Semigroups of Upsilon Transformations

Ole E. Barndorff-Nielsen
Thiele Centre
Department of Mathematical Sciences
University of Aarhus

Makoto Maejima
Department of Mathematics
Keio University

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Abstract

Upsilon transformations satisfying certain regularity conditions are shown to generate semigroups of such transformations. This is based on a general commutativity property of the Upsilon transformations, and uses log infinite divisibility. The existence of random integral representations of Upsilon transformations and of the generated semigroups is also discussed.

Key words: Infinite divisibility; random integral representation.

1 Introduction

Υ (or Upsilon) transformations are a special type of transformations that send Lévy measures to Lévy measures. Closely associated to these are mappings of the set $ID\left(\mathbb{R}^d\right)$ of infinitely divisible laws on $\mathbb{R}^d$ into itself. Furthermore, in many cases, the mappings have a stochastic interpretation in terms of random integrals with respect to Lévy processes.

Various special cases of Υ transformations have been studied in [BNTh02a], [BNTh02b], [BNTh02c], [BNTh04], [BNTh05], [BNTh06a], [BNTh06b], [BNMaSa06], [BNPA06], [BNLi06] and general formulations are given in [BNRoTh07]. The early works cited above arose out of a study of the connection between free and classical probability. That early work was, however, done without the authors being aware of earlier related work by [Ju90] which, at least partly, had its root in studies of selfdecomposability, cf. [Ju85], [Ju88], [JuVer83], and [SaYa84].

In wide generality the Υ transformations are smooth one-to-one transformations with absolutely continuous Lévy measures as images. Moreover, the image measures often have important monotonicity properties.

In the present paper, restricting attention to the case of one-dimensional measures and distributions, we introduce and discuss semigroups $\{\Upsilon_t\}_{t\geq 0}$ of Υ transformations that, in considerable generality, are representable in terms of random integrals with respect to Lévy processes and which interpolate smoothly between some previously studied iterates of specific Υ transformations.

The semigroups of Upsilon transformations are defined and discussed in Section 3, while Section 2 presents a number of relevant background results and examples of Upsilon transformations.

For comprehensive information on the theory of Upsilon transformations and on the relation to random integrals of deterministic functions with respect to Lévy processes, see [BNRoTh07], [Sa06a] and [Sa07], and references given there.

2 Background on Upsilon transformations

We denote the sets of σ-finite Borel measures on $\mathbb{R}$ and on $\mathbb{R}_+ (= (0,\infty))$ by $\mathfrak{M}$ and $\mathfrak{M}^+$, respectively, while $\mathfrak{M}_L$ and $\mathfrak{M}_L^+$ will stand for the sets of Lévy measures in $\mathfrak{M}$ and the set of Lévy measures corresponding to positive infinitely divisible laws, i.e.

$$\mathfrak{M}_L^+ = \left\{ \rho \in \mathfrak{M}^+ : \int_0^\infty (x \wedge 1)\rho\,(\mathrm{d}x) < \infty \right\}.$$

For any γ in $\mathfrak{M}^+$, let $\Upsilon_\gamma : \rho \to \Upsilon_\gamma(\rho)$ be the transformation defined on $\mathfrak{M}$ such that, for any Borel set A of $\mathbb{R}$,

$$\Upsilon_\gamma(\rho)(A) = \int_0^\infty \rho\left(\xi^{-1}A\right)\gamma(\mathrm{d}\xi)\,. \tag{1}$$

We shall also use the notation ρ_γ for $\Upsilon_\gamma(\rho)$. This transformation is called an *Upsilon transformation* and the measure γ is referred to as the *dilation measure*. In case γ is absolutely continuous with density g we sometimes write Υ_g and ρ_g for Υ_γ and ρ_γ, respectively.

The image $\Upsilon_\gamma(\rho)$ is a, not necessarily finite, measure on $\mathbb{R}$. We define the *Lévy domain* of Υ_γ to be the set

$$\mathrm{dom}_L\Upsilon_\gamma = \{\rho \in \mathfrak{M}_L : \Upsilon_\gamma(\rho) \in \mathfrak{M}_L\}$$

and shall also refer to the *positive Lévy domain* as given by

$$\mathrm{dom}_L^+\Upsilon_\gamma = \left\{\rho \in \mathfrak{M}_L^+ : \Upsilon_\gamma(\rho) \in \mathfrak{M}_L^+\right\}$$

Thus $\mathrm{dom}_L\Upsilon_\gamma \subset \mathfrak{M}_L$ and $\mathrm{dom}_L^+\Upsilon_\gamma \subset \mathfrak{M}_L^+$.

Theorem 1 *We have*

$$\mathrm{dom}_L\Upsilon_\gamma = \mathfrak{M}_L \tag{2}$$

if and only if

$$\int_0^\infty (\xi^2 + 1)\gamma(\mathrm{d}\xi) < \infty. \tag{3}$$

Furthermore,

$$\mathrm{dom}_L^+\Upsilon_\gamma = \mathfrak{M}_L^+ \tag{4}$$

if and only if

$$\int_0^\infty (\xi + 1)\gamma(\mathrm{d}\xi) < \infty. \tag{5}$$

A proof of the equivalence of (4) and (5) was given in [Sa05], and a statement similar to the equivalence of (2) and (3) was also proved. For a detailed verification of Theorem 1, see [BNRoTh07].

Note that in both of the cases considered in Theorem 1, the measure γ is necessarily finite.

Example 1 *For any* λ *in* $(-1,\infty)$*, let*

$$g(x) = x^{\lambda-1}\mathrm{e}^{-x}, \qquad x \in (0,\infty).$$

The corresponding Upsilon mappings Υ_λ *were introduced and studied in [BNTh04] (for* $\lambda = 1$*), [Sa05] and [BNPA06]; see moreover [Sa06b]. Here*

$$\mathrm{dom}_L \Upsilon_\lambda = \begin{cases} \mathfrak{M}_L & \text{if } \lambda > 0 \\ \mathfrak{M}_{\log} & \text{if } \lambda = 0 \\ \mathfrak{M}_\lambda & \text{if } \lambda \in (-1,0) \end{cases}$$

where the classes $\mathfrak{M}_{\log}(\mathbb{R})$ *and* $\mathfrak{M}_\lambda(\mathbb{R})$*,* $\lambda \in (-1,0)$*, are defined by:*

$$\mathfrak{M}_{\log} = \left\{\rho \in \mathfrak{M}(\mathbb{R}) \mid \int_1^\infty \log y \, \rho(\mathrm{d}y) < \infty\right\}$$

and

$$\mathfrak{M}_\lambda = \left\{\rho \in \mathfrak{M}(\mathbb{R}) \mid \int_1^\infty y^{-\lambda} \, \rho(\mathrm{d}y) < \infty\right\},$$

respectively. We also note that when $\lambda = 1$ *the range* $\Upsilon(\mathfrak{M}_L)$ *equals the family* $\mathfrak{B}$ *of Lévy measures of the Goldie-Steutel-Bondesson class of infinitely divisible distributions, cf. [BNMaSa06]. For an extension to Upsilon transformations of Lévy measures on the cone of positive definite matrices, see [BNPA06].*

Various other examples are discussed in [Ju90], [BNMaSa06] and [BNTh06b].

Henceforth, in all cases we assume that γ has finite upper tail measure i.e., letting

$$\varepsilon_\gamma(\xi) = \gamma([\xi,\infty)),$$

we require that $\varepsilon_\gamma(\xi) < \infty$ for all $\xi > 0$. Then

$$\gamma(\mathrm{d}\xi) = -\mathrm{d}\varepsilon_\gamma(\xi). \tag{6}$$

The inverse function of ε_γ, denoted ε_γ^*, is defined by

$$\varepsilon_\gamma^*(x) = \inf\{\xi > 0 : \varepsilon_\gamma(\xi) \le x\}. \tag{7}$$

Note that both functions $\xi \to \varepsilon_\gamma(\xi)$ and $x \to \varepsilon_\gamma^*(x)$ are decreasing and càglàd. In case γ is absolutely continuous with density g, we also write ε_g for ε_γ and ε_g^* for ε_γ^*.

Given a $\gamma \in \mathfrak{M}^+$ and a Lévy measure ρ, consider the random integral

$$Y = \int_0^{\varepsilon_\gamma(0)} \varepsilon_\gamma^*(s)\,\mathrm{d}Z_s, \tag{8}$$

where $Z = \{Z_s\}$ is the Lévy process for which the cumulant function of Z_1 is given by

$$\mathrm{C}_\rho(z) = \mathrm{i}\eta z + \int_{\mathbb{R}} \left(e^{\mathrm{i}zt} - 1 - \mathrm{i}zt\left(1+t^2\right)^{-1}\right)\rho(\mathrm{d}t) \tag{9}$$

for some $\eta \in \mathbb{R}$.

We say that (8) is a *random integral representation* (RIR) of Υ_γ at $\rho \in \mathrm{dom}_L\Upsilon_\gamma$ provided the integral (8) exists as the limit in probability of the Riemann sums and the random variable

Y (which is then necessarily infinitely divisible) has the Lévy measure $\rho_\gamma = \Upsilon_\gamma(\rho)$ and the cumulant function

$$\mathrm{C}_{\rho_\gamma}(z) = \mathrm{i}\tilde{\eta}z + \int_{\mathbb{R}} \left(e^{\mathrm{i}zt} - 1 - \mathrm{i}zt\left(1+t^2\right)^{-1}\right)\rho_\gamma(\mathrm{d}t) \tag{10}$$

where

$$\tilde{\eta} = \int_0^\infty x\left(\eta + \int_{\mathbb{R}} y\left(\left(1+(xy)^2\right)^{-1} - \left(1+y^2\right)^{-1}\right)\rho(\mathrm{d}y)\right)\gamma(\mathrm{d}x).$$

Note that, in this case, the cumulant function $\mathrm{C}\{z \ddagger Y\}$ of Y is expressible in terms of C_ρ and ε^*_γ as

$$\mathrm{C}\{z \ddagger Y\} = \int_0^{\varepsilon_\gamma(0)} \mathrm{C}_\rho\left(\varepsilon^*_\gamma(s)z\right)\mathrm{d}s. \tag{11}$$

The RIR property holds, in particular, for all ρ in $\mathrm{dom}_L\Upsilon_\gamma$ provided ε^*_γ is continuous and γ is a probability measure with finite second moment. This follows by direct extension of the proof given by [BNTh04] in the special case $\gamma(\mathrm{d}x) = e^{-x}\mathrm{d}x$. Extensions and ramifications are also discussed in [BNTh06b], [Sa05], [Sa06a] and [Sa07].

The following proposition gives a set of simple sufficient conditions for existence of the random integral (8) in terms of the dilation measure γ. The proof, which relies on a result from the general theory developed in [Sa06a], is given in the Appendix.

Proposition 1 *The following conditions are sufficient for the existence of the random integral in (8):*

$$\int_0^\infty (\xi+\xi^2)\gamma(\mathrm{d}\xi) < \infty, \tag{12}$$

$$\int_0^a \xi^2\gamma(\mathrm{d}\xi) = O(a^2) \quad \textit{as } a \downarrow 0 \tag{13}$$

$$\int_0^a \xi\gamma(\mathrm{d}\xi) = O(a) \quad \textit{as } a \downarrow 0 \tag{14}$$

and

$$\int_{|x|>1} \gamma([1/|x|,1])\rho(\mathrm{d}x) < \infty. \tag{15}$$

Remark 1 *The measures γ in Example* 1 *satisfy the conditions in Proposition* 1*.*

Remark 2 *The conditions in Proposition* 1 *are only sufficient for the existence of (8). However, they are satisfied for many examples found in the literature. To illustrate that the conditions are not necessary, consider the case*

$$\gamma(\mathrm{d}x) = x^{\lambda-1}1_{(0,1]}(x)\mathrm{d}x.$$

When $\lambda \in (-2,-1)$, the random integral (8) exists if and only if $\int_{|x|>1}|x|^\lambda\rho(\mathrm{d}x) < \infty, E[Z_1] = 0$, see [Sa06b], Theorem 2.4*. But, (12) and (14) are not satisfied for this γ.*

Next we recall some results from [BNRoTh07]. Direct calculation shows that the Upsilon transformations commute, that is

$$\Upsilon_\eta \circ \Upsilon_\gamma = \Upsilon_\gamma \circ \Upsilon_\eta.$$

Furthermore, introducing the product convolution $\circledast$ of measures $\eta, \gamma \in \mathfrak{M}^+$, defined by

$$(\eta \circledast \gamma)(\mathrm{d}x) := \int_0^\infty \eta(\xi^{-1}\mathrm{d}x)\,\gamma(\mathrm{d}\xi), \tag{16}$$

one finds that $\eta \circledast \gamma = \Upsilon_\gamma(\eta)$ and

$$\Upsilon_\eta \circ \Upsilon_\gamma = \Upsilon_\gamma \circ \Upsilon_\eta = \Upsilon_{\eta \circledast \gamma}. \tag{17}$$

The latter formula may equivalently be expressed as $\varepsilon_{\gamma \circledast \eta} = \varepsilon_\gamma \circledast \varepsilon_\eta$.

Note that there is no guarantee that the measure $\eta \circledast \gamma$ is finite. However, if γ and η are probability measures then so is $\eta \circledast \gamma$. Note also that the operation $\circledast$ is converted into ordinary convolution by exponential transformation.

3 Upsilon semigroups

Henceforth the dilation measures considered are assumed to be probability measures.

Let γ be such a measure, let $\acute{\gamma}$ be the probability measure obtained from γ by the transformation $\xi \to e^x$, i.e. $\acute{\gamma}\,(\mathrm{d}x) = \gamma\,(e^x \mathrm{d}x)$, and suppose that $\acute{\gamma}$ is infinitely divisible. We then say that γ is *log infinitely divisible*. Let V be a Lévy process such that V_1 has law $\acute{\gamma}$, let $\acute{\gamma}_t$ and γ_t denote the law of V_t and $U_t = e^{V_t}$, respectively, and write ρ_t and Υ_t as a shorthand for ρ_{γ_t} and Υ_{γ_t}; then

$$\rho_t\,(\mathrm{d}x) = \int_0^\infty \rho\left(\xi^{-1}\mathrm{d}x\right)\gamma_t\,(\mathrm{d}\xi)\,. \tag{18}$$

Given such a γ, we shall, for brevity, write ε_t for ε_{γ_t} and ε_t^* for $\varepsilon_{\gamma_t}^*$.

Theorem 2 *We have*

$$\Upsilon_{t+s} = \Upsilon_t \circ \Upsilon_s, \quad t, s \geq 0 \tag{19}$$

i.e. the family $\{\Upsilon_t\}$ *constitutes a semigroup.*

Proof. By (17), $\Upsilon_t \circ \Upsilon_s = \Upsilon_{\gamma_t \circledast \gamma_s}$. Thus, to verify (19) it is enough to show that

$$\gamma_t \circledast \gamma_s = \gamma_{t+s}. \tag{20}$$

Now, for any Borel set A in $\mathbb{R}_+$,

$$\begin{aligned}
(\gamma_t \circledast \gamma_s)(A) &= \int_0^\infty \gamma_t(u^{-1}A)\gamma_s(\mathrm{d}u) \\
&= \int_0^\infty P(e^{V_t} \in u^{-1}A)P(e^{V_s} \in \mathrm{d}u) \\
&= \int_{-\infty}^\infty P(V_t \in \log A - \log u)P(V_s \in \mathrm{d}(\log u)) \\
&= P(V_{t+s} - V_t + V_t \in \log A) \\
&= P(V_{t+s} \in \log A) \\
&= P(e^{V_{t+s}} \in A) \\
&= \gamma_{t+s}(A),
\end{aligned}$$

proving (20). ■

Let $\{\Upsilon_t\}$ denote the semigroup of Upsilon transformations generated by a log infinitely divisible probability measure γ. We proceed to derive a set of conditions ensuring, in particular, that Υ_t has the random integral representation, for all $t > 0$. When this is the case we speak of $\{\Upsilon_t\}$ as an *Upsilon semigroup*.

Note that in the present setting, where $\varepsilon_t(0) = 1$ for all t, (8) becomes

$$Y_t = \int_0^1 \varepsilon_t^*(s)\,\mathrm{d}Z_s. \tag{21}$$

Theorem 3 *Suppose the log infinitely divisible probability measure γ has finite second moment. Then $\mathrm{dom}_L\Upsilon_t = \mathfrak{M}_L$ for all $t \geq 0$, the subsets $\Upsilon_t(\mathfrak{M}_L)$ of $\mathfrak{M}_L$ are decreasing in t, and $\{\Upsilon_t\}$ is an Upsilon semigroup.*

Proof. The first assertion follows from Theorem 1 and the fact that existence of g-moments for submultiplicative measurable function g (here $g(x) = e^x$) is a time independent property of Lévy processes (cf. [Sa99], Theorem 25.3). The second assertion is a consequence of Theorem 2, on noting that

$$\Upsilon_t(\mathfrak{M}_L) = \Upsilon_s \circ \Upsilon_{t-s}(\mathfrak{M}_L) = \Upsilon_s(\Upsilon_{t-s}(\mathfrak{M}_L)) \subset \Upsilon_s(\mathfrak{M}_L),$$

where we have used that $\Upsilon_t(\mathfrak{M}_L) \subset \mathfrak{M}_L$, since $\mathrm{dom}_L\Upsilon_t = \mathfrak{M}_L$. As to the final assertion, since γ is continuous the same holds for $\acute{\gamma}_1$ and hence, by [Sa99] ((b) (1) on page 146), the same is true of $\acute{\gamma}_t$ for all $t > 0$. This in turn implies that ε_t^* is continuous for all $t > 0$. Thus, by the remark following (11), it is enough to show that γ_t has second moment, i.e.

$$\int_{\mathbb{R}} x^2\gamma_t(\mathrm{d}x) = \int_{\mathbb{R}} e^{2x}\acute{\gamma}_t(\mathrm{d}x) < \infty. \tag{22}$$

But, by [Sa99] Proposition 25.3, the latter is the case if and only if it holds for $t = 1$ and is thus implied by the assumption that γ has second moment. ■

Corollary 1 *Suppose that γ is log infinitely divisible with no point mass and that the Lévy measure $\acute{\nu}$ of $\acute{\gamma}$ satisfies*

$$\int_{\{|x|\geq 1\}} e^{2x}\acute{\nu}(\mathrm{d}x) < \infty. \tag{23}$$

Then $\{\Upsilon_t\}$ is an Upsilon semigroup.

Proof. By [Sa99] Proposition 25.3, relation (23) is a necessary and sufficient condition for the validity of (22). ■

Example 2 *Let γ be the log normal law. The conditions of Theorem 3 are satisfied and consequently γ generates an Upsilon semigroup $\{\Upsilon_t\}$, with random integral representation (21).*

Example 3 *Let γ be the gamma law with probability density $g(\xi) = \Gamma(\lambda)^{-1}\xi^{\lambda-1}e^{-\xi}$. The log gamma distribution is infinitely divisible, with the Lévy density*

$$\acute{\nu}(\mathrm{d}x) = |x|^{-1}e^{\lambda x}(1-e^x)^{-1}1_{(-\infty,0)}(x)\,\mathrm{d}x$$

(*see, for instance, [Sa99], Example* 18.19). *Thus condition (23) holds, implying that γ generates an Upsilon semigroup, with random integral representation (21). (Note that the law γ_t is not a gamma distribution unless $t = 1$.)*

4 Appendix

Here we give the proof of Proposition 1.

The following proposition is a direct consequence of the general results in [Sa06a].

Proposition 2 *The existence of the random integral (8) is assured under the following conditions:*

$$\int_0^\infty \varepsilon_\gamma^*(s)^2 \mathrm{d}s < \infty, \tag{24}$$

$$\int_0^\infty \mathrm{d}s \int_{\mathbb{R}} (|\varepsilon_\gamma^*(s)x|^2 \wedge 1)\rho(\mathrm{d}x) < \infty \tag{25}$$

and

$$\int_0^\infty \left(1 + \left| \int_{\mathbb{R}} x \left(\frac{1}{1+|\varepsilon_\gamma^*(s)x|^2} - \frac{1}{1+|x|^2} \right) \rho(\mathrm{d}x) \right| \right) \varepsilon_\gamma^*(s)\mathrm{d}s < \infty. \tag{26}$$

The conditions stated in Proposition 1 imply (24)-(26), as we now show.

Condition (12) entails (24), since

$$\int_0^\infty \varepsilon_\gamma^*(s)^2 \mathrm{d}s = \int_0^\infty \xi^2 \gamma(\mathrm{d}\xi) < \infty.$$

As to (25), we have

$$\begin{aligned} &\int_0^\infty \mathrm{d}s \int_{\mathbb{R}} (|\varepsilon_\gamma^*(s)x|^2 \wedge 1)\rho(\mathrm{d}x) \\ &\quad = -\int_0^\infty d\varepsilon_\gamma(\xi) \int_{\mathbb{R}} (|\xi x|^2 \wedge 1)\rho(\mathrm{d}x) \\ &\quad = \int_0^\infty \gamma(\mathrm{d}\xi) \left(\int_{|x|\le 1/\xi} |\xi x|^2 \rho(\mathrm{d}x) + \int_{|x|>1/\xi} \rho(\mathrm{d}x) \right) \\ &\quad =: I_1 + I_2, \end{aligned}$$

say. Here

$$\begin{aligned} I_1 &= \int_{\mathbb{R}} |x|^2 \rho(\mathrm{d}x) \int_0^{1/|x|} \xi^2 \gamma(\mathrm{d}\xi) \\ &= \left(\int_{|x|\le 1} + \int_{|x|>1} \right) |x|^2 \rho(\mathrm{d}x) \int_0^{1/|x|} \xi^2 \gamma(\mathrm{d}\xi) \\ &=: I_{11} + I_{12}, \end{aligned}$$

say, and

$$\begin{aligned} I_{11} &\le \int_{|x|\le 1} |x|^2 \rho(\mathrm{d}x) \int_0^\infty \xi^2 \gamma(\mathrm{d}\xi) < \infty \quad \text{(by (12))}, \\ I_{12} &= \int_{|x|>1} |x|^2 \rho(\mathrm{d}x) \int_0^{1/|x|} \xi^2 \gamma(\mathrm{d}\xi) < \infty \quad \text{(by (13))}. \end{aligned}$$

Also,

$$I_2 = \int_{\mathbb{R}} \rho(\mathrm{d}x) \int_{1/|x|}^{\infty} \gamma(\mathrm{d}\xi) = \left(\int_{|x|\le 1} + \int_{|x|>1} \right) \rho(\mathrm{d}x) \int_{1/|x|}^{\infty} \gamma(\mathrm{d}\xi)$$
$$=: I_{21} + I_{22},$$

say, and

$$I_{21} \le \int_{|x|\le 1} |x|^2 \rho(\mathrm{d}x) \int_0^{\infty} \xi^2 \gamma(\mathrm{d}\xi) < \infty \quad \text{(by (12))},$$
$$I_{22} = \int_{|x|>1} \rho(\mathrm{d}x) \int_{1/|x|}^{\infty} \gamma(\mathrm{d}\xi) = \int_{|x|>1} \gamma([1/|x|,\infty))\rho(\mathrm{d}x) < \infty, \quad \text{(by (15))}.$$

This shows (25). For (26), we have

$$\int_0^{\infty} \left(1 + \left| \int_{\mathbb{R}} x \left(\frac{1}{1+|\varepsilon_\gamma^*(s)x|^2} - \frac{1}{1+|x|^2} \right) \rho(\mathrm{d}x) \right| \right) \varepsilon_\gamma^*(s)\mathrm{d}s$$
$$\le -\int_0^{\infty} \xi \mathrm{d}\varepsilon_\gamma(\xi) - \int_0^{\infty} \left| \xi \int_{\mathbb{R}} x \left(\frac{1}{1+|\xi x|^2} - \frac{1}{1+|x|^2} \right) \rho(\mathrm{d}x) \right| \mathrm{d}\varepsilon_\gamma(\xi)$$
$$=: I_3 + I_4,$$

say, where

$$I_3 = \int_0^{\infty} \xi \gamma(\mathrm{d}\xi) < \infty, \quad \text{(by (12))}$$
$$I_4 \le \int_0^{\infty} \xi \gamma(\mathrm{d}\xi) \left| \int_{\mathbb{R}} \left(\frac{x|x|^2|\xi^2-1|}{(1+|\xi x|^2)(1+|x|^2)} \right) \rho(\mathrm{d}x) \right|$$
$$\le \int_0^{\infty} \xi|\xi^2-1|\gamma(\mathrm{d}\xi) \int_{\mathbb{R}} \frac{|x|^3}{(1+|\xi x|^2)(1+|x|^2)} \rho(\mathrm{d}x)$$
$$= \int_0^{\infty} \xi|\xi^2-1|\gamma(\mathrm{d}\xi) \left(\int_{|x|\le 1} + \int_{|x|>1} \right) \frac{|x|^3}{(1+|\xi x|^2)(1+|x|^2)} \rho(\mathrm{d}x)$$
$$=: I_{41} + I_{42},$$

say. Here

$$I_{41} = \int_0^{\infty} \xi|\xi^2-1|\gamma(\mathrm{d}\xi) \int_{|x|\le 1} \frac{|x|^3}{1+|x|^2} \rho(\mathrm{d}x)$$
$$= \left(\int_0^1 + \int_1^{\infty} \right) \xi|\xi^2-1|\gamma(\mathrm{d}\xi) \int_{|x|\le 1} \frac{|x|^3}{1+|x|^2} \rho(\mathrm{d}x)$$
$$=: I_{411} + I_{412},$$

say. We have

$$I_{411} \le \int_0^1 \xi \gamma(\mathrm{d}\xi) \int_{|x|\le 1} |x|^2 \rho(\mathrm{d}x) < \infty, \quad \text{(by (12))}$$

and

$$
\begin{aligned}
I_{412} &\le \int_1^\infty (\xi^2+1)\gamma(\mathrm{d}\xi)\int_{|x|\le 1}\frac{|\xi x||x|^2}{1+|\xi x|^2}\rho(\mathrm{d}x)\\
&\le \int_1^\infty (\xi^2+1)\gamma(\mathrm{d}\xi)\int_{|x|\le 1}|x|^2\rho(\mathrm{d}x)<\infty, \quad \text{(by (12))}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
I_{42} &= \int_{|x|>1}\frac{|x|^3}{1+|x|^2}\rho(\mathrm{d}x)\int_0^\infty \frac{\xi|\xi^2-1|}{1+|\xi x|^2}\gamma(\mathrm{d}\xi)\\
&= \int_{|x|>1}\frac{|x|^3}{1+|x|^2}\rho(\mathrm{d}x)\left(\int_0^1+\int_1^\infty\right)\frac{\xi|\xi^2-1|}{1+|\xi x|^2}\gamma(\mathrm{d}\xi)\\
&=: I_{421}+I_{422},
\end{aligned}
$$

say. Furthermore,

$$
\begin{aligned}
I_{421} &= \int_{|x|>1}\frac{|x|^3}{1+|x|^2}\rho(\mathrm{d}x)\left(\int_0^{1/|x|}+\int_{1/|x|}^1\right)\frac{\xi|\xi^2-1|}{1+|\xi x|^2}\gamma(\mathrm{d}\xi)\\
&=: I_{4211}+I_{4212},
\end{aligned}
$$

say. We have

$$
I_{4211}\le \int_{|x|>1}|x|\rho(\mathrm{d}x)\int_0^{1/|x|}\xi\gamma(\mathrm{d}\xi)<\infty, \quad \text{(by (14))},
$$

and

$$
\begin{aligned}
I_{4212} &= \int_{|x|>1}\frac{|x|^2}{1+|x|^2}\rho(\mathrm{d}x)\int_{1/|x|}^1\frac{|\xi x||\xi^2-1|}{1+|\xi x|^2}\gamma(\mathrm{d}\xi)\\
&\le \int_{|x|>1}\gamma([1/|x|,1])\rho(\mathrm{d}x)<\infty, \quad \text{(by (15))}.
\end{aligned}
$$

Also

$$
\begin{aligned}
I_{422} &= \int_{|x|>1}\frac{|x|^2}{1+|x|^2}\rho(\mathrm{d}x)\int_1^\infty\frac{|\xi x||\xi^2-1|}{1+|\xi x|^2}\gamma(\mathrm{d}\xi)\\
&\le \int_{|x|>1}\frac{|x|^2}{1+|x|^2}\rho(\mathrm{d}x)\int_1^\infty(\xi^2+1)\gamma(\mathrm{d}\xi)<\infty \quad \text{(by (12))}.
\end{aligned}
$$

Thus we have (26). This completes the verification.

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