STOCHASTIC INTEGRAL REPRESENTATIONS FOR SUBCLASSES OF SELFDECOMPOSABLE AND SEMI-SELFDECOMPOSABLE DISTRIBUTIONS

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1. Introduction

A distribution μ on $\mathbb{R}^d$ is called selfdecomposable if for any $b > 1$, there exists a distribution ρ_b such that

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho_b}(z), \qquad z \in \mathbb{R}^d, \tag{1.1}$$

where $\widehat{\mu}$ means the characteristic function of μ. μ on $\mathbb{R}^d$ is called b-semi-selfdecomposable if (1.1) holds for some $b > 1$. They are all infinitely divisible. Classes of all selfdecomposable and b-semi-selfdecomposable distributions on $\mathbb{R}^d$ are denoted by $L_0(\mathbb{R}^d)$ and $L_0(b, \mathbb{R}^d)$, respectively. A stochastic process $\{X_t, t \geq 0\}$ on $\mathbb{R}^d$ is called an additive process if it is continuous in probability, has independent increments, has càdlàg sample paths a.s. and $X_0 = 0$ a.s. It is called a Lévy process if, in addition, $X_{t+u} - X_{s+u} \stackrel{\mathcal{L}}{=} X_t - X_s$ for all nonnegative t, s, u, where $\stackrel{\mathcal{L}}{=}$ means equality in distribuiton. It is called a semi-Lévy process with period $p > 0$, if it is an additive process and $X_{t+p} - X_{s+p} \stackrel{\mathcal{L}}{=} X_t - X_s$ for all nonnegative t, s.

Let $\{X_t\}$ be an additive process on $\mathbb{R}^d$ and let $\mu_t = \mathcal{L}(X_t)$, where $\mathcal{L}$ means the ditribution of. Since μ_t is infinitely divisible, there is a generating triplet (A_t, ν_t, γ_t). If the location parameter γ_t is of bounded variation in any finite interval, then $\{X_t\}$ is called *natural*. This notion was intriduced by Sato (2004), and it was shown that an additive process is natural if and only if it is semimartingale. Lévy processes are always natural, but semi-Lévy processes are not necessarily natural.

Let $ID(\mathbb{R}^d)$ be the class of all infinitely divisible distributions on $\mathbb{R}^d$ and let

$$ID_{\log}(\mathbb{R}^d) = \left\{ \mu \in ID(\mathbb{R}^d) : \int_{|x|>1} \log|x| \mu(dx) < \infty \right\}.$$

Let K be a subclass of $ID(\mathbb{R}^d)$, and we call it "completely closed in the strong sense" if the following conditions are satisfied:

(i) $\mu_1, \mu_2 \in K$ implies $\mu_1 * \mu_2 \in K$,

(ii) $\mu_n \in K$ $(n = 1, 2, \ldots)$ and $\mu_n \to \mu$ imply $\mu \in K$,

(iii) if X is an $\mathbb{R}^d$-valued random variable with $\mathcal{L}(X) \in K$, then $\mathcal{L}(aX + c) \in K$ for any $a > 0$ and $c \in \mathbb{R}^d$,

(iv) $\mu \in K$ implies $\mu^{t*} \in K$ for any $t > 0$, where $\widehat{\mu^{t*}} = \widehat{\mu}^t$.

Define $Q(K)$ by

$$Q(K) = \{\mu \in ID(\mathbb{R}^d) : \text{ for any } b > 1, \text{ there exists } \rho_b \in K \\ \text{such that } \widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}_b(z),\ z \in \mathbb{R}^d\},$$

and define, for each $b > 1$, $Q(b, K)$ by

$$Q(b, K) = \{\mu \in ID(\mathbb{R}^d) : \text{ there exists } \rho_b \in K \\ \text{such that } \widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}_b(z),\ z \in \mathbb{R}^d\}.$$

By their definitions, $\mu \in Q(K)$ is selfdecomposable and $\mu \in Q(b, K)$ is b-semi-selfdecomposable.

Example 1.1. Examples of classes comlpetely closed in the strong sense found in the literature are the following. $ID(\mathbb{R}^d)$, $L_0(\mathbb{R}^d) = Q(ID(\mathbb{R}^d))$ (the class of all selfdecomposable disitributions on $\mathbb{R}^d$), $L_m(\mathbb{R}^d) = Q(L_{m-1}(\mathbb{R}^d))$, $L_\infty(\mathbb{R}^d) = \cap_m L_m(\mathbb{R}^d)$; for each $b > 1$, $L_0(b, \mathbb{R}^d) = Q(b, ID(\mathbb{R}^d))$, $L_m(b, \mathbb{R}^d) = Q(b, L_{m-1}(b, \mathbb{R}^d))$, $L_\infty(b, \mathbb{R}^d) = \cap_m L_m(b, \mathbb{R}^d)$. Also, $B(\mathbb{R}^d)$ (Goldie-Steutel-Bondesson class, see Barndorff-Nielsen et al. (2004)), $T_0(\mathbb{R}^d) = Q(B(\mathbb{R}^d))$ (Thorin class, see also Barndorff-Nielsen et al.), $T_m(\mathbb{R}^d) = Q(T_{m-1}(\mathbb{R}^d))$; $TG_0(\mathbb{R}^d)$ (the class of all type G distributions on $\mathbb{R}^d$), $TG_m(\mathbb{R}^d) = Q(TG_{m-1}(\mathbb{R}^d))$, (see Maejima and Rosiński (2001)).

When $K = ID(\mathbb{R}^d)$ or $= L_m(\mathbb{R}^d)$, the following are known.

Theorem 1.2. (i) (Wolfe (1982), Jurek and Vervaat (1983)) $\mu \in L_0(\mathbb{R}^d) = Q(ID(\mathbb{R}^d))$ *if and only if there exists a Lévy process* $\{Z_t\}$ *such that* $\mathcal{L}(Z_1) \in ID_{\log}(\mathbb{R}^d)$ *and*

$$\mu = \mathcal{L}\left(\int_0^\infty e^{-t} dZ_t\right), \tag{1.2}$$

where $\mathcal{L}(X)$ *means the law of* X.

(ii) (Jurek and Vervaat (1983), Sato and Yamazato (1983)) *Let* $m = 1, 2, \ldots, \infty$. $\mu \in L_m(\mathbb{R}^d) = Q(L_{m-1}(\mathbb{R}^d))$ *if and only if there exists a Lévy process* $\{Z_t\}$ *such that* $\mathcal{L}(Z_1) \in L_{m-1}(\mathbb{R}^d) \cap ID_{\log}(\mathbb{R}^d)$ *and* (1.2) *holds.*

Theorem 1.3. (Maejima and Sato (1999)) *Fix* $b > 1$. $\mu \in L_0(b, \mathbb{R}^d) = Q(b, ID(\mathbb{R}^d))$ *if and only if there exists a natural semi-Lévy process* $\{Z_t\}$ *with period* $p = \log b$ *such that* $\mathcal{L}(Z_p) \in ID_{\log}(\mathbb{R}^d)$ *and* (1.2) *holds.*

The purpose of this paper is to ponit out or to show that Theorems 1.2 and 1.3 remain true for any completely closed in the strong sense set $K \subset ID(\mathbb{R}^d)$. As a corollary of the second result (Theorem 3.1 below), we get a "semi"-version of Theorem 1.2 (ii), which was the first motivation of our present study.

2. The first result.

Theorem 2.1. *Let $K \subset ID(\mathbb{R}^d)$ be completely closed in the strong sense. Then $\mu \in Q(K)$ if and only if there exists Lévy process $\{Z_t\}$ such that*

$$\mathcal{L}(Z_1) \in K \cap ID_{\log}$$

and

$$\mu = \mathcal{L}\left(\int_0^\infty e^{-t} dZ_t\right).$$

This theorem is Lemma 4.1 (iii) of Barndorff-Nielsen et al. (2004).

3. The second result and its proof.

Theorem 3.1. *Fix $b > 1$ and let $K \subset ID(\mathbb{R}^d)$ be completely closed in the strong sense. Then $\mu \in Q(b, K)$ if and only if there exists a natural semi-Lévy process $\{Z_t\}$ with period $p = \log b$ such that*

$$\mathcal{L}(Z_t - Z_s) \in K \quad \text{for any} \quad 0 \le s < t \le p, \tag{3.1}$$

$$\mathcal{L}(Z_p) \in ID_{\log}(\mathbb{R}^d) \tag{3.2}$$

and

$$\mu = \mathcal{L}\left(\int_0^\infty e^{-t} dZ_t\right). \tag{3.3}$$

As a special case of this theorem, we get a semi-version of Theorem 1.2 (ii) as follows.

Corollary 3.2. *Let $b > 1$ and $m = 1, 2, \cdots, \infty$. $\mu \in L_m(b, \mathbb{R}^d)$ if and only if there exists a natural semi-Lévy process $\{Z_t\}$ with period $p = \log b$ such that*

$$\mathcal{L}(Z_t - Z_s) \in L_{m-1}(b, \mathbb{R}^d) \quad \text{for any} \quad 0 \le s < t \le p,$$

$$\mathcal{L}(Z_p) \in ID_{\log}(\mathbb{R}^d)$$

$$\mu = \mathcal{L}\left(\int_0^\infty e^{-t} dZ_t\right).$$

To prove Theorem 3.1, we first state two lemmas. Let $H > 0$. $\{X_t, t \geq 0\}$ is called H-semi-selfsimilar on $\mathbb{R}^d$ if for some $a > 1$

$$\{X_{at}, t \geq 0\} \stackrel{\mathrm{d}}{=} \{a^H X_t, t \geq 0\}. \tag{3.4}$$

The number a and a^H in (3.4) are called an epoch and a span of $\{X_t\}$, respectively.

Lemma 3.3. (Theorem 1.1 of Maejima and Sato (2003)) *Let $b > 1$. Suppose that $\{X_t, t \geq 0\}$ is a 1-semi-selfsimilar natural additive process on $\mathbb{R}^d$ with epoch b, and put*

$$Z_t = \int_1^{e^t} s^{-1} dX_s. \tag{3.5}$$

Then $\{Z_t\}$ is a natural semi-Lévy process with period $p = \log b$ and $\mathcal{L}(Z_p) \in ID_{\log}(\mathbb{R}^d)$. Also,

$$X_t = \int_{-\infty}^{\log t} e^s dZ_s, \quad t > 0 \quad \text{a.s.} \tag{3.6}$$

Lemma 3.4. *Let $b > 1$. If K is completely closed in the strong sense, then so is $Q(b, K)$.*

Proof. Recall the definition of $Q(b, K)$. If $\mu \in Q(b, K)$, then $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho_b}(z)$.

(i) $\widehat{\mu_1}(z) = \widehat{\mu_1}(b^{-1}z)\widehat{\rho_{1,b}}(z)$ and $\widehat{\mu_2}(z) = \widehat{\mu_2}(b^{-1}z)\widehat{\rho_{2,b}}(z)$ imply that $\widehat{\mu_1 * \mu_2}(z) = \widehat{\mu_1 * \mu_2}(b^{-1}z)\widehat{\rho_{1,b}}(z)\widehat{\rho_{2,b}}(z)$, where $\widehat{\rho_{1,b}}(z)\widehat{\rho_{2,b}}(z)$ belongs to K.

(ii) $\widehat{\mu_n}(z) = \widehat{\mu_n}(b^{-1}z)\widehat{\rho_{n,b}}(z)$ and $\mu_n \to \mu$ imply that $\rho_{n,b} \to \rho_b$, say, and ρ_b belongs to K. Thus $\mu \in Q(b, K)$.

(iii) $\widehat{\mathcal{L}(aX + c)}(z) = \widehat{\mu}(az)e^{i\langle c,z\rangle} = \widehat{\mu}(ab^{-1}z)\widehat{\rho_b}(az)e^{i\langle c,z\rangle}$
$= \widehat{\mathcal{L}(b^{-1}(aX + c))}(z)\widehat{\rho_b}(az)e^{i\langle c(1-b^{-1}),z\rangle}$, where $\widehat{\rho_b}(az)e^{i\langle c(1-b^{-1}),z\rangle}$ belongs to K. Thus $\mathcal{L}(aX + c)$ belongs to $Q(b, K)$.

(iv) $\widehat{\mu}(z)^t = \widehat{\mu}(b^{-1}z)^t\widehat{\rho_b}(z)^t$, where $\widehat{\rho_b}(z)^t$ belongs to K. □

Proof of Theorem 3.1.

(The "if"part.) Put

$$X = \int_0^\infty e^{-t} dZ_t.$$

Here this stochastic integral is definiable, because $\{Z_t\}$ is natural additive and $\mathcal{L}(Z_p) \in ID_{\log}(\mathbb{R}^d)$, (see Sato(2004)). Then

$$\begin{aligned} X &= \int_p^\infty e^{-t} dZ_t + \int_0^p e^{-t} dZ_t \\ &= e^{-p} \int_0^\infty e^{-s} dZ_{p+s} + \int_0^p e^{-t} dZ_t \\ &\overset{\mathcal{L}}{=} e^{-p} \int_0^\infty e^{-s} d\widetilde{Z}_s + \int_0^p e^{-t} dZ_t, \end{aligned}$$

where $\widetilde{Z}_s = Z_{p+s}$. Thus, $\{\widetilde{Z}_s : s \geq 0\}$ is another semi-Lévy process with period p independent of $\{Z_t, 0 \leq t \leq p\}$. Then we have

$$X \overset{\mathcal{L}}{=} e^{-p} X + \int_0^p e^{-t} dZ_t,$$

where $e^p = b$. In order to conclude that $\mathcal{L}(X) \in Q(b, K)$, it is enough to show that

$$\mathcal{L}\Big(\int_0^p e^{-t} dZ_t\Big) \in K$$

by the definition of $Q(b, K)$. By the definition of stochastic integrals, for some positive step functions $f_n(t) = \sum_{j=1}^n a_{n,j} 1_{\{t_{n,j-1} < t \leq t_{n,j}\}}$,

$$\int_0^p e^{-t} dZ_t = \text{p-}\lim_{n\to\infty} \int_0^p f_n(t) dZ_t = \text{p-}\lim_{n\to\infty} \sum_{j=1}^n a_{nj} (Z_{t_{n,j}} - Z_{t_{n,j-1}}),$$

where p-$\lim_{n\to\infty}$ means limit in probability. By the condition (3.1), $\mathcal{L}(Z_{t_{n,j}} - Z_{t_{n,j-1}}) \in K$, and thus $\int_0^p e^{-t} dZ_t \in K$ since K is completely closed in the strong sense. This completes the proof of the "if" part.

(The "only if" part.) Suppose $\mu \in Q(b, K)$. Since μ is b-semi-selfdecomposable, there exists a 1-semi-selfsimilar natural additive process $\{X_t, t \geq 0\}$ with span b on $\mathbb{R}^d$ such that

$$\mu = \mathcal{L}(X_1), \tag{3.7}$$

$$\mathcal{L}(X_t) \in Q(b, K), \quad t \geq 0, \tag{3.8}$$

$$\mathcal{L}(X_t - X_s) \in K, \quad 0 \leq s < t. \tag{3.9}$$

The reason for the existence of such a semi-selfsimilar natural additive process is the following.

Since $\mu \in Q(b, K)$, by the same reasoning as for Theorem 8 of Maejima and Sato (1999), there exists a 1-semi-selfsimilar additive process $\{X_t\}$ with span b on $\mathbb{R}^d$

satisfying (3.7). In their proof, they construct $\{X_t\}$ by choosing $\mu_t = \mathcal{L}(X_t)$ as

$$\widehat{\mu}_t(z) = \widehat{\mu}(z)^{1-h(t)}\widehat{\mu}(bz)^{h(t)}, \quad 1 \le t < b, \tag{3.10}$$

where $h(t)$ is a continuous increasing function such that $h(1) = 0$ and $h(b) = 1$. Since $\mu \in Q(b, K)$ and $Q(b, K)$ is completely closed in the storng sense (by Lemma 3.4), we have $\mu_t \in Q(b, K)$, $1 \le t < b$. By the semi-selsimilarity, we have (3.8). Also, since $\mu \in Q(b, K)$, there exists an infinitely divisible distribution $\rho \in K$ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}(z)$. Thus it follows from (3.10) that

$$\widehat{\mu}_t(z) = \widehat{\mu}_s(z)\widehat{\rho}(bz)^{h(t)-h(s)} \quad \text{for } 1 \le s \le t < b,$$

and thus if we put $\mu_{t,s} = \mathcal{L}(X_t - X_s)$, then

$$\widehat{\mu}_{t,s}(z) = \widehat{\mu}_t(z)\left(\widehat{\mu}_s(z)\right)^{-1} = \widehat{\rho}(bz)^{h(t)-h(s)} \quad \text{for } 1 \le s \le t < b,$$

by the uniquness of $\rho \in K$. Hence $\mathcal{L}(X_t - X_s) \in K$ for $1 \le s \le t < b$. By the semi-selfsimilarilty, we conclude (3.9). Furhtermore, the location parameter γ_t of μ_t is

$$\gamma_t = (1 - h(t))\gamma_1 + h(t)\gamma_{b^{1/H}},$$

which is of bounded variation on $[1, b^{1/H}]$. Thus by Remark before Proposition 5.1 of Sato (2004), $\{X_t\}$ is natural.

Now put

$$Z_t = \int_1^{e^t} s^{-1} dX_s$$

as in (3.5). Then by Lemma 3.3, this $\{Z_t\}$ satisfies (3.2). It also follows from (3.6) that

$$\mathcal{L}(X_1) = \mathcal{L}\Big(\int_{-\infty}^0 e^s dZ_s\Big) = \mathcal{L}\Big(\int_0^\infty e^{-s} dZ_s\Big).$$

This shows (3.3). It remains to show (3.1). By the definition of stochastic integrals, there exist positive step functions $f_n(u)$ such that

$$Z_t - Z_s = \int_{e^s}^{e^t} u^{-1} dX_u = \text{p-}\lim_{n\to\infty} \int_{e^s}^{e^t} f_n(u) dX_u.$$

Futhermore, for $e^s = u_{n,0} < u_{n,1} < \cdots < u_{n,k} = e^t, a_{n,j} > 0$,

$$\int_{e^s}^{e^t} f_n(u) dX_u = \sum_{j=1}^k a_{n,j}\Big(X_{u_{n,j}} - X_{u_{n,j-1}}\Big).$$

Here $\mathcal{L}(X_{u_{n,j}} - X_{u_{n,j-1}}) \in K$ by (3.9) and thus the distribution on the right hand side belongs to K, because K is completely closed in the strong sense. This completes the proof of the "only if" part.

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