A NOTE ON NEW CLASSES OF INFINITELY DIVISIBLE DISTRIBUTIONS ON $\mathbb{R}^d$

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Abstract
This paper introduces and studies a family of new classes of infinitely divisible distributions on $\mathbb{R}^d$ with two parameters. Depending on parameters, these classes connect the Goldie–Steutel–Bondesson class and the class of generalized type $G$ distributions, connect the Thorin class and the class $M$, connect the class $M$ and the class of generalized type $G$ distributions. These classes are characterized by stochastic integral representations with respect to Lévy processes.

1. Introduction

Let $I(\mathbb{R}^d)$ be the class of all infinitely divisible distributions on $\mathbb{R}^d$. $\hat{\mu}(z), z \in \mathbb{R}^d,$ denotes the characteristic function of $\mu \in I(\mathbb{R}^d)$ and $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$. We use the Lévy-Khintchine triplet $(A, \nu, \gamma)$ of $\mu \in I(\mathbb{R}^d)$ in the sense that

$$\hat{\mu}(z) = \exp \left\{ -2^{-1} \langle z, Az \rangle + i \langle \gamma, z \rangle + \int_{\mathbb{R}^d} \left( e^{i \langle z, x \rangle} - 1 - i \langle z, x \rangle (1 + |x|^2)^{-1} \right) \nu(dx) \right\},$$

$z \in \mathbb{R}^d$, where $A$ is a symmetric nonnegative-definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$ and $\nu$ is a measure (called the Lévy measure) on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(dx) < \infty.$$

The following polar decomposition is a basic result on the Lévy measure of $\mu \in I(\mathbb{R}^d)$. Let $\nu$ be the Lévy measure of some $\mu \in I(\mathbb{R}^d)$ with $0 < \nu(\mathbb{R}^d) \leq \infty$. Then there exist a measure $\lambda$ on $S = \{x \in \mathbb{R}^d : |x| = 1\}$ with $0 < \lambda(S) \leq \infty$ and a family $\{\nu_\xi : \xi \in S\}$ of measures on $\mathbb{R}$ such that $\nu_\xi(B)$ is measurable in $\xi$ for each
\(B \in \mathcal{B}((0, \infty)), \ 0 < \nu_\xi((0, \infty)) \leq \infty \) for each \(\xi \in S\), and

\[
\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).
\] (1.1)

Here \(\lambda\) and \(\{\nu_\xi\}\) are uniquely determined by \(\nu\) up to multiplication of measurable functions \(c(\xi)\) and \(\frac{1}{c(\xi)}\), respectively, with \(0 < c(\xi) < \infty\). We say that \(\nu\) has the polar decomposition \((\lambda, \nu_\xi)\) and \(\nu_\xi\) is called the radial component of \(\nu\). (See, e.g., Barndorff-Nielsen et al. (2006), Lemma 2.1.)

A real-valued function \(f\) defined on \((0, \infty)\) is said to be completely monotone if it has derivatives \(f^{(n)}\) of all orders and for each \(n = 0, 1, 2, ..., (-1)^nf^{(n)}(r) \geq 0, r > 0\). Bernstein’s theorem says that \(f\) on \((0, \infty)\) is completely monotone if and only if there exists a (not necessarily finite) measure \(Q\) on \([0, \infty)\) such that \(f(r) = \int_{(0,\infty)} e^{-ru}Q(du)\). (See, e.g., Feller (1966), p.439.)

In this paper, we introduce and study the following classes.

**Definition 1.1.** (The class \(J_{\alpha,\beta}(\mathbb{R}^d)\).) Let \(\alpha < 2\) and \(\beta > 0\). We say that \(\mu \in I(\mathbb{R}^d)\) belongs to the class \(J_{\alpha,\beta}(\mathbb{R}^d)\) if \(\nu = 0\) or \(\nu \neq 0\) and, in case \(\nu \neq 0\), \(\nu_\xi\) in (1.1) has expression

\[
\nu_\xi(dr) = r^{-\alpha - 1}g_\xi(r^\beta)dr, \quad r > 0,
\] (1.2)

where \(g_\xi(x)\) is measurable in \(\xi\), is completely monotone in \(x\) on \((0, \infty)\) \(\lambda\)-a.e. \(\xi\), not identically zero and \(\lim_{x \to \infty} g_\xi(x) = 0\) \(\lambda\)-a.e. \(\xi\).

**Remark 1.2.** If \(\alpha \leq 0\), then automatically \(\lim_{x \to -\infty} g_\xi(x) = 0\) \(\lambda\)-a.e. \(\xi\), because of the finiteness of \(\int_{|x| > 1} \nu(dx)\). So, when we consider the classes \(B(\mathbb{R}^d), G(\mathbb{R}^d), T(\mathbb{R}^d)\) and \(M(\mathbb{R}^d)\) appearing later, we do not have to write this condition explicitly.

**Remark 1.3.** The integrability condition of the Lévy measure \(\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty\) implies that

\[
\int_0^\infty (r^2 \wedge 1)r^{-\alpha - 1}g_\xi(r^\beta)dr < \infty, \quad \lambda\text{-a.e. } \xi,
\] (1.3)

so we do not have to assume (1.3) in the definition. It is automatically satisfied.

**Remark 1.4.** The classes \(J_{\alpha,1}(\mathbb{R}^d), \alpha < 2\), are studied in Sato (2006b).

Before mentioning our motivation of this study, we state a general result on the relations among the classes \(J_{\alpha,\beta}(\mathbb{R}^d), \alpha < 2, \beta > 0\).

**Theorem 1.5.** (i) Fix \(\alpha < 2\) and let \(0 < \beta_1 < \beta_2\). Then

\[J_{\alpha,\beta_1}(\mathbb{R}^d) \subset J_{\alpha,\beta_2}(\mathbb{R}^d)\].
(ii) Fix $\beta > 0$ and let $\alpha_1 < \alpha_2 < 2$. Then

$$J_{\alpha_2,\beta}(\mathbb{R}^d) \subset J_{\alpha_1,\beta}(\mathbb{R}^d).$$

Proof. For the proof of (i), we need the following lemma.

**Lemma 1.6.** (See Feller (1966), p.441, Corollary 2.) Let $\phi$ be a completely monotone function on $(0, \infty)$ and let $\psi$ be a nonnegative function on $(0, \infty)$ whose derivative is completely monotone. Then $\phi(\psi)$ is completely monotone.

Let $h_\xi(x) = g_\xi(x^{\beta_1/\beta_2}), x > 0$, where $g_\xi$ is the one in (1.2), which is completely monotone on $(0, \infty)$. Since $\psi(x) = x^{\beta_1/\beta_2}, x > 0$, has a completely monotone derivative, it follows from Lemma 1.6 that $h_\xi(x)$ is completely monotone. Suppose $\mu \in J_{\alpha,\beta}(\mathbb{R}^d)$ and let $g_\xi$ be the one in (1.2). Since $g_\xi(r^{\beta_1}) = h_\xi(r^{\beta_2})$, where $h_\xi$ is completely monotone as has been just shown above, we have $\mu \in J_{\alpha_1,\beta_2}(\mathbb{R}^d)$. This proves (i).

To prove (ii), suppose that $\mu \in J_{\alpha_2,\beta}(\mathbb{R}^d)$. Then $\nu_\xi(dr) = r^{-\alpha_2-1}g_\xi(r^\beta)dr, r > 0$, as in (1.2), where $g_\xi$ is completely monotone on $(0, \infty)$ $\lambda$-a.e. $\xi$. Note that

$$h_\xi(x) = x^{-(\alpha_2-\alpha_1)/\beta}g_\xi(x)$$

is completely monotone, because $x^{-p}, p > 0$, is completely monotone and the product of two completely monotone functions is also completely monotone. We now have

$$\nu_\xi(dr) = r^{-\alpha_2-1}g_\xi(r^\beta)dr = r^{-\alpha_1-1}h_\xi(r^\beta)dr,$$

and thus $\mu$ also belongs to $J_{\alpha_1,\beta_2}(\mathbb{R}^d)$. This proves (ii). \qed

The motivations for studying the classes $J_{\alpha,\beta}(\mathbb{R}^d)$ are the following.

I. The classes connecting the Goldie–Steutel–Bondesson class and the class of generalized type $G$ distributions.

Let $\alpha = -1$ and consider the classes $J_{-1,\beta}(\mathbb{R}^d), \beta > 0$.

A distribution $\mu \in I(\mathbb{R}^d)$ is said to be of generalized type $G$ if $\nu_\xi$ in (1.2) has expression $\nu_\xi(dr) = g_\xi(r^2)dr$ for some completely monotone function $g_\xi$ on $(0, \infty)$, and denote by $G(\mathbb{R}^d)$ the class of all generalized type $G$ distributions on $\mathbb{R}^d$. Let $I_{\text{sym}}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) \mid \mu \text{ is symmetric in the sense that } \mu(B) = \mu(-B), B \in \mathcal{B}(\mathbb{R}^d)\}$.

**Remark 1.7.** A distribution $\mu \in G(\mathbb{R}^d) \cap I_{\text{sym}}(\mathbb{R}^d)$ is a so-called type $G$ distribution, which is, in one dimension, a variance mixture of the standard normal distribution with a positive infinitely divisible mixing distribution.
Remark 1.8. $G(\mathbb{R}^d) = J_{-1,2}(\mathbb{R}^d)$.

Remark 1.9. The Goldie-Steutel-Bondesson class denoted by $B(\mathbb{R}^d)$ is $J_{-1,1}(\mathbb{R}^d)$. (For details on $B(\mathbb{R}^d)$, see Barndorff-Nielsen et al. (2006).)

Therefore, by Theorem 1.5 (i) with $\alpha = -1$, for $1 < \beta < 2$,
$$B(\mathbb{R}^d) \subset J_{-1,\beta}(\mathbb{R}^d) \subset G(\mathbb{R}^d),$$
and hence $\{J_{-1,\beta}(\mathbb{R}^d), 1 \leq \beta \leq 2\}$ is a family of classes of infinitely divisible distributions on $\mathbb{R}^d$ connecting $B(\mathbb{R}^d)$ and $G(\mathbb{R}^d)$ with continuous parameter $\beta \in [1, 2]$.

II. The classes connecting the Thorin class and the class $M(\mathbb{R}^d)$.

Let $\alpha = 0$ and consider the classes $J_{0,\beta}(\mathbb{R}^d)$, $\beta > 0$.

Remark 1.10. The Thorin class denoted by $T(\mathbb{R}^d)$ is $J_{0,1}(\mathbb{R}^d)$. (For details on $T(\mathbb{R}^d)$, see also Barndorff-Nielsen et al. (2006).)

Remark 1.11. The class $M(\mathbb{R}^d)$ is defined by $J_{0,2}(\mathbb{R}^d)$. (The class $M(\mathbb{R}^d) \cap I_{\text{sym}}(\mathbb{R}^d)$ is studied in Aoyama et al. (2008).)

By Theorem 1.5 (i) with $\alpha = 0$, for $1 < \beta < 2$,
$$T(\mathbb{R}^d) \subset J_{0,\beta}(\mathbb{R}^d) \subset M(\mathbb{R}^d),$$
and hence $\{J_{0,\beta}(\mathbb{R}^d), 1 \leq \beta \leq 2\}$ is a family of classes of infinitely divisible distributions on $\mathbb{R}^d$ connecting $T(\mathbb{R}^d)$ and $M(\mathbb{R}^d)$ with continuous parameter $\beta \in [1, 2]$.

III. The classes connecting the classes $M(\mathbb{R}^d)$ and $G(\mathbb{R}^d)$.

Let $\beta = 2$ and consider the classes $J_{\alpha,2}(\mathbb{R}^d)$, $\alpha < 2$. Then, by Theorem 1.5 (ii) with $\beta = 2$, for $-1 \leq \alpha \leq 0$
$$M(\mathbb{R}^d) \subset J_{\alpha,2}(\mathbb{R}^d) \subset G(\mathbb{R}^d),$$
and hence $\{J_{\alpha,2}(\mathbb{R}^d), -1 \leq \alpha \leq 0\}$ is a family of classes of infinitely divisible distributions on $\mathbb{R}^d$ connecting $M(\mathbb{R}^d)$ and $G(\mathbb{R}^d)$ with continuous parameter $\alpha \in [-1, 0]$.

IV. The classes connecting the classes $T(\mathbb{R}^d)$ and $B(\mathbb{R}^d)$.

Let $\beta = 1$ and consider the classes $J_{\alpha,1}(\mathbb{R}^d)$, $\alpha < 2$. Then, by Theorem 1.5 (ii) with $\beta = 1$, for $-1 \leq \alpha \leq 0$
$$T(\mathbb{R}^d) \subset J_{\alpha,1}(\mathbb{R}^d) \subset B(\mathbb{R}^d),$$
and hence \( \{J_{\alpha,1}(\mathbb{R}^d), -1 \leq \alpha \leq 0\} \) is a family of classes of infinitely divisible distributions on \( \mathbb{R}^d \) connecting \( T(\mathbb{R}^d) \) and \( B(\mathbb{R}^d) \) with continuous parameter \( \alpha \in [-1, 0] \). (This fact is already mentioned in Sato (2006b).)

2. Stochastic integral characterizations for \( J_{\alpha,\beta}(\mathbb{R}^d) \)

The purpose of this paper is to characterize the classes \( J_{\alpha,\beta}(\mathbb{R}^d) \) by stochastic integral representations. For that, we first define mappings from \( I(\mathbb{R}^d) \) into \( I(\mathbb{R}^d) \) and investigate the domains of those mappings.

We introduce the following function \( G_{\alpha,\beta}(u) \). For \( \alpha < 2 \) and \( \beta > 0 \), let

\[
G_{\alpha,\beta}(u) = \int_u^\infty x^{-\alpha-1}e^{-x^\beta}dx, \quad u \geq 0,
\]

and let \( G^*_{\alpha,\beta}(t) \) be the inverse function of \( G_{\alpha,\beta}(u) \), that is, \( t = G_{\alpha,\beta}(u) \) if and only if \( u = G^*_{\alpha,\beta}(t) \). Let \( \{X_t(\mu)\} \) be a Lévy process on \( \mathbb{R}^d \) with the law \( \mu \in I(\mathbb{R}^d) \) at \( t = 1 \). We consider the stochastic integrals

\[
\int_0^{G_{\alpha,\beta}(0)} G^*_{\alpha,\beta}(t)dX_t(\mu), \quad \text{where} \quad G_{\alpha,\beta}(0) = \begin{cases} \beta^{-1}\Gamma(-\alpha\beta^{-1}), & \text{if } \alpha < 0, \\ \infty, & \text{if } \alpha \geq 0. \end{cases}
\]

As to the definition of stochastic integrals of non-random measurable functions \( f \) which are \( \int_0^T f(t)dX_t(\mu), T < \infty, \mu \in I(\mathbb{R}^d) \), we follow the definition in Sato (2004, 2006a), whose idea is to define a stochastic integral with respect to \( \mathbb{R}^d \)-valued independently scattered random measure induced by a Lévy process on \( \mathbb{R}^d \). The improper stochastic integral \( \int_0^\infty f(t)dX_t(\mu) \) is defined as the limit in probability of \( \int_0^T f(t)dX_t(\mu) \) as \( T \to \infty \) whenever the limit exists. See also Sato (2006b). In the following, \( \mathcal{L}(X) \) stands for “the law of \( X \)”. If we write

\[
\Psi_{\alpha,\beta}(\mu) = \mathcal{L}\left( \int_0^{G_{\alpha,\beta}(0)} G^*_{\alpha,\beta}(t)dX_t(\mu) \right),
\]

then \( \Psi_{\alpha,\beta} \) can be considered as a mapping with domain \( \mathcal{D}(\Psi_{\alpha,\beta}) \) being the class of \( \mu \in I(\mathbb{R}^d) \) for which \( \int_0^{G_{\alpha,\beta}(0)} G^*_{\alpha,\beta}(t)dX_t(\mu) \) is definable.

**Theorem 2.1.** If \( \alpha < 0 \), then \( \mathcal{D}(\Psi_{\alpha,\beta}) = I(\mathbb{R}^d) \).

**Proof.** By Proposition 3.4 in Sato (2006a), since \( G_{\alpha,\beta}(0) < \infty \) for \( \alpha < 0 \), if \( \int_0^{G_{\alpha,\beta}(0)} (G^*_{\alpha,\beta}(t))^2 dt < \infty \), then \( \int_0^{G_{\alpha,\beta}(0)} G^*_{\alpha,\beta}(t)dX_t(\mu) \) is well-defined. Actually,

\[
\int_0^{G_{\alpha,\beta}(0)} (G^*_{\alpha,\beta}(t))^2 dt = \int_0^{\infty} u^2dG_{\alpha,\beta}(u) = \int_0^{\infty} u^{1-\alpha}e^{-u^\beta}du < \infty.
\]
To determine the domain of $\Psi_{\alpha,\beta}$, $\alpha \geq 0$, we need the following result by Sato (2006b). In the following, $a(t) \sim b(t)$ means that $\lim_{t \to \infty} a(t)/b(t) = 1$, $a(t) \asymp b(t)$ means that $0 < \liminf_{t \to \infty} a(t)/b(t) \leq \limsup_{t \to \infty} a(t)/b(t) < \infty$ and $I_{\log}(\mathbb{R}^d) = \{ \mu \in I(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} \log^+ |x| \mu(dx) < \infty \}$, where $\log^+ |x| = (\log |x|) \vee 0$.

**Proposition 2.2.** (Sato (2006b), Theorems 2.4 and 2.8.) Let $p \geq 0$. Denote

$$\Phi_{\varphi_p}(\mu) = \mathcal{L} \left( \int_0^\infty \varphi_p(t) dX^{(\mu)}_t \right).$$

Suppose that $\varphi_p$ is locally square-integrable with respect to Lebesgue measure on $[0, \infty)$ and satisfies

1. $\varphi_0(t) \asymp e^{-ct}$ as $t \to \infty$ with some $c > 0$,
2. $\varphi_p(t) \asymp t^{-1/p}$ as $t \to \infty$ for $p \in (0, 1) \cup (1, \infty)$,
3. $\varphi_1(t) \asymp t^{-1}$ as $t \to \infty$ and for some $t_0 > 0$, $c > 0$ and $\psi(t)$, $\varphi_1(t) = t^{-1} \psi(t)$ for $t > t_0$ with $\int_0^\infty t^{-1} |\psi(t)| - c|dt < \infty$.

Then

1. If $p = 0$, then $\mathcal{D}(\Phi_{\varphi_0}) = I_{\log}(\mathbb{R}^d)$.
2. If $0 < p < 1$, then $\mathcal{D}(\Phi_{\varphi_p}) = \{ \mu \in I(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^p \mu(dx) < \infty \} =: I_p(\mathbb{R}^d)$.
3. If $p = 1$, then $\mathcal{D}(\Phi_{\varphi_1}) = \{ \mu \in I(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x| \mu(dx) < \infty \}$

$$\lim_{T \to \infty} \int_0^T t^{-1} dt \int_{|x| > t} x v(dx) \text{ exists in } \mathbb{R}^d, \int_{\mathbb{R}^d} x \mu(dx) = 0 \} =: I_1^*(\mathbb{R}^d).$$
4. If $1 < p < 2$, then $\mathcal{D}(\Phi_{\varphi_p}) = \{ \mu \in I(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^p \mu(dx) < \infty, \int_{\mathbb{R}^d} x \mu(dx) = 0 \}$

$$= : I_p^*(\mathbb{R}^d).$$
5. If $p \geq 2$, then $\mathcal{D}(\Phi_{\varphi_p}) = \{ \delta_0 \}$, where $\delta_0$ is the distribution with the total mass at $0$.

We apply Proposition 2.2 to our problem. First we note that when $\alpha < 2$, $G_{\alpha,\beta}^*(t)$ is locally square-integrable with respect to Lebesgue measure on $[0, \infty)$.

**Theorem 2.3.** (Case $\alpha = 0$.) $\mathcal{D}(\Psi_{0,\beta}) = I_{\log}(\mathbb{R}^d)$.

**Proof.** Note that $t(= G_{\alpha,\beta}(u)) \uparrow \infty$ if and only if $u(= G_{\alpha,\beta}(t)) \downarrow 0$, when $\alpha \geq 0$. It is enough to show that for some $C_1 \in (0, \infty)$, $u \sim C_1e^{-t}$ as $t \to \infty$. We have

$$u \quad \frac{u}{e^{-t}} = \exp \{ -G_{0,\beta}(u) \} = \exp \{ G_{0,\beta}(u) + \log u \} = \exp \left\{ \int_u^\infty x^{-1}e^{-x^\alpha} dx + \log u \right\}$$

$$= \exp \left\{ \beta^{-1} \int_{u^\alpha}^\infty y^{-1}e^{-y} dy - \beta^{-1} \int_{u^\alpha}^1 y^{-1} dy \right\}$$
By substituting (2.2) into (2.1), we have
\[
\begin{aligned}
(2.2) & \quad u \\
(2.1) & \quad \text{Therefore,}
\end{aligned}
\]
say, as \( u \downarrow 0 \). Hence \( u \sim C_1 e^{-t} \) as \( t \to \infty \), and the condition (1) of Proposition 2.2 is satisfied. Thus Proposition 2.2 (i) gives us the assertion. \( \square \)

**Theorem 2.4.** (Case \( \alpha \in (0, \infty) \).)

(i) If \( 0 < \alpha < 1 \), then \( \mathfrak{D} (\Psi_{\alpha, \beta}) = I_\alpha (\mathbb{R}^d) \).

(ii) If \( \alpha = 1 \), then \( \mathfrak{D} (\Psi_{1, \beta}) = I_1^* (\mathbb{R}^d) \).

(iii) If \( 1 < \alpha < 2 \), then \( \mathfrak{D} (\Psi_{\alpha, \beta}) = I_\alpha^0 (\mathbb{R}^d) \).

(iv) If \( \alpha \geq 2 \), then \( \mathfrak{D} (\Psi_{\alpha, \beta}) = \{ \delta_0 \} \).

**Proof.** (i) and (iii). It is enough to show that \( u \sim C_2 t^{-1/\alpha} \) as \( t \to \infty \) for some \( C_2 \in (0, \infty) \). We have, as \( t \to \infty \) (equivalently \( u \downarrow 0 \)), for some \( C_3 \in (0, \infty) \),
\[
\frac{u}{t^{-1/\alpha}} = \frac{u}{(G_{\alpha, \beta} (u))^{-1/\alpha}} = \frac{1}{(\beta^{-1} \int_{\alpha/\beta}^{\alpha} y^{-\alpha/\beta} e^{-y} dy)^{-1/\alpha}} \sim \frac{u}{(C_3 u^{-\alpha})^{-1/\alpha}} = C_3^{1/\alpha} =: C_2,
\]
and the condition (2) of Proposition 2.3 is satisfied. Thus Proposition 2.3 (ii) and (iv) give us the assertions.

(ii). Suppose \( \beta \neq 1 \). (The case \( \beta = 1 \) is proved in Sato (2006b).) We first have
\[
G_{1, \beta} (u) = \int_u^\infty x^{-2} e^{-x^\beta} dx = \int_u^\infty x^{-2} dx + \int_u^\infty x^{-2} (e^{-x^\beta} - 1) dx
\]
\[
= \int_u^\infty x^{-2} dx + \int_u^1 x^{-2} (e^{-x^\beta} - 1 + x^\beta) dx - \int_u^1 x^{-2 + \beta} dx + \int_1^\infty x^{-2} (e^{-x^\beta} - 1) dx
\]
\[
= u^{-1} + (\beta - 1)^{-1} u^{-1 + \beta} + O(1), \; u \downarrow 0.
\]
Thus
\[
t = G_{1, \beta}^* (t)^{-1} + (\beta - 1)^{-1} G_{1, \beta}^* (t)^{-1 + \beta} + O(1), \; t \to \infty.
\]
Therefore,
\[
G_{1, \beta}^* (t) = t^{-1} + (\beta - 1)^{-1} t^{-1} G_{1, \beta}^* (t)^{\beta} + O(t^{-1} G_{1, \beta}^* (t)), \; t \to \infty.
\]
We have shown in (i) and (iii) that \( u \sim C_2 t^{-1/\alpha} \), but this is also true for \( \alpha = 1 \). Hence
\[
u = G_{1, \beta}^* (t) = C_2 t^{-1} (1 + o(1)), \; t \to \infty.
\]
By substituting (2.2) into (2.1), we have
\[
G_{1, \beta}^* (t) = t^{-1} + C_2^3 (\beta - 1)^{-1} t^{-1 - \beta} + t^{-1} a(t), \; t \to \infty,
\]
\[
= t^{-1} \left( 1 + C_2^3 (\beta - 1)^{-1} t^{-\beta} + a(t) \right), \; t \to \infty,
\]
\[
\text{(2.1)}
\]
\[
\text{(2.2)}
\]
\[
\text{(2.3)}
\]
\[
\text{(2.4)}
\]
\[
\text{(2.5)}
\]
\[
\text{(2.6)}
\]
where
\[ a(t) = \begin{cases} 
    o(t^{-\beta}), & t \to \infty, \quad \text{when } 0 < \beta < 1, \\
    O(t^{-1}), & t \to \infty, \quad \text{when } \beta > 1. 
\end{cases} \]

Thus
\[ G_{1,\beta}^*(t) = t^{-1} \psi(t), \]
where
\[ \psi(t) := 1 + C_2^\beta(\beta - 1)^{-1} t^{-\beta} + a(t), \]
and
\[ \int_1^\infty t^{-1} |\psi(t) - 1| dt = \int_1^\infty t^{-1} |C_2^\beta(\beta - 1)^{-1} t^{-\beta} + a(t)| dt < \infty. \]
Thus the condition (3) of Proposition 2.2 is satisfied with \( t_0 = 1 \) and \( c = 1 \), and Proposition 2.2 (iii) gives us the assertion (iii).

(iv) The same as in Sato (2006b).

We now calculate the Lévy measure of \( \tilde{\mu} = \Psi_{\alpha,\beta}(\mu) \), and note that the mapping \( \Psi_{\alpha,\beta} \) is one-to-one.

**Lemma 2.5.** Let \( \alpha < 2 \) and \( \beta > 0 \). Let \( \mu \in \mathcal{D}(\Psi_{\alpha,\beta}) \) and \( \tilde{\mu} = \Psi_{\alpha,\beta}(\mu) \), and let \( \nu \) and \( \tilde{\nu} \) be the Lévy measures of \( \mu \) and \( \tilde{\mu} \), respectively.

1. We have
   \[ \tilde{\nu}(B) = \int_0^\infty \nu(s^{-1} B)s^{-\alpha-1}e^{-s^\beta} ds, \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}). \]  

2. If \( \nu \neq 0 \), and \( \nu \) has polar decomposition \( (\lambda, \nu_\xi) \), then a polar decomposition of \( \tilde{\nu} = (\tilde{\lambda}, \tilde{\nu}_\xi) \) is given by \( \tilde{\lambda} = \lambda \) and \( \tilde{\nu}_\xi(dr) = r^{-\alpha-1} \tilde{g}_\xi(r^\beta) dr \), where
   \[ \tilde{g}_\xi(u) = \int_0^\infty r^{\alpha} e^{-u/r^\beta} \nu_\xi(dr). \]

3. \( \tilde{g}_\xi \) in (2.4) satisfies the requirements of \( g_\xi \) in (1.2).

**Proof.** Suppose \( \mu \in \mathcal{D}(\Psi_{\alpha,\beta}) \) and \( \tilde{\mu} = \Psi_{\alpha,\beta}(\mu) \).

1. We see that (by using Proposition 2.6 of Sato (2006b)),
   \[
   \tilde{\nu}(B) = \int_0^{G_{\alpha,\beta}(0)} dt \int_{\mathbb{R}^d} 1_B(xG_{\alpha,\beta}(t)) \nu(dx) = -\int_0^\infty dG_{\alpha,\beta}(s) \int_{\mathbb{R}^d} 1_B(xs) \nu(dx)
   = \int_0^\infty s^{-\alpha-1}e^{-s^\beta} ds \int_{\mathbb{R}^d} 1_s^{-1}B(x) \nu(dx) = \int_0^\infty \nu(s^{-1}B)s^{-\alpha-1}e^{-s^\beta} ds,
   \]
   which is (2.3).
Next assume that \( \nu \neq 0 \) and \( \nu \) has polar decomposition \((\lambda, \nu_\xi)\). Then, we have

\[
\tilde{\nu}(B) = \int_0^\infty s^{-\alpha-1} e^{-s^\beta} ds \lambda(d\xi) \int_0^\infty 1_{s^{-1}B}(r\xi) \nu_\xi(dr)
\]

\[
= \int_S \lambda(d\xi) \int_0^\infty \nu_\xi(dr) r^{-1} \int_0^\infty (u/r)^{-\alpha-1} e^{-(u/r)^\beta} 1_B(u\xi)du
\]

\[
= \int_S \lambda(d\xi) \int_0^\infty 1_B(u\xi) u^{-\alpha-1} \tilde{g}_\xi(u^\beta)du,
\]

where \( \tilde{\lambda} = \lambda \) and

\[
(2.5) \quad \tilde{g}_\xi(u) = \int_0^\infty r^\alpha e^{-u/r^\beta} \nu_\xi(dr),
\]

which is (2.4). The finiteness of \( \tilde{g}_\xi \) is trivial for \( \alpha \leq 0 \). For \( \alpha > 0 \), since \( \int_{\mathbb{R}^d} |x|^\alpha \mu(dx) < \infty \). When \( \alpha > 0 \), note that \( \int_{\mathbb{R}^d} |x|^\alpha \mu(dx) < \infty \) implies \( \int_1^\infty r^\alpha \nu_\xi(dr) < \infty \), (see, e.g. Sato (1999), Theorem 25.3). Hence the integral \( \tilde{g}_\xi \) exists.

(3) If we put

\[
\tilde{Q}(B) = \int_0^\infty r^\alpha 1_B(r^{-\beta}) \nu_\xi(dr),
\]

then it follows that \( \tilde{g}_\xi(u) = \int_0^\infty e^{-uy} \tilde{Q}(dy) \), and thus \( \tilde{g}_\xi \) is completely monotone by Bernstein’s theorem. If \( \alpha \leq 0 \), then automatically \( \lim_{u \to \infty} \tilde{g}_\xi(u) = 0 \) \( \lambda \)-a.e. \( \xi \), since

\[
\int_{|x|>1} \tilde{\nu}(dx) = \int_S \lambda(d\xi) \int_1^\infty u^{-\alpha-1} \tilde{g}_\xi(u^\beta)du.
\]

When \( \alpha > 0 \), since \( \int_1^\infty r^\alpha \nu_\xi(dr) < \infty \), the assertion that \( \lim_{u \to \infty} \tilde{g}_\xi(u) = 0 \) \( \lambda \)-a.e. \( \xi \) also follows from (2.5) by the dominated convergence theorem.

The proof of the lemma is thus concluded.

\[\square\]

**Remark 2.6.** (2.3) can be written as, by introducing a transformation \( \Upsilon_{\alpha,\beta} \) of Lévy measures as \( \tilde{\nu} = \Upsilon_{\alpha,\beta}(\nu) \). Then this \( \Upsilon_{\alpha,\beta} \) is a generalized Upsilon transformation discussed in Barndorff-Nielsen et al. (2008) with the dilation measure \( \tau(ds) = s^{-\alpha-1} e^{-s^\beta} ds. \)

**Theorem 2.7.** For each \( \alpha < 2 \) and \( \beta > 0 \), the mapping \( \Psi_{\alpha,\beta} \) is one-to-one.

The proof is carried out in the same way as for Proposition 4.1 of Sato (2006b).
would be needed. (See, e.g., Sato (2006b) and Maejima et al. (2009).) Also, the
classes appearing in our motivation of introducing the classes \(J_{\alpha,\beta}(\mathbb{R}^d)\) are restricted
to the case \(\alpha \leq 0\).

**Theorem 2.8.** Let \(\alpha < 1\) and \(\beta > 0\). The range of the mapping \(\Psi_{\alpha,\beta}\) equals \(J_{\alpha,\beta}(\mathbb{R}^d)\),
that is,

\[
J_{\alpha,\beta}(\mathbb{R}^d) = \Psi_{\alpha,\beta}(\mathcal{D}(\Psi_{\alpha,\beta})).
\]

**Remark 2.9.** This theorem is already known for \(\alpha = -1, 0\) and \(\beta = 1\) in Theorems A and C of Barndorff-Nielsen et al. (2006) and for \(\alpha < 1\) and \(\beta = 1\) in Theorem 4.2
of Sato (2006b).

**Proof of Theorem 2.8.** We first show that \(\Psi_{\alpha,\beta}(\mathcal{D}(\Psi_{\alpha,\beta})) \subset J_{\alpha,\beta}(\mathbb{R}^d)\). Suppose
\(\mu \in \mathcal{D}(\Psi_{\alpha,\beta})\) and \(\bar{\mu} = \Psi_{\alpha,\beta}(\mu)\), and let \(\nu\) and \(\bar{\nu}\) be the
Lévy measures of \(\mu\) and \(\bar{\mu}\), respectively. Thus, if \(\nu = 0\), then \(\bar{\nu} = 0\) and \(\bar{\mu} \in J_{\alpha,\beta}(\mathbb{R}^d)\). When \(\nu \neq 0\), it follows
from Lemma 2.5 that \(\bar{\mu} \in J_{\alpha,\beta}(\mathbb{R}^d)\).

Next we show that \(J_{\alpha,\beta}(\mathbb{R}^d) \subset \Psi_{\alpha,\beta}(\mathcal{D}(\Psi_{\alpha,\beta}))\). Suppose \(\bar{\mu} \in J_{\alpha,\beta}(\mathbb{R}^d)\) with
the Lévy-Khintchine triplet \((\bar{A}, \bar{\nu}, \bar{\gamma})\). If \(\bar{\nu} = 0\), then \(\bar{\mu} = \Psi_{\alpha,\beta}(\mu)\) for some \(\mu \in \mathcal{D}(\Psi_{\alpha,\beta})\).
Thus, suppose that \(\bar{\nu} \neq 0\). Then, in a polar decomposition \((\bar{\lambda}, \bar{\nu}(\xi))\) of \(\bar{\nu}\), we have
\(\bar{\nu}(dr) = r^{-\alpha-1}\bar{g}_\xi(r^\beta)dr\), where \(\bar{g}_\xi(v)\) is completely monotone in \(v > 0\) \(\bar{\lambda}\)-a.e. \(\xi\), and is
measurable in \(\xi\). Thus by Bernstein’s theorem, there are measures \(\bar{Q}_\xi\) on \([0, \infty)\) such that

\[
\bar{g}_\xi(v) = \int_{[0,\infty)} e^{-v u} \bar{Q}_\xi(du).
\]

In general, \(\bar{Q}_\xi\) is a measure on \([0, \infty)\), but since \(\lim_{v \to \infty} \bar{g}_\xi(v) = 0\) \(\bar{\lambda}\)-a.e. \(\xi\), \(\bar{Q}_\xi\) does
not have a point mass at 0, and hence \(\bar{Q}_\xi\) is a measure on \((0, \infty)\). We see that

\[
\bar{\nu}(B) = \int_{S} \bar{\lambda}(d\xi) \int_{0}^{\infty} 1_{B}(r\xi) r^{-\alpha-1} \bar{g}_\xi(r^\beta) dr
\]

\[
= \int_{S} \bar{\lambda}(d\xi) \int_{0}^{\infty} 1_{B}(r\xi) r^{-\alpha-1} dr \int_{0}^{\infty} e^{-ru} \bar{Q}_\xi(du).
\]

Since \(\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\bar{\nu}(dx) < \infty\), we have

\[
\int_{S} \bar{\lambda}(d\xi) \int_{0}^{1} r^{1-\alpha} dr \int_{1}^{\infty} e^{-ru} \bar{Q}_\xi(du) + \int_{S} \bar{\lambda}(d\xi) \int_{0}^{\infty} r^{-\alpha-1} dr \int_{0}^{1} e^{-ru} \bar{Q}_\xi(du) < \infty.
\]

Hence, we have, by the change of variables \(r \to v\) by \(r^\beta u = v\),

\[
\int_{0}^{1} r^{1-\alpha} dr \int_{1}^{\infty} e^{-ru} \bar{Q}_\xi(du) = \int_{1}^{\infty} \bar{Q}_\xi(du) \int_{0}^{1} r^{1-\alpha} e^{-ru} dr.
\]
\[
\beta^{-1} \int_{1}^{\infty} u^{(\alpha-2)/\beta} \bar{Q}_\xi(du) \int_{0}^{u} v^{-1+(2-\alpha)/\beta} e^{-v} dv \geq C_4 \int_{1}^{\infty} u^{(\alpha-2)/\beta} \bar{Q}_\xi(du),
\]
where
\[
C_4 = \beta^{-1} \int_{0}^{1} v^{-1+(2-\alpha)/\beta} e^{-v} dv \in (0, \infty).
\]
Thus
\[
(2.7) \quad \int_{S} \tilde{\lambda}(d\xi) \int_{1}^{\infty} u^{(\alpha-2)/\beta} \bar{Q}_\xi(du) < \infty.
\]

We also have for any \( \alpha < 1 \),
\[
(2.8) \quad \int_{1}^{\infty} r^{-a-1} dr \int_{0}^{1} e^{-r^\beta} u \bar{Q}_\xi(du) = \int_{0}^{1} \tilde{Q}_\xi(du) \int_{1}^{\infty} r^{-a-1} e^{-r^\beta} u dr
\]
\[
= \beta^{-1} \int_{0}^{1} u^{a/\beta} \bar{Q}_\xi(du) \int_{1}^{\infty} v^{-1-(a/\beta)} e^{-v} dv \geq C_5 \int_{0}^{1} u^{a/\beta} \bar{Q}_\xi(du),
\]
where
\[
C_5 = \beta^{-1} \int_{1}^{\infty} v^{-1-(a/\beta)} e^{-v} dv \in (0, \infty).
\]
Thus
\[
(2.9) \quad \int_{S} \tilde{\lambda}(d\xi) \int_{0}^{1} u^{a/\beta} \bar{Q}_\xi(du) < \infty.
\]

In addition, if \( \alpha = 0 \), (2.8) is turned out to be
\[
\int_{1}^{\infty} r^{-1} dr \int_{0}^{1} e^{-r^\beta} u \bar{Q}_\xi(du) = \beta^{-1} \int_{0}^{1} \tilde{Q}_\xi(du) \int_{1}^{\infty} v^{-1} e^{-v} dv
\]
\[
\geq (\beta e)^{-1} \int_{0}^{1} \tilde{Q}_\xi(du) \int_{1}^{\infty} v^{-1} dv = (\beta e)^{-1} \int_{0}^{1} (-\log u) \bar{Q}_\xi(du).
\]
Thus, when \( \alpha = 0 \),
\[
(2.10) \quad \int_{S} \tilde{\lambda}(d\xi) \int_{0}^{1} (-\log u) \bar{Q}_\xi(du) < \infty.
\]

Furthermore,
\[
\int_{1}^{\infty} r^{-a-1} dr \int_{0}^{1} e^{-r^\beta} Q(du) \geq \int_{1}^{\infty} r^{-a-1} e^{-r^\beta} dr \int_{0}^{1} \tilde{Q}_\xi(du) = C_6 \int_{0}^{1} \tilde{Q}_\xi(du),
\]
where
\[
C_6 := \int_{1}^{\infty} r^{-a-1} e^{-r^\beta} dr \in (0, \infty).
\]
Thus we have
\[
(2.11) \quad \int_{S} \tilde{\lambda}(d\xi) \int_{0}^{1} \tilde{Q}_\xi(du) < \infty.
\]

Define
\[
(2.12) \quad \nu_\xi(B) = \int_{0}^{\infty} u^{a/\beta} 1_B (u^{-1/\beta}) \bar{Q}_\xi(du), \quad B \in B((0, \infty)).
\]
Then, it follows from (2.7) and (2.9) that

\begin{equation}
\int_S \tilde{\lambda}(d\xi) \int_0^\infty (r^2 \wedge 1) \nu_\xi(dr) = \int_S \tilde{\lambda}(d\xi) \int_0^\infty u^{\alpha/\beta} (u^{-2/\beta} \wedge 1) \tilde{Q}_\xi(du)
\end{equation}

\begin{equation}
= \int_S \tilde{\lambda}(d\xi) \left( \int_0^1 u^{\alpha/\beta} \tilde{Q}_\xi(du) + \int_1^\infty u^{(\alpha-2)/\beta} \tilde{Q}_\xi(du) \right) < \infty.
\end{equation}

Define \( \nu \) by

\begin{equation}
\nu(B) = \int_S \tilde{\lambda}(d\xi) \int_0^1 1_B(r\xi) \nu_\xi(dr).
\end{equation}

Then, by (2.13), \( \nu \) is the Lévy measure of some infinitely divisible distribution \( \mu \), and \( \mu \) belongs to \( \mathcal{D}(\Psi_{\alpha,\beta}) \) and satisfies

\begin{equation}
\tilde{\nu}(B) = \int_0^{G_{\alpha,\beta}(0)} \nu((G_{\alpha,\beta}^*(t))^{-1}B)dt.
\end{equation}

To show (2.15), by (2.6), (2.12) and (2.14), we have

\begin{align*}
\tilde{\nu}(B) &= \int_S \tilde{\lambda}(d\xi) \int_0^\infty 1_B(r\xi) r^{-\alpha-1} dr \int_0^\infty e^{-r^\beta u} \tilde{Q}_\xi(du) \\
&= \int_S \tilde{\lambda}(d\xi) \int_0^\infty 1_B(u^{-1/\beta}s\xi) s^{-\alpha-1} e^{-s^\beta} ds \int_0^\infty u^{\alpha/\beta} \tilde{Q}_\xi(du) \\
&= \int_0^\infty s^{-\alpha-1} e^{-s^\beta} ds \int_S \tilde{\lambda}(d\xi) \int_0^\infty 1_B(u^{-1/\beta}s\xi) u^{\alpha/\beta} \tilde{Q}_\xi(du) \\
&= \int_0^\infty s^{-\alpha-1} e^{-s^\beta} ds \int_S \tilde{\lambda}(d\xi) \int_0^\infty 1_B(rs\xi) \nu_\xi(dr) \\
&= \int_0^\infty s^{-\alpha-1} e^{-s^\beta} ds \int_S \tilde{\lambda}(d\xi) \int_0^\infty 1_{s^{-1}B}(r\xi) \nu_\xi(dr) \\
&= \int_0^\infty \nu(s^{-1}B) s^{-\alpha-1} e^{-s^\beta} ds = -\int_0^\infty \nu(s^{-1}B) dG_{\alpha,\beta}(s) \\
&= \int_0^{G_{\alpha,\beta}(0)} \nu((G_{\alpha,\beta}^*(t))^{-1}B)dt.
\end{align*}

To show that \( \mu \in \mathcal{D}(\Psi_{\alpha,\beta}) \), it is enough to show that \( \int_{|x|>1} |x|^{\alpha} \nu(dx) < \infty \), which is if and only if \( \mu \in I_\alpha(\mathbb{R}^d) \), when \( 0 < \alpha < 1 \), and \( \int_{|x|>1} \log |x| \nu(dx) < \infty \), which is if and only if \( \mu \in I_{\log}(\mathbb{R}^d) \), when \( \alpha = 0 \), (see Sato (1999), Theorem 25.3). Note that by (2.12) we see, for any nonnegative measurable function \( f \) on \( (0, \infty) \),

\begin{equation}
\int_0^\infty f(r) \nu_\xi(dr) = \int_0^\infty u^{\alpha/\beta} f(u^{-1/\beta}) \tilde{Q}_\xi(du).
\end{equation}
Thus if we choose \( f(r) = I[r > 1]r^\alpha \), where \( I[A] \) is the indicator function of the set \( A \), then \( \nu \) in (2.14) satisfies that for \( \alpha > 0 \)

\[
(2.16) \quad \int_{|x|>1} |x|^\alpha \nu(dx) = \int_S \tilde{\lambda}(d\xi) \int_1^\infty r^\alpha \nu_\xi(dr) = \int_S \tilde{\lambda}(d\xi) \int_0^1 \tilde{Q}_\xi(du) < \infty
\]
due to (2.11). When \( \alpha = 0 \),

\[
(2.17) \quad \int_{|x|>1} \log |x| \nu(dx) = \int_S \tilde{\lambda}(d\xi) \int_1^\infty \log r \nu_\xi(dr) = \int_S \tilde{\lambda}(d\xi) \int_0^1 \log u^{-1/\beta} \tilde{Q}_\xi(du) = \beta^{-1} \int S \tilde{\lambda}(d\xi) \int_0^1 (-\log u) \tilde{Q}_\xi(du) < \infty
\]
due to (2.10).

Notice again that

\[
\int_0^{\tilde{A}} (G_{\alpha,\beta}^*(t))^2 dt = -\int_0^\infty u^2 dG_{\alpha,\beta}(u) = \int_0^\infty u^{1-\alpha} e^{-u^\beta} du < \infty.
\]

Define \( A \) and \( \gamma \) by

\[
(2.18) \quad \tilde{A} = \left( \int_0^{G_{\alpha,\beta}(t)} G_{\alpha,\beta}^*(t)^2 dt \right) A
\]
and

\[
(2.19) \quad \tilde{\gamma} = \int_0^{G_{\alpha,\beta}(t)} G_{\alpha,\beta}^*(t) dt \left( \gamma + \int_{\mathbb{R}^d} x \left( \frac{1}{1 + |G_{\alpha,\beta}^*(t)x|^2} - \frac{1}{1 + |x|^2} \right) \nu(dx) \right).
\]

Here we have to check the finiteness of this integral. We first have

\[
\int_0^{G_{\alpha,\beta}(t)} G_{\alpha,\beta}^*(t) dt = -\int_0^\infty udG_{\alpha,\beta}(u) = \int_0^\infty u^{-\alpha} e^{-u^\beta} du < \infty,
\]
since \( \alpha < 1 \). Below, \( C_\gamma, C_8 \in (0, \infty) \) are suitable constants. Recall \( \alpha < 1 \). When \( \alpha \neq 0 \), we have

\[
\int_0^{G_{\alpha,\beta}(t)} G_{\alpha,\beta}^*(t) dt \int_{\mathbb{R}^d} |x| \left( \frac{1}{1 + |G_{\alpha,\beta}^*(t)x|^2} - \frac{1}{1 + |x|^2} \right) \nu(dx)
\]
\[
= \int_0^\infty u^{-\alpha} e^{-u^\beta} du \int_{\mathbb{R}^d} |x| \left( \frac{1}{1 + |ux|^2} - \frac{1}{1 + |x|^2} \right) \nu(dx)
\]
\[
\leq \int_0^\infty u^{-\alpha}(1 + u^2) e^{-u^\beta} du \int_{\mathbb{R}^d} \left( \frac{|x|^3}{(1 + |ux|^2)(1 + |x|^2)} \right) \nu(dx)
\]
\[
\leq \int_0^\infty u^{-\alpha}(1 + u^2) e^{-u^\beta} du
\]
\[
\times \left( \int_{|x| \leq 1} |x|^2 \nu(dx) + \int_{|x| > 1, |ux| \leq 1} |x| \nu(dx) + \int_{|x| > 1, |ux| > 1} \frac{|x|}{|ux|^2} \nu(dx) \right)
\]
\[
\begin{align*}
&= C_7 + \int_{|x|>1} |x|\nu(dx) \int_0^{1/|x|} u^{-\alpha} e^{-u^2} du \\
&\quad + \int_{|x|>1} \nu(dx) \int_0^{\infty} u^{-\alpha-1} (1 + u^2) e^{-u^2} du \\
&\leq C_7 + \int_{|x|>1} |x|\nu(dx) \int_0^{1/|x|} 2u^{-\alpha} du \\
&\quad + \int_{|x|>1} \nu(dx) \left\{ \left( \int_0^1 + \int_1^\infty \right) u^{-\alpha-1} (1 + u^2) e^{-u^2} du \right\} \\
&\leq C_7 + 2(1 - \alpha)^{-1} \int_{|x|>1} |x|^\alpha \nu(dx) \\
&\quad + \int_{|x|>1} \nu(dx) \left\{ -2\alpha^{-1} (1 - |x|^\alpha) + C_8 \right\} \\
&= C_7 + 2(1 - \alpha)^{-1} \int_{|x|>1} |x|^\alpha \nu(dx) \\
&\quad + 2\alpha^{-1} \int_{|x|>1} |x|^\alpha \nu(dx) + (C_8 - 2\alpha^{-1}) \int_{|x|>1} \nu(dx) < \infty,
\end{align*}
\]

by (2.16). When \(\alpha = 0\), since
\[
\int_0^1 u^{-\alpha-1} du = \int_0^1 u^{-1} du = \log |x|,
\]
in (2.20), we have
\[
\int_{|x|>1} \log |x| \nu(dx)
\]
instead of \(\int_{|x|>1} |x|^\alpha \nu(dx)\) in (2.21) in the calculation above. The finiteness of (2.22) is assured by (2.17).

Thus \(\gamma\) can be defined. Hence, if we denote by \(\mu\) an infinitely divisible distribution having the Lévy-Khintchine triplet \((A, \nu, \gamma)\) above, then by (2.15), (2.18) and (2.19), we see that
\[
\tilde{\mu} = \mathcal{L} \left( \int_0^{G_{\alpha,\beta}(0)} G_{\alpha,\beta}^* (t) dX_t^{(\mu)} \right),
\]
concluding that \(\tilde{\mu} \in \Psi_{\alpha,\beta}(\mathcal{D}(\Psi_{\alpha,\beta}))\). This completes the proof. \(\Box\)
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References


