribution (necessarily infinitely divisible) μ_b on $\mathbb{R}^d$ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\mu}_b(z)$, (where $\widehat{\mu}$ is the characteristic function of μ), and it is called semi-selfdecomposable with span $b > 1$ if there exists an infinitely divisible distribution ρ on $\mathbb{R}^d$ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}(z)$. Lévy processes are connected to selfdecomposable distributions in the sense that μ is selfdecomposable if and only if there exists a Lévy process $\{X_t\}_{t\geq 0}$ with $E[\log(1 + |X_1|)] < \infty$ satisfying

$$\mu = \mathcal{L}\left(\int_0^\infty e^{-t} dX_t\right), \tag{1.1}$$

where $\mathcal{L}(Y)$ denotes the law of a random variable Y. In the same fashion, semi-Lévy processes are related to semi-selfdecomposable distributions in the sense that μ is semi-selfdecomposable with span $b > 1$ if and only if there exists a semi-Lévy process $\{X_t\}_{t\geq 0}$ with period $p = \log b$ which is a semimartingale with $E[\log(1 + |X_1|)] < \infty$ and which satisfies (1.1) (see Maejima and Sato [4]).

Furthermore, as shown in Maejima and Sato [4] and Becker-Kern [1], semi-Lévy processes are also related to semi-selfsimilar additive processes which were introduced and deeply studied in Maejima and Sato [3]. More precisely, semi-Lévy processes and semi-selfsimilar additive processes can mutually be represented by stochastic integrals with respect to each other if they are semimartingales.

Example 1.1. The following example of semi-Lévy process was given by Ken-iti Sato to the authors through a private communication. Let $\{Y_t\}_{t\geq 0}$ and $\{Z_t\}_{t\geq 0}$ be two independent Lévy processes and let $0 < q < p$ be arbitrary. A stochastic process $\{X_t\}_{t\geq 0}$ defined by

$$X_0 = 0 \quad a.s.,$$
$$X_t = \begin{cases} X_{np} + Y_t - Y_{np} & (np < t \leq np + q), \\ X_{np+q} + Z_t - Z_{np+q} & (np + q < t \leq (n+1)p) \end{cases}$$

is a semi-Lévy process with period p. The recurrence and transience property of this process is discussed in Section 4.

Throughout this paper, $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$, $\mathbb{Z}_+$ denotes the set of nonnegative integers and $B_a := \{x \in \mathbb{R}^d \colon |x| < a\}$ denotes the open ball of radius $a > 0$ around the origin.

Definition 1.2 (Sato and Yamamuro [7])**.** Let $s \geq 0$. An additive process $\{X_t\}_{t\geq 0}$ on $\mathbb{R}^d$ is called *s-recurrent* if

$$\liminf_{t\to\infty} |X_t - X_s| = 0 \quad \text{a.s.},$$

and it is called *recurrent* if it is s-recurrent for any $s \geq 0$. It is called *transient* if

$$\lim_{t\to\infty} |X_t| = \infty \quad \text{a.s.}$$

Remark 1.3. Note that, for Lévy processes, by the stationary increment property, recurrence and 0-recurrence are equivalent. This is also the case for semi-Lévy processes as seen in Theorem 1.4 (i) below.

Now, we state the main results of this paper.

Theorem 1.4. *Let $\{X_t\}_{t\geq 0}$ be a semi-Lévy process on $\mathbb{R}^d$ with period $p > 0$. Then we have the following.*

(i) *It is recurrent if it is 0-recurrent.*

(ii) *It is either recurrent or transient.*

(iii) *It is recurrent if and only if*

$$\int_0^\infty P(X_t \in B_a)dt = \infty \quad \textit{for every } a > 0. \tag{1.2}$$

(iv) *It is recurrent if and only if*

$$\int_0^\infty \mathbb{1}_{B_a}(X_t)dt = \infty \quad \textit{a.s.} \quad \textit{for every } a > 0.$$

(v) *It is transient if and only if*

$$\int_0^\infty P(X_t \in B_a)dt < \infty \quad \textit{for every } a > 0. \tag{1.3}$$

(vi) *It is transient if and only if*

$$\int_0^\infty \mathbb{1}_{B_a}(X_t)dt < \infty \quad \textit{a.s.} \quad \textit{for every } a > 0. \tag{1.4}$$

(vii) *Fix $h \in (0,\infty) \cap p\mathbb{Q}$ arbitrarily. Then, the semi-Lévy process $\{X_t\}_{t\geq 0}$ is recurrent if and only if the semi-random walk $\{X_{nh}\}_{n\in\mathbb{Z}_+}$ is recurrent.* (*See Section* 2 *for the definition of semi-random walk.*)

Further, let $\{X_t\}_{t\geq 0}$ be a semi-Lévy process with period $p>0$. Since X_p is an infinitely divisible random variable, then there is a Lévy process $\{Y_t\}_{t\geq 0}$ such that $Y_1 \overset{\mathrm{d}}{=} X_p$. Now, as a consequence of Theorem 1.4, we deduce the following.

Theorem 1.5. *$\{X_t\}_{t\geq 0}$ is recurrent if and only if $\{Y_t\}_{t\geq 0}$ is recurrent.*

Finally, as a consequence of Theorem 1.5, the criterion of Chung-Fuchs type immediately follows.

Corollary 1.6. *Let $\{X_t\}_{t\geq 0}$ be a semi-Lévy process on $\mathbb{R}^d$ with period $p>0$ and let $\psi(z)=\log\widehat{\mathcal{L}(X_p)}(z)$. Fix $a>0$. Then the following three statements are equivalent.*

(i) *$\{X_t\}_{t\geq 0}$ is recurrent.*
(ii) $\lim_{q\downarrow 0}\int_{B_a}\mathrm{Re}\left(\frac{1}{q-\psi(z)}\right)dz=\infty$.
(iii) $\limsup_{q\downarrow 0}\int_{B_a}\mathrm{Re}\left(\frac{1}{q-\psi(z)}\right)dz=\infty$.

Proof. The claim of this corollary follows from the corresponding result for Lévy processes (see, e.g., Theorem 37.5 of Sato [5]) and Theorem 1.5. □

This paper is organized as follows. In Section 2, we introduce the notion of semi-random walk and prove its recurrence and transience dichotomy. In Section 3, we give proofs of Theorems 1.4 and 1.5. In Section 4, we investigate the recurrence and transience property of the semi-Lévy process defined in Example 1.1. Once we have investigated the behavior of sample paths as $t\to\infty$, it would be natural to consider the laws of large numbers. Therefore, in Section 5, we discuss the laws of large numbers for semi-Lévy processes.

2. Semi-random walks

Crucial step in proving the recurrence and transience dichotomy of Lévy processes is the recurrence and transience dichotomy of random walks. In order to prove Theorem 1.4, we follow this idea, and, in this section, we introduce the notion of semi-random walk and its recurrence and transience properties.

Definition 2.1. Fix $p\in\mathbb{N}$. A sequence of $\mathbb{R}^d$-valued random variables $\{S_n\}_{n\in\mathbb{Z}_+}$ is called a semi-random walk with period $p\in\mathbb{N}$, if $S_0=0$ a.s., S_n-S_m and S_k-S_l are independent for any $n,m,k,l\in\mathbb{Z}_+$ satisfying $n>m\geq k>l$, and

$$S_{n+p}-S_{m+p}\overset{\mathrm{d}}{=}S_n-S_m \quad \text{for any } n,m\in\mathbb{Z}_+.$$

Let us remark that if $\{S_n\}_{n\in\mathbb{Z}_+}$ is a semi-random walk with period p, then $\{S_{np}\}_{n\in\mathbb{Z}_+}$ is a random walk.

Definition 2.2. Let $m \in \mathbb{Z}_+$. A semi-random walk $\{S_n\}_{n\in\mathbb{Z}_+}$ on $\mathbb{R}^d$ is called *m-recurrent* if

$$\liminf_{n\to\infty} |S_n - S_m| = 0 \quad \text{a.s.},$$

and it is called *recurrent* if it is m-recurrent for any $m \in \mathbb{Z}_+$. It is called *transient* if

$$\lim_{n\to\infty} |S_n| = \infty \quad \text{a.s.}$$

Remark 2.3. Note that, for random walks, by the stationary increment property, recurrence and 0-recurrence are equivalent. In Theorem 2.5 (i) below we show that this is also the case for semi-random walks.

The following is an example of semi-random walk, which is constructed from a semi-Lévy process.

Example 2.4. Suppose that $\{X_t\}_{t\geq 0}$ is a semi-Lévy process with period $p > 0$. Let $h \in (0,\infty) \cap p\mathbb{Q}$. Then $\{X_{nh}\}_{n\in\mathbb{Z}_+}$ is a semi-random walk with period $n_2 \in \mathbb{N}$, where n_2 is such that $h = pn_1/n_2$, $(n_1, n_2 \in \mathbb{N})$. Indeed, for any $n, m \in \mathbb{Z}_+$,

$$X_{(n+n_2)h} - X_{(m+n_2)h} = X_{nh+pn_1} - X_{mh+pn_1} \stackrel{\mathrm{d}}{=} X_{nh} - X_{mh},$$

since $\{X_t\}_{t\geq 0}$ is a semi-Lévy process with period p.

Theorem 2.5. *Let $\{S_n\}_{n\in\mathbb{Z}_+}$ be a semi-random walk on $\mathbb{R}^d$ with period $p \in \mathbb{N}$. Then the following hold.*

(i) *It is recurrent if it is 0-recurrent.*

(ii) *It is either recurrent or transient.*

(iii) *It is recurrent if and only if*

$$\sum_{n=1}^{\infty} P(S_n \in B_a) = \infty \quad \textit{for every } a > 0. \tag{2.1}$$

(iv) *It is transient if and only if*

$$\sum_{n=1}^{\infty} P(S_n \in B_a) < \infty \quad \textit{for every } a > 0. \tag{2.2}$$

(v) *It is recurrent if and only if the random walk $\{S_{np}\}_{n\in\mathbb{Z}_+}$ is recurrent.*

In order to prove Theorem 2.5, we need the following lemma.

Lemma 2.6. *Let $\{S_n\}_{n\in\mathbb{Z}_+}$ be a semi-random walk on $\mathbb{R}^d$ with period $p \in \mathbb{N}$. The following three statements are equivalent.*

(1) (2.2) *holds.*

(2) $\{S_n\}_{n\in\mathbb{Z}_+}$ *is transient.*

(3) $\{S_{np}\}_{n\in\mathbb{Z}_+}$ *is transient.*

Proof. (1)⇒(2). Suppose (2.2). Then by the Borel-Cantelli lemma, we have $P(\limsup_{n\to\infty}\{|S_n| < a\}) = 0$, namely, $P(\exists m$ such that $|S_n| \geq a$ for all $n \geq m) = 1$. Since a is arbitrary, $\{S_n\}_{n\in\mathbb{Z}_+}$ is transient.

(2)⇒(3). This follows from the definition of transience of semi-random walk in Definition 2.2, since $\{S_{np}\}_{n\in\mathbb{Z}_+}$ is a subsequence of $\{S_n\}_{n\in\mathbb{Z}_+}$.

(3)⇒(1). Suppose

$$\sum_{n=1}^{\infty} P(S_n \in B_a) = \infty \quad \text{for some } a > 0.$$

It follows that

$$\sum_{l=0}^{p-1}\sum_{n=0}^{\infty} P(S_{np+l} \in B_a) = \sum_{n=0}^{\infty} P(S_n \in B_a) = \infty.$$

Hence there is a number $l_1 \in \{0, 1, \dots, p-1\}$ such that

$$\sum_{n=0}^{\infty} P(S_{np+l_1} \in B_a) = \infty.$$

Let $K = \{x \in \mathbb{R}^d : |x| \leq a\}$. Since $B_a \subset K$,

$$\sum_{n=0}^{\infty} P(S_{np+l_1} \in K) \geq \sum_{n=0}^{\infty} P(S_{np+l_1} \in B_a) = \infty.$$

Fix $\eta > 0$ arbitrarily. Since K is compact, it is covered by a finite number of open balls with radius $\eta/2$. Hence, there is an open ball B with radius $\eta/2$ such that

$$\sum_{n=0}^{\infty} P(S_{np+l_1} \in B) = \infty. \tag{2.3}$$

We have

$$\begin{aligned}
1 &\geq P\left(\bigcup_{k=0}^{\infty}\{S_{kp+l_1} \in B,\ S_{(k+n)p+l_1} \notin B \text{ for all } n \in \mathbb{N}\}\right) \\
&= \sum_{k=0}^{\infty} P(S_{kp+l_1} \in B,\ S_{(k+n)p+l_1} \notin B \text{ for all } n \in \mathbb{N}) \\
&\geq \sum_{k=0}^{\infty} P(S_{kp+l_1} \in B,\ |S_{(k+n)p+l_1} - S_{kp+l_1}| \geq \eta \text{ for all } n \in \mathbb{N})
\end{aligned}$$

$$= P(|S_{np+l_1} - S_{l_1}| \geq \eta \text{ for all } n \in \mathbb{N}) \sum_{k=0}^{\infty} P(S_{kp+l_1} \in B)$$

by the definition of semi-random walk. By (2.3), we have

$$P(|S_{np+l_1} - S_{l_1}| \geq \eta \text{ for all } n \in \mathbb{N}) = 0. \tag{2.4}$$

Since $S_{p+l_1} - S_p$, $S_p - S_{l_1}$ and S_{l_1} are independent and $S_{p+l_1} - S_p \stackrel{\mathrm{d}}{=} S_{l_1}$, we have

$$\begin{aligned} S_{p+l_1} - S_{l_1} &= (S_{p+l_1} - S_p) + (S_p - S_{l_1}) \\ &\stackrel{\mathrm{d}}{=} S_{l_1} + (S_p - S_{l_1}) = S_p. \end{aligned}$$

Since $\{S_{np+l_1} - S_{l_1}\}_{n\in\mathbb{Z}_+}$ and $\{S_{np}\}_{n\in\mathbb{Z}_+}$ are random walks with the same law at $n = 1$, we have $\{S_{np+l_1} - S_{l_1}\}_{n\in\mathbb{Z}_+} \stackrel{\mathrm{d}}{=} \{S_{np}\}_{n\in\mathbb{Z}_+}$. Therefore, by (2.4),

$$P(|S_{np}| \geq \eta \text{ for all } n \in \mathbb{N}) = 0. \tag{2.5}$$

Then for $k \in \mathbb{Z}_+$ and for $0 < \eta < \varepsilon$,

$$\begin{aligned} &P(|S_{kp}| < \varepsilon - \eta, \ |S_{(k+n)p}| \geq \varepsilon \text{ for all } n \in \mathbb{N}) \\ &\quad \leq P(|S_{kp}| < \varepsilon - \eta, \ |S_{(k+n)p} - S_{kp}| \geq \eta \text{ for all } n \in \mathbb{N}) \\ &\quad = P(|S_{kp}| < \varepsilon - \eta) P(|S_{np}| \geq \eta \text{ for all } n \in \mathbb{N}) = 0, \end{aligned}$$

by the definition of semi-random walk and (2.5). Letting $\eta \downarrow 0$, we have

$$P(|S_{kp}| < \varepsilon, \ |S_{(k+n)p}| \geq \varepsilon \text{ for all } n \in \mathbb{N}) = 0 \tag{2.6}$$

for all $k \in \mathbb{Z}_+$. Using (2.6), we have

$$\begin{aligned} &P(\exists m \in \mathbb{N} \text{ such that } |S_{np}| \geq \varepsilon \text{ for all } n \geq m) \\ &\qquad = \sum_{k=0}^{\infty} P(|S_{kp}| < \varepsilon, \ |S_{(k+n)p}| \geq \varepsilon \text{ for all } n \in \mathbb{N}) = 0, \end{aligned}$$

namely, $P(\forall m \in \mathbb{N}, \ \exists n \geq m \text{ such that } |S_{np}| < \varepsilon) = 1$. Hence $\{S_{np}\}_{n\in\mathbb{Z}_+}$ is 0-recurrent, implying that $\{S_{np}\}_{n\in\mathbb{Z}_+}$ is not transient. This completes the proof of the equivalence of (1)–(3). □

Proof of Theorem 2.5. (i) Suppose that $\{S_n\}_{n\in\mathbb{Z}_+}$ is not 0-recurrent. Then the subsequence $\{S_{np}\}_{n\in\mathbb{Z}_+}$ is not 0-recurrent. Then by Theorem 35.3 (i) of Sato [5], $\{S_{np}\}_{n\in\mathbb{Z}_+}$ is transient. By the equivalence of (2) and (3) of Lemma 2.6, $\{S_n\}_{n\in\mathbb{Z}_+}$ is transient. Hence $\{S_n\}_{n\in\mathbb{Z}_+}$ is either 0-recurrent or transient. If $\{S_n\}_{n\in\mathbb{Z}_+}$ is 0-recurrent, then, for every $m \in \mathbb{Z}_+$,

$$\liminf_{n\to\infty} |S_{n+m} - S_m| \leq \liminf_{n\to\infty} |S_{n+m}| + |S_m| = |S_m| < \infty \quad \text{a.s.} \tag{2.7}$$

Since $\{S_{n+m} - S_m\}_{n \in \mathbb{Z}_+}$ is a semi-random walk with period p, it is either 0-recurrent or transient. By (2.7), we have $\liminf_{n\to\infty} |S_{n+m} - S_m| = 0$ a.s. Thus $\{S_n\}_{n\in\mathbb{Z}_+}$ is recurrent.

(ii) This is proved in the proof of (i).

(iv) The claim follows from the equivalence of (1) and (2) of Lemma 2.6.

(iii) Suppose $\sum_{n=1}^{\infty} P(S_n \in B_a) < \infty$ for some $a > 0$. Then by the Borel-Cantelli lemma, $P(\exists m \text{ such that } |S_n| \geq a \text{ for all } n \geq m) = 1$, implying that $\{S_n\}_{n\in\mathbb{Z}_+}$ is not recurrent. Next suppose (2.1). Then by (iv) and (ii), $\{S_n\}_{n\in\mathbb{Z}_+}$ is recurrent.

(v) The claim follows from (ii) and the equivalence of (2) and (3) of Lemma 2.6. □

3. Proofs of main results

In this section, we prove Theorems 1.4 and 1.5. We start with the following lemma, which is a generalization of Lemma 35.5 of Sato [5].

Lemma 3.1. *For any semi-Lévy process $\{X_t\}_{t\geq 0}$ and any $a > 0$, there is a function $\gamma(a,\varepsilon)$ satisfying $\gamma(a,\varepsilon) \to 1$ as $\varepsilon \downarrow 0$, such that, for every $t > 0$ and $\varepsilon > 0$,*

$$P\left(\int_t^{\infty} \mathbb{1}_{B_{2a}}(X_s)ds > \varepsilon\right) \geq \gamma(a,\varepsilon) P\left(|X_{t+s}| < a \text{ for some } s > 0\right).$$

Proof. Let $\mathcal{F}_t$ be the σ-field generated by $\{X_s\}_{s\in[0,t]}$ and take $\Lambda \in \mathcal{F}_t$ such that $\Lambda \subset \{|X_t| < a\}$ and $P(\Lambda) > 0$. Consider

$$Y := \frac{1}{2\varepsilon}\int_t^{t+2\varepsilon} \mathbb{1}_{B_{2a}}(X_s)ds.$$

Then $0 \leq Y \leq 1$ and $2E[Y|\Lambda] \leq P(Y > 1/2 \,|\, \Lambda) + 1$. Hence, we have

$$\begin{aligned}
&P\left(\int_t^{t+2\varepsilon} \mathbb{1}_{B_{2a}}(X_s)ds > \varepsilon \,\middle|\, \Lambda\right) \\
&\qquad \geq \frac{1}{\varepsilon}E\left[\int_t^{t+2\varepsilon} \mathbb{1}_{B_{2a}}(X_s)ds \,\middle|\, \Lambda\right] - 1 \\
&\qquad = 2\int_0^1 P\left(|X_{t+2\varepsilon s}| < 2a \,|\, \Lambda\right) ds - 1 \\
&\qquad \geq 2\int_0^1 P\left(|X_{t+2\varepsilon s} - X_t| < a \,|\, \Lambda\right) ds - 1 \\
&\qquad = 2\int_0^1 P\left(|X_{t+2\varepsilon s} - X_t| < a\right) ds - 1
\end{aligned}$$

$$= 2\int_0^1 P\left(|X_{t-[t/p]p+2\varepsilon s} - X_{t-[t/p]p}| < a\right) ds - 1$$

$$\geq 2\int_0^1 \inf_{t\in[0,p)} P\left(|X_{t+2\varepsilon s} - X_t| < a\right) ds - 1 =: \gamma(a,\varepsilon),$$

where $[x]$ denotes the largest integer not greater than $x \in \mathbb{R}$. Note that a stochastically continuous process is uniformly stochastically continuous on any compact interval $[0, t_0]$, i.e., $\lim_{\delta\downarrow 0} \sup_{u,v\in[0,t_0],|u-v|<\delta} P(|X_u - X_v| > a) = 0$ for each $a > 0$ (see, e.g., Lemma 9.6 of Sato [5]). Hence, as $(0,1) \ni \delta \downarrow 0$,

$$\inf_{t\in[0,p)} P\left(|X_{t+\delta} - X_t| < a\right) \geq \inf_{u,v\in[0,p+1],|u-v|\leq\delta} P(|X_u - X_v| < a) \to 1.$$

Therefore $\lim_{\delta\downarrow 0} \inf_{t\in[0,p)} P\left(|X_{t+\delta} - X_t| < a\right) = 1$. Using the bounded convergence theorem, we thus have $\lim_{\varepsilon\downarrow 0} \gamma(a,\varepsilon) = 1$. Hence

$$P\left(\Lambda \cap \left\{\int_t^\infty \mathbb{1}_{B_{2a}}(X_s)ds > \varepsilon\right\}\right) \geq \gamma(a,\varepsilon)P(\Lambda) \tag{3.1}$$

for any $\Lambda \in \mathcal{F}_t$ satisfying $\Lambda \subset \{|X_t| < a\}$. For each $k \in \mathbb{N}$,

$$\begin{aligned}
&P\left(\int_t^\infty \mathbb{1}_{B_{2a}}(X_s)ds > \varepsilon\right)\\
&\geq \sum_{n=0}^\infty P\left(|X_{t+2^{-k}j}| \geq a \text{ for } 0 \leq j < n, |X_{t+2^{-k}n}| < a, \int_{t+2^{-k}n}^\infty \mathbb{1}_{B_{2a}}(X_s)ds > \varepsilon\right)\\
&\geq \gamma(a,\varepsilon)\sum_{n=0}^\infty P\left(|X_{t+2^{-k}j}| \geq a \text{ for } 0 \leq j < n, |X_{t+2^{-k}n}| < a\right)\\
&= \gamma(a,\varepsilon)P\left(|X_{t+2^{-k}n}| < a \text{ for some } n \in \mathbb{Z}_+\right),
\end{aligned}$$

where we have used the inequality (3.1). Letting $k \to \infty$, we have

$$\begin{aligned}
P\left(\int_t^\infty \mathbb{1}_{B_{2a}}(X_s)ds > \varepsilon\right) &\geq \gamma(a,\varepsilon)P\left(|X_{t+2^{-k}n}| < a \text{ for some } k, n \in \mathbb{Z}_+\right)\\
&= \gamma(a,\varepsilon)P\left(|X_{t+s}| < a \text{ for some } s > 0\right),
\end{aligned}$$

where we have used the denseness of $\{2^{-k}n\colon k, n \in \mathbb{Z}_+\}$ in $[0,\infty)$ and the right-continuity of the sample paths of $\{X_t\}_{t\geq 0}$. □

We also need the following lemma.

Lemma 3.2. *Let $\{X_t\}_{t\geq 0}$ be a semi-Lévy process on $\mathbb{R}^d$ with period $p > 0$. Fix $a > 0$. Then the following statements are equivalent.*

(1) $\{X_t\}_{t\geq 0}$ *is 0-recurrent.*
(2) $\int_0^\infty \mathbb{1}_{B_a}(X_t)dt = \infty$ *a.s.*
(3) $\int_0^\infty P(X_t \in B_a)dt = \infty$.

(4) *There exists* $h_0 > 0$ *such that, for any* $h \in (0, h_0] \cap p\mathbb{Q}$, *the semi-random walk* $\{X_{nh}\}_{n \in \mathbb{Z}_+}$ *is recurrent.*

Proof. (1)⇒(2). Fix $\varepsilon > 0$. By (1), for every $t > 0$, $P(|X_{t+s}| < a/2 \text{ for some } s > 0) = 1$. Using Lemma 3.1, we have for any $\varepsilon > 0$

$$P\Big(\int_0^\infty \mathbb{1}_{B_a}(X_t)dt = \infty\Big) \geq P\Big(\int_n^\infty \mathbb{1}_{B_a}(X_t)dt > \varepsilon \text{ for all } n \in \mathbb{N}\Big)$$
$$= \lim_{n\to\infty} P\Big(\int_n^\infty \mathbb{1}_{B_a}(X_t)dt > \varepsilon\Big) \geq \gamma(a/2, \varepsilon).$$

Then, letting $\varepsilon \downarrow 0$, we have (2).

(2)⇒(3). Obvious by taking expectation.

(3)⇒(4). Recall again that a stochastically continuous process is uniformly stochastically continuous on any compact interval. Hence there is $h_0 > 0$ such that, for $u, v \in [0, p+1]$ satisfying $|u - v| < h_0$, $P(X_u - X_v \in B_a) > 1/2$. Without loss of generality, we may assume $h_0 < 1$. Let $h \in (0, h_0] \cap p\mathbb{Q}$. Let $n \in \mathbb{N}$, $(n-1)h \leq t \leq nh$ and $x \in B_a$. Then $t - [t/p]p \in [0, p)$, $nh - [t/p]p = (nh - t) + (t - [t/p]p) \in [0, h+p) \subset [0, p+1)$ and $|nh - t| \leq h \leq h_0$. Therefore

$$P(x + X_{nh} - X_t \in B_{2a}) \geq P(X_{nh} - X_t \in B_a)$$
$$= P(X_{nh-[t/p]p} - X_{t-[t/p]p} \in B_a) > 1/2.$$

Thus

$$P(X_{nh} \in B_{2a}) \geq P(X_t \in B_a, X_t + (X_{nh} - X_t) \in B_{2a})$$
$$= E\Big[\mathbb{1}_{B_a}(X_t) \cdot P(x + X_{nh} - X_t \in B_{2a})\big|_{x=X_t}\Big] \geq \frac{1}{2}P(X_t \in B_a).$$

Hence we have

$$P(X_{nh} \in B_{2a}) \geq \frac{1}{2h}\int_{(n-1)h}^{nh} P(X_t \in B_a)dt.$$

Thus (3) implies that $\sum_{n=1}^\infty P(X_{nh} \in B_{2a}) = \infty$. By Theorem 2.5 (ii) and (iv), we have (4).

(4)⇒(1). If (4) holds, then

$$\liminf_{t\to\infty} |X_t| \leq \liminf_{n\to\infty} |X_{nh}| = 0 \text{ a.s.} \tag{3.2}$$

Hence (1) holds. □

Remark 3.3. Note that each of the statements (2) and (3) in Lemma 3.2 holds for some $a > 0$ if and only if it holds for every $a > 0$. This follows from the independence of the statements (1) and (4) from $a > 0$. Indeed, if (2) [resp. (3)] holds for some $a > 0$, then (1) holds, implying that (2) [resp. (3)] holds for every $a > 0$.

Proof of Theorem 1.4. (v) If $\{X_t\}_{t\geq 0}$ is transient, then it is not 0-recurrent and hence (1.3) holds by the equivalence of (1) and (3) of Lemma 3.2. Conversely, assume (1.3). For $a > 0$, choose $\varepsilon > 0$ such that $\gamma(a, \varepsilon) > 1/2$, where γ is the function in Lemma 3.1. Then, for every $t > 0$,

$$\int_t^\infty P(X_s \in B_{2a})ds = E\left[\int_t^\infty \mathbb{1}_{B_{2a}}(X_s)ds\right]$$
$$\geq \varepsilon P\left(\int_t^\infty \mathbb{1}_{B_{2a}}(X_s)ds > \varepsilon\right) \geq \frac{\varepsilon}{2} P(|X_{t+s}| < a \text{ for some } s > 0).$$

Letting $t \to \infty$ and using (1.3), we have

$$\lim_{t\to\infty} P(|X_{t+s}| < a \text{ for some } s > 0) = 0.$$

Hence

$$P\left(\lim_{t\to\infty} |X_t| = \infty\right) = P\left(\bigcap_{k=1}^\infty \bigcup_{n=1}^\infty \{|X_{n+s}| \geq k \text{ for all } s > 0\}\right)$$
$$= \lim_{k\to\infty} \lim_{n\to\infty} P(|X_{n+s}| \geq k \text{ for all } s > 0) = 1,$$

that is, $\{X_t\}_{t\geq 0}$ is transient.

(i) Assume that $\{X_t\}_{t\geq 0}$ is not 0-recurrent. Then, by the equivalence of (1) and (3) of Lemma 3.2, (1.3) holds. Thus $\{X_t\}_{t\geq 0}$ is transient by (v). Hence $\{X_t\}_{t\geq 0}$ is either 0-recurrent or transient. If $\{X_t\}_{t\geq 0}$ is 0-recurrent, then, for every $s \geq 0$,

$$\liminf_{t\to\infty} |X_{t+s} - X_s| \leq \liminf_{t\to\infty} |X_{t+s}| + |X_s| = |X_s| < \infty \quad \text{a.s.} \tag{3.3}$$

Since $\{X_{t+s} - X_s\}_{t\geq 0}$ is a semi-Lévy process with period p, it is either 0-recurrent or transient. By (3.3), we have $\liminf_{t\to\infty} |X_{t+s} - X_s| = 0$ a.s. Thus $\{X_t\}_{t\geq 0}$ is recurrent.

(ii) The claim has been proved in (i).

(iii) By (i) and the equivalence of (1) and (3) of Lemma 3.2, we have (iii).

(iv) By (i) and the equivalence of (1) and (2) of Lemma 3.2, we have (iv).

(vi) If (1.4) holds, then $\{X_t\}_{t\geq 0}$ is transient by (iv) and (ii). If it is transient, then (1.3) holds by (v) and (1.3) implies (1.4).

(vii) Let us assume that the semi-random walk $\{X_{nh}\}_{n\in\mathbb{Z}_+}$ is recurrent. Then, the claim follows from (3.2) and (i). Conversely, if we suppose the recurrence of $\{X_t\}_{t\geq 0}$, then, by (iii), (1.2) holds. Fix $h \in (0, \infty) \cap p\mathbb{Q}$. Note that $\lim_{n\to\infty} P(\sup_{s\in[0,p+h]} |X_s| < n) = P(\sup_{s\in[0,p+h]} |X_s| < \infty) = 1$ by the right-continuity with left limits of the sample paths of $\{X_s\}_{s\geq 0}$. Hence $P(\sup_{s\in[0,p+h]} |X_s| < a/2) > 1/2$ for some $a > 0$. Let

$n \in \mathbb{N}$, $(n-1)h \le t \le nh$ and $x \in B_a$. Then $t - [t/p]p \in [0, p)$ and $nh - [t/p]p = (nh - t) + (t - [t/p]p) \in [0, h+p)$. Therefore

$$\begin{aligned} P(x + X_{nh} - X_t \in B_{2a}) &\ge P(X_{nh} - X_t \in B_a) \\ &= P(X_{nh-[t/p]p} - X_{t-[t/p]p} \in B_a) \\ &\ge P(|X_{nh-[t/p]p}| < a/2, |X_{t-[t/p]p}| < a/2) \\ &\ge P\left(\sup_{s \in [0,p+h]} |X_s| < a/2 \right) > 1/2. \end{aligned}$$

Recalling the proof that (3) implies (4) in Lemma 3.2, we have $\sum_{n=1}^{\infty} P(X_{nh} \in B_{2a}) = \infty$. Thus, by Theorem 2.5 (ii) and (iv), $\{X_{nh}\}_{n \in \mathbb{Z}_+}$ is recurrent. □

Using Theorem 1.4, we prove Theorem 1.5.

Proof of Theorem 1.5. Let $\mu_t = \mathcal{L}(X_t)$ and $\mu_{s,t} = \mathcal{L}(X_t - X_s)$ for $0 \le s \le t$.

(i) Let us prove the "only if" part. By Theorem 1.4 (vii), if $\{X_t\}_{t \ge 0}$ is recurrent, then the random walk $\{X_{np}\}_{n \in \mathbb{Z}_+}$ is recurrent, and $\sum_{n=1}^{\infty} P(X_{np} \in B_a) = \infty$ for every $a > 0$ by Theorem 35.3 (ii) of Sato [5]. For every $n \in \mathbb{N}$,

$$\mathcal{L}(X_{np}) = \mu_{np} = \mu_p * \mu_{p,2p} * \cdots * \mu_{(n-1)p,np} = \mu_p^n = \mathcal{L}(Y_n)$$

by the definition of semi-Lévy process. Hence, we also have $\sum_{n=1}^{\infty} P(Y_n \in B_a) = \infty$ for every $a > 0$ and again by Theorem 35.3 (ii) of Sato [5], the random walk $\{Y_n\}_{n \in \mathbb{Z}_+}$ is recurrent. Thus $\{Y_t\}_{t \ge 0}$ is recurrent by Theorem 35.4 (vi) of Sato [5]. The "if" part can be proved in the same way.

(ii) By Theorem 1.4 (ii), (ii) follows from (i). □

4. An example

Recall the example of semi-Lévy process defined in Example 1.1. Let $\{X_t\}_{t \ge 0}$, $\{Y_t\}_{t \ge 0}$ and $\{Z_t\}_{t \ge 0}$ be as in Example 1.1. Then the following holds.

Proposition 4.1. *The recurrence of the semi-Lévy process $\{X_t\}_{t \ge 0}$ is equivalent to that of the Lévy process $\{Y_{qt} + Z_{(p-q)t}\}_{t \ge 0}$.*

Proof. By Theorem 1.5, the recurrence of semi-Lévy process $\{X_t\}_{t \ge 0}$ is reduced to that of the Lévy process $\{Y_{qt} + Z_{(p-q)t}\}_{t \ge 0}$, since $X_p = Y_q + Z_p - Z_q$. □

In the rest of this section we assume $d = 1$. Further, recall that if $\{L_t\}_{t \ge 0}$ is a Lévy process on $\mathbb{R}$ with $E[|L_1|] < \infty$, then

(4.1) $\{L_t\}_{t \ge 0}$ is recurrent if and only if $E[L_1] = 0$

(see Theorem 36.7 of Sato [5]). Now, as a consequence of Proposition 4.1 and (4.1), the following holds.

Proposition 4.2. *Assume $E[|Y_1|] < \infty$ and $E[|Z_1|] < \infty$. Then $\{X_t\}_{t\geq 0}$ is recurrent if and only if $E[Y_q + Z_{p-q}] = 0$.*

Due to Proposition 4.2 and (4.1), we have the following.

Corollary 4.3. *Assume $E[|Y_1|] < \infty$ and $E[|Z_1|] < \infty$.*

(i) *$\{Y_t\}_{t\geq 0}$ and $\{Z_t\}_{t\geq 0}$ are recurrent $\Leftrightarrow E[Y_1] = E[Z_1] = 0 \Rightarrow E[Y_q + Z_{p-q}] = 0 \Leftrightarrow$ $\{X_t\}_{t\geq 0}$ is recurrent.*

(ii) *$\{Y_t\}_{t\geq 0}$ is recurrent and $\{Z_t\}_{t\geq 0}$ is transient $\Leftrightarrow E[Y_1] = 0$ and $E[Z_1] \neq 0 \Rightarrow$ $E[Y_q + Z_{p-q}] \neq 0 \Leftrightarrow \{X_t\}_{t\geq 0}$ is transient.*

(iii) *$\{Y_t\}_{t\geq 0}$ and $\{Z_t\}_{t\geq 0}$ are transient $\Rightarrow \{X_t\}_{t\geq 0}$ is recurrent if $E[Y_q] = -E[Z_{p-q}]$ and it is transient if $E[Y_q] \neq -E[Z_{p-q}]$.*

Remark 4.4. (i) Let us remark that in the case when $E[|Y_1|] = \infty$, (ii) of Corollary 4.3 does not have to hold. For example, let $\{Y_t\}_{t\geq 0}$ be a symmetric Cauchy process and let $\{Z_t\}_{t\geq 0}$ be an arbitrary Lévy process with $E[|Z_1|] < \infty$. Recall that $\{Y_t\}_{t\geq 0}$ is recurrent (see Example 35.7 of Sato [5]) and $E[|Y_1|] = \infty$. Now, by Exercise 39.8 (i) and (iv) of Sato [5] and Proposition 4.1, $\{X_t\}_{t\geq 0}$ is recurrent.

(ii) A simple example which satisfies Corollary 4.3 (iii) is when $Y_t = Z_t = t$, in the transient case, and $Y_t = (p-q)t$ and $Z_t = -qt$, in the recurrent case.

5. Laws of large numbers

We conclude this paper by discussing laws of large numbers for semi-Lévy processes. We need the following lemma.

Lemma 5.1. *Let $\{X_t\}_{t\geq 0}$ be an additive process on $\mathbb{R}$ and let $T > 0$. If $E[|X_t|] < \infty$ and $E[X_t] = 0$ for all $t \leq T$, then $E[\sup_{t\in[0,T]} |X_t|] \leq 8E[|X_T|]$.*

Proof. This lemma follows from Theorem 5.1 in Chapter VII of Doob [2] and the right continuity of the sample paths of additive processes. □

Theorem 5.2 (Strong law of large numbers)**.** *Let $\{X_t\}_{t\geq 0}$ be a semi-Lévy process on $\mathbb{R}^d$ with period $p > 0$.*

(i) *If $E[|X_p|] < \infty$, then $\lim_{t\to\infty} t^{-1} X_t = p^{-1} E[X_p]$ a.s.*

(ii) *If* $E[|X_p|] = \infty$*, then* $\limsup_{t\to\infty} t^{-1}|X_t| = \infty$ *a.s.*

Proof. (i) Obviously, it is sufficient to prove the assertion for each component of the process $\{X_t\}_{t\geq 0}$ on $\mathbb{R}^d$, and hence we assume $d = 1$. We use the Lévy-Khintchine representation of the characteristic function of X_t in the following form:

$$E[e^{izX_t}] = \exp\left\{-\frac{1}{2}a_t z^2 + i\gamma_t z + \int_{\mathbb{R}} \left(e^{izx} - 1 - izx\mathbb{1}_{[-1,1]}(x)\right)\nu_t(dx)\right\},\ z \in \mathbb{R},$$

where $a_t \geq 0$, $\gamma_t \in \mathbb{R}$ and ν_t is the Lévy measure of X_t for $t \geq 0$. Note that, if $E[|X_p|] < \infty$, then $E[|X_t|] < \infty$ for all $t \geq 0$. Indeed we have $\int_{|x|>1} |x|\nu_t(dx) \leq \int_{|x|>1} |x|\nu_p(dx) < \infty$ for $t \leq p$, because $\nu_t(B) \leq \nu_p(B), B \in \mathcal{B}(\mathbb{R})$, for $t \leq p$ by Theorem 9.8 of Sato [5], and we have $E[|X_t|] < \infty$ for $t \leq p$. Hence, for $t > 0$, $E[|X_t|] = E[|X_{[t/p]p+(t-[t/p]p)}|] \leq [t/p]E[|X_p|] + E[|X_{t-[t/p]p}|] < \infty$ by Proposition 2.1 of Maejima and Sato [4]. Let $Y_n = \sup_{t\in[0,p]} |X_{t+np} - X_{np}|$. Since $\{X_{t+np} - X_{np}\}_{t\geq 0}$ is a right-continuous process having the same finite-dimensional distributions as $\{X_t\}_{t\geq 0}$ for each $n \in \mathbb{N}$, we have $Y_n \overset{\mathrm{d}}{=} \sup_{t\in[0,p]} |X_t|$ for every $n \in \mathbb{N}$. By the independent increment property of $\{X_t\}_{t\geq 0}$, $Y_n, n \in \mathbb{N}$, are independent. Also, we have $E[X_t] = \int_{|x|>1} x\nu_t(dx) + \gamma_t$. Once again by Theorem 9.8 of Sato [5], $t \mapsto \gamma_t$ is continuous, $\nu_s(B) \leq \nu_t(B), B \in \mathcal{B}(\mathbb{R})$, for $s \leq t$, and $\nu_s(B) \to \nu_t(B)$ as $s \to t$ for $B \in \mathcal{B}(\mathbb{R})$ satisfying $B \subset \{x\colon |x| > \varepsilon\}$ with some $\varepsilon > 0$. By the Radon-Nikodym theorem, for $t \leq t_0$, $\nu_t(dx) = g_t(x)\nu_{t_0}(dx)$ with some nonnegative-valued measurable function g_t. This g_t satisfies that $g_s \leq g_t \leq 1$ ν_{t_0}-a.e. for $s \leq t \leq t_0$. Also, for any $\varepsilon > 0$, by the dominated convergence theorem, $\int_B \lim_{s\to t} g_s(x)\nu_{t_0}(dx) = \lim_{s\to t}\int_B g_s(x)\nu_{t_0}(dx) = \lim_{s\to t}\nu_s(B) = \nu_t(B) = \int_B g_t(x)\nu_{t_0}(dx)$ for $B \in \mathcal{B}(\mathbb{R})$ satisfying $B \subset \{x\colon |x| > \varepsilon\}$, which yields that $\lim_{s\to t} g_s(x) = g_t(x)$ ν_{t_0}-a.e. $x \in \mathbb{R}$. Then by the dominated convergence theorem again, for $t < t_0$, $\lim_{(0,t_0]\ni s\to t}\int_{|x|>1} x\nu_s(dx) = \lim_{(0,t_0]\ni s\to t}\int_{|x|>1} xg_s(x)\nu_{t_0}(dx) = \int_{|x|>1} xg_t(x)\nu_{t_0}(dx) = \int_{|x|>1} x\nu_t(dx)$. Thus $t \mapsto \int_{|x|>1} x\nu_t(dx)$ is continuous. Therefore $t \mapsto E[X_t]$ is continuous. Hence $\{X_t - E[X_t]\}_{t\geq 0}$ is an additive process with $E[X_t - E[X_t]] = 0$ and $E[\sup_{t\in[0,p]} |X_t - E[X_t]|] \leq 8E[|X_p - E[X_p]|] \leq 16E[|X_p|]$ by Lemma 5.1. Also, $\sup_{t\in[0,p]} |E[X_t]| < \infty$ by the continuity of $t \mapsto E[X_t]$. Thus

$$\begin{aligned} E[|Y_1|] = E\left[\sup_{t\in[0,p]} |X_t|\right] &\leq E\left[\sup_{t\in[0,p]} |X_t - E[X_t]|\right] + \sup_{t\in[0,p]} |E[X_t]| \\ &\leq 16E[|X_p|] + \sup_{t\in[0,p]} |E[X_t]| < \infty. \end{aligned}$$

Applying the strong law of large numbers to $\sum_{k=1}^{n} Y_k$, we have $n^{-1}Y_n = n^{-1}\sum_{k=1}^{n} Y_k - n^{-1}\sum_{k=1}^{n-1} Y_k \to E[Y_1]-E[Y_1] = 0$ as $n \to \infty$ with probability 1. Also, since $\{X_{np}\}_{n\in\mathbb{Z}_+}$ is a random walk, $\lim_{n\to\infty} n^{-1}X_{np} = E[X_p]$ a.s. Then we have, almost surely,

$$\left|\frac{X_t}{t} - \frac{E[X_p]}{p}\right| \le \left|\frac{X_{[t/p]p}}{[t/p]}\frac{[t/p]}{t} - \frac{E[X_p]}{p}\right| + \frac{Y_{[t/p]}}{[t/p]}\frac{[t/p]}{t} \to 0$$

as $t \to \infty$.

(ii) If $E[|X_p|] = \infty$, then $\limsup_{n\to\infty} n^{-1}|X_{np}| = \infty$ a.s. Thus $\limsup_{t\to\infty} t^{-1}|X_t| = \infty$ a.s. □

Finally, we prove the weak law of large numbers for semi-Lévy processes.

Theorem 5.3 (Weak law of large numbers). *Let $\{X_t\}_{t\ge 0}$ be a semi-Lévy process on $\mathbb{R}^d$ with period $p > 0$ and let $c \in \mathbb{R}^d$. Then the following are equivalent.*

(i) $t^{-1}X_t \to c$ *as* $t \to \infty$ *in probability.*

(ii) $\lim_{t\to\infty} tP(|X_p| > t) = 0$ *and* $\lim_{t\to\infty} E[X_p \mathbb{1}_{(0,t]}(|X_p|)] = cp$.

Proof. (i)⇒(ii). Since $\{X_{np}\}_{n\in\mathbb{Z}_+}$ is a random walk and (i) implies that $n^{-1}X_{np} \to cp$ as $n \to \infty$ in probability, we have $\lim_{t\to\infty} tP(|X_p| > t) = 0$ and $\lim_{t\to\infty} E[X_p \mathbb{1}_{(0,t]}(|X_p|)] = cp$ by Theorem 36.4 of Sato [5].

(ii)⇒(i). The statement (ii) implies $n^{-1}X_{np} \to cp$ as $n \to \infty$ in probability. Hence

$$\frac{X_{[t/p]p}}{[t/p]}\frac{[t/p]}{t} \to c$$

in probability. Also,

$$\frac{|X_t - X_{[t/p]p}|}{t} \overset{\mathrm{d}}{=} \frac{|X_{t-[t/p]p}|}{t} \le \frac{\sup_{s\in[0,p]} |X_s|}{t} \overset{\text{a.s.}}{\to} 0,$$

implying that $t^{-1}(X_t - X_{[t/p]p}) \to 0$ in probability. Hence

$$\frac{X_t}{t} = \frac{X_{[t/p]p}}{[t/p]}\frac{[t/p]}{t} + \frac{X_t - X_{[t/p]p}}{t} \to c$$

in probability. □

Remark 5.4. Recently, the notion of mean of infinitely divisible distributions has been generalized to that of weak mean (see Sato [6]). More precisely, let μ be an infinitely divisible distribution with Lévy measure ν. If the limit $\lim_{a\to\infty}\int_{1<|x|\le a} x\nu(dx)$ exists in $\mathbb{R}^d$, then there exists $c \in \mathbb{R}^d$ such that $\widehat{\mu}(z)$ can be represented as

$$\widehat{\mu}(z) = \exp\left\{-\frac{1}{2}\langle z, Az\rangle + \lim_{a\to\infty}\int_{|x|\le a}\left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle\right)\nu(dx) + i\langle c, z\rangle\right\},$$

where A is the Gaussian covariance matrix of μ. The vector c is called the weak mean of μ. If we use the notion of weak mean, by Theorem 5.2 of Sato and Ueda [8], the statement (ii) of Theorem 5.3 is equivalent to that

$$X_p \text{ has weak mean } cp \text{ and } \lim_{t\to\infty} t \int_{|x|>t} \nu_p(dx) = 0,$$

where ν_p is the Lévy measure of X_p.

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References

[1] Becker-Kern, P. (2004), Random integral representation of operator-semi-self-similar processes with independent increments. *Stoch. Proc. Appl.* **109**, 327–344.

[2] Doob, J.L. (1953). *Stochastic Processes.* Wiley, New York.

[3] Maejima, M. and Sato, K. (1999), Semi-selfsimilar processes. *J. Theoret. Probab.* **11**, 347–373.

[4] Maejima, M. and Sato, K. (2003), Semi-Lévy processes, semi-selfsimilar additive processes, and semi-stationary Ornstein-Uhlenbeck type processes. *J. Math. Kyoto Univ.* **43**, 609–639.

[5] Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge.

[6] Sato, K. (2010). Fractional integrals and extensions of selfdecomposability. Lecture Notes in Mathematics 2001, Lévy matters I. 1-91, Springer.

[7] Sato. K. and Yamamuro, K. (1998). On selfsimilar and semi-selfsimilar processes with independent increments. *J. Korean Math. Soc.* **35**, 207–224.

[8] Sato, K. and Ueda, Y. (2012). Weak drifts of infinitely divisible distributions and their applications. To appear in *J. Theoret. Probab.* DOI: 10.1007/s10959-012-0419-2