α-selfdecomposable distributions and related Ornstein-Uhlenbeck type processes

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Abstract

The concept of selfdecomposability has been generalized to that of α-selfdecomposability, $\alpha \in \mathbb{R}$, by many authors. We first mention existing results on the class of α-selfdecomposable distributions and investigate remaining problems. We give complete characterizations by stochastic integrals with respect to Lévy processes for the case $1 \le \alpha < 2$. The main topic of this paper is Langevin type equations and the corresponding Ornstein-Uhlenbeck type processes related to α-selfdecomposable distributions.

Keywords: infinitely divisible distribution, Lévy process, selfdecomposable distribution, stochastic integral representation, Langevin type equation, Ornstein-Uhlenbeck type process
2000 MSC: 60E07, 60G51

1. Introduction

The purpose of this paper is to generalize the concept of selfdecomposability and to study related topics. Throughout this paper, we use the following notation: $\mathscr{P}(\mathbb{R}^d)$ is the class of all probability distributions on $\mathbb{R}^d$, $I(\mathbb{R}^d) = \{\mu \in \mathscr{P}(\mathbb{R}^d)\colon \mu \text{ is infinitely divisible}\}$, $I_{\log}(\mathbb{R}^d) = \left\{\mu \in I(\mathbb{R}^d)\colon \int_{\mathbb{R}^d} \log^+ |x|\mu(dx) < \infty\right\}$, where $\log^+ |x| = (\log |x|) \vee 0$ and $|x|$ is the Euclidean norm of $x \in \mathbb{R}^d$, $\widehat{\mu}(z), z \in \mathbb{R}^d$, is the characteristic function of $\mu \in \mathscr{P}(\mathbb{R}^d)$, $\mathcal{L}(X)$ is the law

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of X, $\widehat{\mathcal{L}}(X)$ is the characteristic function of $\mathcal{L}(X)$, δ_γ is the distribution with total mass at $\gamma \in \mathbb{R}^d$, $\{X_t^{(\mu)}\}$ is a Lévy process with $\mathcal{L}(X_1^{(\mu)}) = \mu$ and $\mathcal{B}_0(\mathbb{R}^d)$ is the totality of $B \in \mathcal{B}(\mathbb{R}^d)$ satisfying $\inf_{x\in B}|x| > 0$. Let $S = \{x \in \mathbb{R}^d \colon |x| = 1\}$ and we write, for $E \in \mathcal{B}((0,\infty))$ and $C \in \mathcal{B}(S)$, $EC = \{x \in \mathbb{R}^d \setminus \{0\} \colon |x| \in E \text{ and } x/|x| \in C\}$.

We use the Lévy-Khintchine representation of the characteristic function of $\mu \in I(\mathbb{R}^d)$ in the following form:

$$\widehat{\mu}(z) = \exp\left\{-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z, x\rangle}{1+|x|^2}\right)\nu(dx)\right\}, \quad z \in \mathbb{R}^d,$$

where $\langle\cdot,\cdot\rangle$ denotes Euclidean inner product on $\mathbb{R}^d$, A is a nonnegative-definite symmetric $d\times d$ matrix, $\gamma \in \mathbb{R}^d$, and ν is a measure, called Lévy measure, satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$. We call (A, ν, γ) the Lévy-Khintchine triplet of μ and we write $\mu = \mu_{(A,\nu,\gamma)}$ when we want to emphasize the Lévy-Khintchine triplet. The cumulant function $C_\mu(z), z \in \mathbb{R}^d$, of $\mu \in I(\mathbb{R}^d)$ is defined as the unique continuous function satisfying $\widehat{\mu}(z) = e^{C_\mu(z)}$ and $C_\mu(0) = 0$. For a random variable X with $\mathcal{L}(X) = \mu \in I(\mathbb{R}^d)$, we also write $C_X(z)$ for $C_\mu(z)$. For $t \geq 0$ and $\mu \in I(\mathbb{R}^d)$, we write μ^t for the distribution with characteristic function $\widehat{\mu}(z)^t := e^{tC_\mu(z)}$ and call μ^t the tth convolution of μ.

We also use the polar decomposition (1.1) of the Lévy measure ν of $\mu \in I(\mathbb{R}^d)$. Namely, if $0 < \nu(\mathbb{R}^d) \leq \infty$, then there exist a measure λ on S with $0 < \lambda(S) \leq \infty$ and a family $\{\nu_\xi, \xi \in S\}$ of measures on $(0,\infty)$ such that $\nu_\xi(B)$ is measurable in ξ for each $B \in \mathcal{B}((0,\infty))$, $0 < \nu_\xi((0,\infty)) \leq \infty$ for each $\xi \in S$ and

$$\nu(B) = \int_S \lambda(d\xi)\int_0^\infty 1\!\!1_B(r\xi)\nu_\xi(dr), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}). \tag{1.1}$$

Here λ and $\{\nu_\xi\}$ are uniquely determined by ν up to multiplication of measurable functions $c(\xi)$ and $c(\xi)^{-1}$, respectively, with $0 < c(\xi) < \infty$. (See, e.g., Lemma 2.1 of [2].) If $\nu = 0$, then we take $\lambda = 0$ and $\nu_\xi = 0$ for all $\xi \in S$. The measures λ and ν_ξ are called the spherical component and the radial component of ν, respectively. When ν has the polar decomposition (1.1), we may simply write $\nu = (\lambda, \nu_\xi)$.

A distribution $\mu \in I(\mathbb{R}^d)$ is selfdecomposable if for any $b > 1$, there exists $\rho_b \in I(\mathbb{R}^d)$ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}_b(z)$. The class of selfdecomposable distributions on $\mathbb{R}^d$, denoted by $L(\mathbb{R}^d)$, has a long history in the study of subclasses of $I(\mathbb{R}^d)$. The class $L(\mathbb{R}^d)$ has many characterizations. A distribution $\mu \in L(\mathbb{R}^d)$ is the limiting distribution of some normalized partial sums of independent random variables under the infinitesimal condition. It is also known that $\mu \in L(\mathbb{R}^d)$ if and only if the radial component ν_ξ of the Lévy measure of μ (whose precise definition will be explained in the next section) satisfies

$$\nu_\xi(dr) = r^{-1}\ell_\xi(r)dr, \quad r > 0, \tag{1.2}$$

where $\ell_\xi(r)$ is a nonnegative function which is measurable in ξ, and nonincreasing and right-continuous in r, (see, e.g., Theorem 15.10 of [22]). In the following, the totality of $\ell = \{\ell_\xi(r), \xi \in S, r \in (0,\infty)\}$ with the conditions above is denoted by $\mathcal{H}$. Furthermore, $\mu \in L(\mathbb{R}^d)$ has the stochastic integral representation $\mu = \mathcal{L}\left(\int_0^\infty e^{-t}dX_t\right)$ with respect to a Lévy process $\{X_t\}$ satisfying $\mathcal{L}(X_1) \in I_{\log}(\mathbb{R}^d)$, (see [30, 12, 28]). In addition, any selfdecomposable distribution is the limiting distribution of the solution of a Langevin type equation with Lévy noise. More precisely, let $\{X_t, t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$, $c \in \mathbb{R}$, and let M be an $\mathbb{R}^d$-valued random variable. Consider a Langevin type equation

$$Z_t = M + X_t - c\int_0^t Z_s ds, \quad t \geq 0. \tag{1.3}$$

The following is known, (see, e.g., [30, 20]):

$$Z_t = e^{-ct}\left(M + \int_0^t e^{cs}dX_s\right), \quad t \geq 0, \tag{1.4}$$

is an almost surely unique solution of (1.3). Let $c > 0$ and let M be independent of $\{X_t\}$. Then, $\mathcal{L}(Z_t)$ tends to some distribution μ as $t \to \infty$ if and only if $\mathcal{L}(X_1) \in I_{\log}(\mathbb{R}^d)$. In this case, $\mu \in L(\mathbb{R}^d)$. Also, if $c > 0$ and $\mu \in L(\mathbb{R}^d)$, then there exists a Lévy process $\{X_t\}$ with $\mathcal{L}(X_1) \in I_{\log}(\mathbb{R}^d)$ such that for any M independent of $\{X_t\}$, the law of Z_t in (1.4) tends to μ as $t \to \infty$. We also know that selfdecomposable distributions are related to stationary Ornstein-Uhlenbeck type processes, (see, e.g., [20, 14]). That is, let $c > 0$ and consider the Langevin equation

$$Z_t - Z_s = \int_s^t X(du) - c\int_s^t Z_u du, \quad -\infty < s \leq t < \infty, \tag{1.5}$$

where X is an $\mathbb{R}^d$-valued homogeneous independently scattered random measure over $\mathbb{R}$. Then, (1.5) has a stationary process solution called a stationary

Ornstein-Uhlenbeck type process if and only if $\mathcal{L}(X((0,1])) \in I_{\log}(\mathbb{R}^d)$, in which case, this stationary Ornstein-Uhlenbeck type process $\{Z_t, t \in \mathbb{R}\}$ is almost surely unique with the form

$$Z_t = e^{-ct} \int_{-\infty}^{t} e^{cu} X(du), \quad t \in \mathbb{R}, \tag{1.6}$$

and fulfills that $\mathcal{L}(Z_t) = \mathcal{L}\left(\int_0^\infty e^{-cu} X(du)\right) \in L(\mathbb{R}^d)$ for all $t \in \mathbb{R}$. Furthermore, selfdecomposable distributions are marginal distributions of selfsimilar additive processes, (see [21]). Finally, it has recently been recognized that some selfdecomposable distributions on $\mathbb{R}$ are important in the area of mathematical finance, (see [3]). Also, the selfdecomposability is related to the autoregressive processes, (see, e.g., [29] and references therein).

One of the purposes of this paper is to generalize the concept of selfdecomposability, which we have mentioned at the beginning of this paper, as follows.

Definition 1.1 (α-selfdecomposable distributions). Let $\alpha \in \mathbb{R}$. We say that $\mu \in I(\mathbb{R}^d)$ is *α-selfdecomposable*, if for any $b > 1$, there exists $\rho_b \in I(\mathbb{R}^d)$ satisfying

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)^{b^\alpha} \widehat{\rho}_b(z), \quad z \in \mathbb{R}^d. \tag{1.7}$$

We denote the totality of α-selfdecomposable distributions on $\mathbb{R}^d$ by $L^{\langle \alpha \rangle}(\mathbb{R}^d)$.

Notice that (1.7) implies that the Lévy measure ν of μ satisfies

$$\nu(B) \geq b^\alpha \nu(bB), \quad \text{for all } B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}). \tag{1.8}$$

Note also that $L^{\langle 0 \rangle}(\mathbb{R}^d) = L(\mathbb{R}^d)$, and thus the definition above is a generalization of selfdecomposability. The reason why we want to call $\mu \in L^{\langle \alpha \rangle}(\mathbb{R}^d)$ α-selfdecomposable will be explained in Section 3.

These classes $L^{\langle \alpha \rangle}(\mathbb{R}^d), \alpha \in \mathbb{R}$, have already been studied by O'Connor, Jurek, Maejima and others, and we survey some existing results related to our present work. Alf and O'Connor [1] and O'Connor [18] investigated the class of all infinitely divisible distributions on $\mathbb{R}$ with unimodal Lévy measures with mode 0, and showed that the class is equal to $L^{\langle -1 \rangle}(\mathbb{R})$. As to this class, Alf and O'Connor [1] studied stochastic integral characterizations with respect to Lévy processes. O'Connor [18] studied the decomposability (1.7) for $d = 1$ and $\alpha = -1$, and characterized this class by some limit theorem. O'Connor [17, 19] also studied the classes $L_\alpha, \alpha \in (0, 3)$, in his notation, which is equal to $L^{\langle \alpha-1\rangle}(\mathbb{R})$ in our notation in this paper. He defined these classes by a condition of radial components of Lévy measures, and characterized these classes by stochastic integrals with respect to Lévy processes, by the decomposability (1.7) for $d = 1$, and by similar limit theorems to that in the case $L^{\langle -1\rangle}(\mathbb{R})$. Jurek [5, 6, 10], Iksanov et al. [4] defined and studied so-called s-selfdecomposable distributions on a real separable Hilbert space H. The totality of s-selfdecomposable distributions, denoted by $\mathscr{U}(H)$ in their papers, is equal to $L^{\langle -1\rangle}(\mathbb{R}^d)$ when $H = \mathbb{R}^d$. Jurek [7, 8, 9], Jurek and Schreiber [11] studied the classes $\mathscr{U}_\beta(Q), \beta \in \mathbb{R}$, of distributions on a real separable Banach space E, where Q is a linear operator on E with certain properties. These classes are equal to $L^{\langle -\beta\rangle}(\mathbb{R}^d)$ if $E = \mathbb{R}^d$ and Q is the identity operator. They defined the classes $\mathscr{U}_\beta(Q)$ by some limit theorems. As to these classes, they studied the decomposability similar to (1.7) and stochastic integral characterizations, although some results are only for the case that Q is the identity operator. Maejima et al. [13] studied the classes $K_\alpha(\mathbb{R}^d), \alpha < 2$, which turns out to be equal to $L^{\langle \alpha\rangle}(\mathbb{R}^d) \cap \mathcal{C}_\alpha(\mathbb{R}^d)$, where $\mathcal{C}_\alpha(\mathbb{R}^d)$ is the totality of $\mu \in I(\mathbb{R}^d)$ whose Lévy measure ν satisfies $\lim_{r\to\infty} r^\alpha \int_{|x|>r} \nu(dx) = 0$.

In spite of this summarized research, there are still remaining problems. For example, any Langevin type equation similar to (1.3) or (1.5) related to the classes $L^{\langle \alpha\rangle}(\mathbb{R}^d)$ has not been studied. In this paper, we construct those. The forthcoming papers [16, 15] study nested subclasses of $L^{\langle \alpha\rangle}(\mathbb{R}^d)$ and an analogue of selfsimilar additive processes in the case of selfdecomposable distributions.

2. Stochastic integrals

In this section, we explain the notion of stochastic integrals of nonrandom measurable integrands used in this paper. Let J be an interval in $\mathbb{R}$ and let $\mathcal{B}_J^0$ denote the totality of $B \in \mathcal{B}(J)$ whose closure in the relative topology on J is compact. An $\mathbb{R}^d$-valued independently scattered random measure (i.s.r.m.) X over J is said to be homogeneous if $\mathcal{L}(X(B)) = \mathcal{L}(X(B + a))$ for all $B \in \mathcal{B}_J^0$ and $a \in \mathbb{R}$ satisfying $B + a \in \mathcal{B}_J^0$. See [14, 23, 24, 27], for the definition and the deep study of stochastic integrals of nonrandom measurable functions $f\colon J \to \mathbb{R}$ with respect to $\mathbb{R}^d$-valued i.s.r.m.'s X over

J, denoted by $\int_B f(s)X(ds), B \in \mathcal{B}_J^0$. For $s,t \in J$, we use the symbol

$$\int_s^t f(u)X(du) = \begin{cases} \int_{(s,t]} f(u)X(du), & \text{for } s < t, \\ 0, & \text{for } t = s, \\ -\int_{(t,s]} f(u)X(du), & \text{for } t < s, \end{cases}$$

which is understood to be càdlàg in $s \in J$ for each fixed $t \in J$ and càdlàg in $t \in J$ for each fixed $s \in J$, since such a modification always exists. Indeed, by Remark 3.16 of [14], for a fixed $t_0 \in J$, $Y_t := \int_{t_0}^t f(s)X(ds), t \in J$ has a càdlàg modification $\{\widetilde{Y}_t, t \in J\}$, and $\{\widetilde{Y}_t - \widetilde{Y}_s, s, t \in J\}$ is a desired modification of $\left\{\int_s^t f(u)X(du), s, t \in J\right\}$. If J is infinite to the right, then the improper stochastic integral $\int_t^\infty f(s)X(ds), t \in J$, is defined as the limit in probability of $\int_{(t,u]} f(s)X(ds)$ as $u \to \infty$ whenever the limit exists. Then we understand $\{\int_t^\infty f(s)X(ds), t \in J\}$ to be a càdlàg process, since such a modification always exists. If J is infinite to the left, then $\int_{-\infty}^t f(s)X(ds)$ for $t \in J$ is defined in a similar way and $\left\{\int_{-\infty}^t f(s)X(ds), t \in J\right\}$ is regarded as a càdlàg process. Similarly, when J is infinite to the left, for a nonrandom continuous function $q\colon J \to \mathbb{R}^d$, we regard $\left\{\text{p-lim}_{s\downarrow -\infty}\left(\int_s^t f(u)X(du) - q(s)\right), t \in J\right\}$ as a càdlàg process if the limit in probability exists, where p-lim means limit in probability, (which will be seen in Lemma 8.2).

For any Lévy process $\{X_t, t \geq 0\}$ on $\mathbb{R}^d$, there exists a unique $\mathbb{R}^d$-valued homogeneous i.s.r.m. X over $[0,\infty)$ satisfying $X_t = X([0,t])$ a.s. for each $t \geq 0$. Then, stochastic integrals $\int_B f(s)dX_s$ of nonrandom measurable functions $f\colon [0,\infty) \to \mathbb{R}$ with respect to Lévy processes $\{X_t\}$ are defined by $\int_B f(s)X(ds)$ for $B \in \mathcal{B}^0_{[0,\infty)}$. We say that the essential improper integral of a nonrandom measurable function $f\colon [0,\infty) \to \mathbb{R}$ with respect to a Lévy process $\{X_t\}$ on $\mathbb{R}^d$ is definable, if there exists a nonrandom function $q\colon [0,\infty) \to \mathbb{R}^d$ such that $\int_0^t f(s)dX_s - q(t)$ is convergent in probability as $t \to \infty$. For details on essential improper integrals, see [24, 25, 26].

Using stochastic integrals with respect to Lévy processes, we can define stochastic integral mappings as follows. Let $f\colon [0,\infty) \to \mathbb{R}$ be a nonrandom measurable function and $\mathfrak{D}(\Phi_f)$ the totality of $\mu \in I(\mathbb{R}^d)$ for which $\int_0^\infty f(t)dX_t^{(\mu)}$ is definable in the sense above. Define a mapping Φ_f from the domain $\mathfrak{D}(\Phi_f)$ into $I(\mathbb{R}^d)$ by

$$\Phi_f(\mu) = \mathcal{L}\left(\int_0^\infty f(t)dX_t^{(\mu)}\right), \quad \mu \in \mathfrak{D}(\Phi_f).$$

See also [25, 26]. The range of Φ_f denoted by $\mathfrak{R}(\Phi_f)$ is defined as $\Phi_f(\mathfrak{D}(\Phi_f))$. As to essential improper stochastic integrals, we use the following symbols. $\mathfrak{D}(\Phi_{f,\mathrm{es}})$ denotes the totality of $\mu \in I(\mathbb{R}^d)$ such that the essential improper integral of f with respect to $\{X_t^{(\mu)}\}$ is definable. Also, for $\mu \in \mathfrak{D}(\Phi_{f,\mathrm{es}})$, let

$$\Phi_{f,\mathrm{es}}(\mu) := \left\{ \mathcal{L}\left(\operatorname*{p-lim}_{t\to\infty} \left(\int_0^t f(s)dX_s^{(\mu)} - q(t) \right) \right) : q \text{ is an } \mathbb{R}^d\text{-valued nonrandom function} \right.$$
$$\left. \text{such that } \int_0^t f(s)dX_s^{(\mu)} - q(t) \text{ converges in probability as } t \to \infty \right\}.$$

3. Basic properties of the classes $L^{\langle\alpha\rangle}(\mathbb{R}^d), \alpha \in \mathbb{R}$

In this section, we study several properties of the classes $L^{\langle\alpha\rangle}(\mathbb{R}^d), \alpha \in \mathbb{R}$. We start with the following, which follows from Definition 1.1.

Proposition 3.1. *$L^{\langle\alpha\rangle}(\mathbb{R}^d)$ is decreasing in $\alpha \in \mathbb{R}$ with respect to set inclusion, that is, $L^{\langle\alpha_1\rangle}(\mathbb{R}^d) \supset L^{\langle\alpha_2\rangle}(\mathbb{R}^d)$ for $\alpha_1 < \alpha_2$.*

The following proposition is about $L^{\langle\alpha\rangle}(\mathbb{R}^d), \alpha \geq 2$.

Proposition 3.2. *If $\alpha > 2$, then $L^{\langle\alpha\rangle}(\mathbb{R}^d) = \{\delta_\gamma \colon \gamma \in \mathbb{R}^d\}$, and $L^{\langle 2\rangle}(\mathbb{R}^d)$ is the class of all Gaussian distributions.*

Proof. Let $\alpha \geq 2$. Fix any $b > 1$. If $\mu = \mu_{(A,\nu,\gamma)} \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$, then for any $n \in \mathbb{N}$,

$$\infty > \int_{|x|\leq b^{-n+1}} |x|^2\nu(dx) = \sum_{k=n}^{\infty} \int_{(b^{-k},b^{-k+1}]} |x|^2\nu(dx) \geq \sum_{k=n}^{\infty} b^{-2k}\nu\left((b^{-k}, b^{-k+1}]S\right)$$
$$\geq \sum_{k=n}^{\infty} b^{-2k}b^{(k-n)\alpha}\nu\left((b^{-n}, b^{-n+1}]S\right) = b^{-n\alpha}\sum_{k=n}^{\infty} b^{(\alpha-2)k}\nu\left((b^{-n}, b^{-n+1}]S\right),$$

where we have used (1.8). Hence $\nu\left((b^{-n}, b^{-n+1}]S\right) = 0$ if $\alpha \geq 2$. Thus

$$\nu\left(\mathbb{R}^d \setminus \{0\}\right) = \sum_{n=1}^{\infty} \nu\left((b^{-n}, b^{-n+1}]S\right) + \sum_{n=0}^{\infty} \nu\left((b^{n}, b^{n+1}]S\right)$$
$$\leq \sum_{n=1}^{\infty} \nu\left((b^{-n}, b^{-n+1}]S\right) + \sum_{n=0}^{\infty} b^{-(n+1)\alpha}\nu\left((b^{-1}, 1]S\right) = 0,$$

namely, $\nu = 0$. Furthermore, if $\alpha > 2$, then $A = 0$, since $A - b^{\alpha-2}A$ must be nonnegative-definite by (1.7). Conversely, δ-distributions and Gaussian distributions obviously belong to $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ with $\alpha > 2$ and $L^{\langle 2\rangle}(\mathbb{R}^d)$, respectively. □

By Proposition 3.2, trivially $\lim_{\alpha\uparrow\infty} L^{\langle\alpha\rangle}(\mathbb{R}^d) = \bigcap_{\alpha\in\mathbb{R}} L^{\langle\alpha\rangle}(\mathbb{R}^d) = \{\delta_\gamma \colon \gamma \in \mathbb{R}^d\}$. On the other hand, finding the limit $\lim_{\alpha\downarrow-\infty} L^{\langle\alpha\rangle}(\mathbb{R}^d) = \bigcup_{\alpha\in\mathbb{R}} L^{\langle\alpha\rangle}(\mathbb{R}^d)$ is a natural question. Actually, Jurek [9] showed that the closure under weak convergence of this limit is $I(\mathbb{R}^d)$. It seems an open question what the limit $\lim_{\alpha\downarrow-\infty} L^{\langle\alpha\rangle}(\mathbb{R}^d)$ itself is.

The following is a relation between α-selfdecomposable distributions and stable distributions, which is a reason why we want to call $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ α-selfdecomposable.

Proposition 3.3. *Let $0 < \alpha \le 2$. Then for $\beta \in [\alpha, 2]$, $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ contains all β-stable distributions. However, for $\beta \in (0, \alpha)$, $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ does not contain any β-stable distribution except δ-distributions.*

Proof. Let $0 < \alpha \le 2$. We first show that $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ includes all α-stable distributions. If μ is α-stable, then for any $a > 0$ there is $c(a) \in \mathbb{R}^d$ satisfying $\widehat{\mu}(z)^a = \widehat{\mu}(a^{1/\alpha}z)e^{i\langle c(a),z\rangle}$. It follows that for any $b > 1$, $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)^{b^\alpha}e^{i\langle -b^{-1}c(b^\alpha),z\rangle}$. Letting $\rho_b = \delta_{-b^{-1}c(b^\alpha)}$, we have (1.7). Thus $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$. Then, Proposition 3.1 yields that $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ includes all β-stable distributions for $\alpha \le \beta \le 2$.

Let $0 < \beta < \alpha$ and suppose that μ is β-stable. Then for any $a > 0$ there is $c(a) \in \mathbb{R}^d$ satisfying $\widehat{\mu}(z)^a = \widehat{\mu}(a^{1/\beta}z)e^{i\langle c(a),z\rangle}$, and thus $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)^{b^\beta}e^{i\langle -b^{-1}c(b^\beta),z\rangle}$ for $b > 1$. If $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$, then $\widehat{\mu}(b^{-1}z)^{b^\beta-b^\alpha}e^{i\langle -b^{-1}c(b^\beta),z\rangle}$ is the characteristic function of some $\rho_b \in I(\mathbb{R}^d)$. Then $1 \le |\widehat{\mu}(b^{-1}z)|^{b^\beta-b^\alpha} = |\widehat{\rho}_b(z)| \le 1$, which yields that μ is a δ-distribution. □

Also, as will be seen in Proposition 4.3, when $0 < \alpha < 2$, any $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ belongs to the normal domain of attraction of some α-stable distribution. This fact for $1 \le \alpha < 2$ was already shown by Jurek and Schreiber [11], but our proof is quite different from theirs. This is another reason why we call $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ an α-selfdecomposable distribution.

Moreover, for $\alpha > 0$, $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ satisfies $\int_{\mathbb{R}^d} |x|^\beta \mu(dx) < \infty$ for $0 < \beta < \alpha$, which will be proved in Proposition 4.4.

We conclude this section with the following proposition, which is about the continuity of $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ in α.

Proposition 3.4. $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ *is left-continuous in* $\alpha \in \mathbb{R}$*, namely,*

$$\bigcap_{\beta<\alpha} L^{\langle\beta\rangle}(\mathbb{R}^d) = L^{\langle\alpha\rangle}(\mathbb{R}^d) \quad \text{for all } \alpha \in \mathbb{R}.$$

Proof. Let $\alpha \in \mathbb{R}$. The inclusion $\bigcap_{\beta<\alpha} L^{\langle\beta\rangle}(\mathbb{R}^d) \supset L^{\langle\alpha\rangle}(\mathbb{R}^d)$ follows from Proposition 3.1. If $\mu \in \bigcap_{\beta<\alpha} L^{\langle\beta\rangle}(\mathbb{R}^d)$, then for each $\beta < \alpha$ and any $b > 1$, there exists $\rho_{b,\beta} \in I(\mathbb{R}^d)$ satisfying $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)^{b^\beta}\widehat{\rho}_{b,\beta}(z)$. Since $\mu \in I(\mathbb{R}^d)$, it holds that $\widehat{\mu}(z) \neq 0$ for all $z \in \mathbb{R}^d$. Therefore $\widehat{\rho}_{b,\beta}(z) = \widehat{\mu}(z)/\widehat{\mu}(b^{-1}z)^{b^\beta} \to \widehat{\mu}(z)/\widehat{\mu}(b^{-1}z)^{b^\alpha}$ as $\beta \uparrow \alpha$. Since $\widehat{\mu}(z)/\widehat{\mu}(b^{-1}z)^{b^\alpha}$ is continuous in z, it is the characteristic function of some $\rho_{b,\alpha} \in I(\mathbb{R}^d)$. Hence $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)^{b^\alpha}\widehat{\rho}_{b,\alpha}(z)$ for all $b > 1$. Thus $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$. □

The case $\alpha = 0$ is that of selfdecomposable distributions, which are well known. Because of this and Proposition 3.2, we do not consider the cases $\alpha = 0$ and $\alpha \geq 2$ from now on in this paper, unless otherwise stated.

4. Characterization of α-selfdecomposable distributions in terms of radial components of Lévy measures

Our first problem is to characterize the classes $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ in terms of radial components of Lévy measures. The answer is the following.

Theorem 4.1. *Let* $\alpha \in (-\infty, 0) \cup (0, 2)$*. Then,* $\mu \in I(\mathbb{R}^d)$ *with Lévy measure* $\nu = (\lambda, \nu_\xi)$ *belongs to* $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ *if and only if*

$$\nu_\xi(dr) = r^{-\alpha-1}\ell_\xi(r)dr, \quad r > 0, \tag{4.1}$$

for some $\ell \in \mathcal{H}$*.*

Proof. The case $\alpha < 0$ is Theorem 3.1 of [13].

Let $0 < \alpha < 2$. We first show the "if" part. Suppose that $\mu = \mu_{(A,\nu,\gamma)} \in I(\mathbb{R}^d)$, $\nu = (\lambda, \nu_\xi)$ and ν_ξ satisfies (4.1) for some $\ell \in \mathcal{H}$. Then, for any $b > 1$ and $B \in \mathcal{B}((0,\infty))$,

$$b^\alpha \nu_\xi(bB) = b^\alpha \int_{bB} r^{-\alpha-1}\ell_\xi(r)dr = \int_B u^{-\alpha-1}\ell_\xi(br)du \leq \int_B u^{-\alpha-1}\ell_\xi(r)du = \nu_\xi(B),$$

which implies that, if we let $\nu_b(B) := \nu(B) - b^\alpha\nu(bB) \geq 0$ for $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$, then ν_b is a Lévy measure. Letting $A_b := (1 - b^{\alpha-2})A$ and $\gamma_b := (1 - b^{\alpha-1})\gamma +$

$b^\alpha \int_{\mathbb{R}^d} x\big\{(1+|bx|^2)^{-1} - (1+|x|^2)^{-1}\big\}\nu(b\,dx)$, we have that $\rho_b = \rho_{b(A_b,\nu_b,\gamma_b)} \in I(\mathbb{R}^d)$ satisfies (1.7).

We next show the "only if" part. Suppose that $\mu = \mu_{(A,\nu,\gamma)} \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ and $\nu = (\lambda, \nu_\xi)$. Since $0 < \alpha < 2$, we have $L^{\langle\alpha\rangle}(\mathbb{R}^d) \subset L^{\langle 0\rangle}(\mathbb{R}^d) = L(\mathbb{R}^d)$ by Proposition 3.1, and thus, by (1.2), $\nu_\xi(dr) = r^{-1}k_\xi(r)dr$ for some $k \in \mathcal{H}$. Then, $\ell_\xi(r) := r^\alpha k_\xi(r)$ is right-continuous in r and measurable in ξ and satisfies (4.1). The property (1.8) for any $b > 1$ implies that $\nu_\xi(B) \geq b^\alpha \nu_\xi(bB)$ for any $b > 1$ and $B \in \mathcal{B}((0,\infty))$ λ-a.e. $\xi \in S$, which yields that

$$\int_B r^{-\alpha-1}\ell_\xi(r)dr \geq b^\alpha \int_{bB} r^{-\alpha-1}\ell_\xi(r)dr = \int_B u^{-\alpha-1}\ell_\xi(bu)du.$$

Thus $\ell_\xi(r)$ is nonincreasing in r for λ-a.e. $\xi \in S$. Hence $\ell \in \mathcal{H}$. □

The following Lemma 4.2 will be needed later.

Lemma 4.2. *Let $\alpha \in (-\infty, 0) \cup (0, 2)$ and $\mu \in I(\mathbb{R}^d)$ with Lévy measure $\nu = (\lambda, \nu_\xi)$. Suppose that ν_ξ satisfies* (4.1) *for some $\ell \in \mathcal{H}$. Then,*

$$\lim_{r\to\infty} \ell_\xi(r) = 0 \quad \lambda\text{-a.e. } \xi \in S \tag{4.2}$$

if and only if $\mu \in \mathcal{C}_\alpha(\mathbb{R}^d)$.

Proof. Using the dominated convergence theorem, we have

$$\begin{aligned}\lim_{r\to\infty} r^\alpha \int_{|x|>r} \nu(dx) &= \lim_{r\to\infty} r^\alpha \int_S \lambda(d\xi) \int_r^\infty u^{-\alpha-1}\ell_\xi(u)du \\ &= \lim_{r\to\infty} \int_S \lambda(d\xi) \int_1^\infty v^{-\alpha-1}\ell_\xi(rv)dv = \int_S \lambda(d\xi) \int_1^\infty v^{-\alpha-1} \lim_{r\to\infty} \ell_\xi(rv)dv.\end{aligned}$$

This implies the assertion. □

Let $0 < \alpha < 2$. We are now going to show, as mentioned in Section 3, that any element of $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ belongs to the normal domain of attraction of some α-stable distribution. In this paper, δ-distributions are understood to be α-stable for all $\alpha \in (0, 2)$. For a sequence $\{c_n\} \subset \mathbb{R}^d$ and an α-stable distribution σ_α on $\mathbb{R}^d$, let $\mathrm{NDA}(c_n, \sigma_\alpha)$ be the totality of $\mu \in \mathscr{P}(\mathbb{R}^d)$ such that

$$\lim_{n\to\infty} \widehat{\mu}\left(n^{-1/\alpha}z\right)^n e^{i\langle c_n, z\rangle} = \widehat{\sigma}_\alpha(z), \quad z \in \mathbb{R}^d.$$

We also write $\mathrm{NDA}(\sigma_\alpha)$ for $\bigcup_{\{c_n\}\subset\mathbb{R}^d} \mathrm{NDA}(c_n, \sigma_\alpha)$, and call it the normal domain of attraction of σ_α.

Proposition 4.3. *Let $0 < \alpha < 2$. Then we have*

$$L^{\langle \alpha \rangle}(\mathbb{R}^d) \subset \bigcup_{\sigma_\alpha \textit{ is } \alpha\textit{-stable}} \mathrm{NDA}(\sigma_\alpha).$$

Proof. Let $\mu = \mu_{(A,\nu,\gamma)} \in L^{\langle \alpha \rangle}(\mathbb{R}^d)$ with $\nu = (\lambda, \nu_\xi)$. Then, ν_ξ satisfies (4.1) for some $\ell \in \mathcal{H}$ by Theorem 4.1. By virtue of the properties of $\ell_\xi(r)$, $\ell_\xi(\infty) := \lim_{r\to\infty} \ell_\xi(r) \geq 0$ exists and is measurable in ξ. Defining Lévy measures $\nu^{(1)}$ and $\nu^{(2)}$ by

$$\begin{aligned}
\nu^{(1)}(B) &:= \int_S \lambda(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) r^{-\alpha-1} \left\{ \ell_\xi(r) - \ell_\xi(\infty) \right\} dr, \\
\nu^{(2)}(B) &:= \int_S \ell_\xi(\infty) \lambda(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) r^{-\alpha-1} dr,
\end{aligned}$$

and letting $\mu^{(1)} = \mu^{(1)}{}_{(A,\nu^{(1)},\gamma)}$ and $\mu^{(2)} = \mu^{(2)}{}_{(0,\nu^{(2)},0)}$, we have that $\mu = \mu^{(1)} * \mu^{(2)}$ and that $\mu^{(2)}$ is α-stable due to Theorem 14.3 of [22]. Then, in order to prove this proposition, it suffices to show that $\lim_{n\to\infty} \widehat{\mu}^{(1)}(n^{-1/\alpha} z)^n e^{i\langle c_n, z\rangle} = 1$ for some $\{c_n\} \subset \mathbb{R}^d$. Putting $c_n := -n^{1-1/\alpha}\gamma - n \int_{\mathbb{R}^d} x \left\{ (1+|x|^2)^{-1} - (1+|n^{1/\alpha}x|^2)^{-1} \right\} \nu^{(1)}(n^{1/\alpha}$ we have

$$nC_{\mu^{(1)}}\left(n^{-1/\alpha}z\right) + i\langle c_n, z\rangle = -\frac{1}{2} n^{1-2/\alpha} \langle z, Az\rangle + n \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z,x\rangle}{1+|x|^2} \right) \nu^{(1)}\left(n^{1/\alpha} dx\right).$$

For any bounded continuous function $f\colon \mathbb{R}^d \to \mathbb{R}$ vanishing on a neighborhood of 0, we have

$$\lim_{n\to\infty} n \int_{\mathbb{R}^d} f(x) \nu^{(1)}\left(n^{1/\alpha} dx\right) = 0,$$

since $\mu^{(1)} \in \mathcal{C}_\alpha(\mathbb{R}^d)$ due to Lemma 4.2. Noting that $\nu^{(1)}(B) \geq n\nu^{(1)}(n^{1/\alpha}B)$ for $B \in \mathcal{B}(\mathbb{R}^d)$ by the definition of $\nu^{(1)}$, we have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \overline{\lim_{n\to\infty}} \left| n^{1-2/\alpha}\langle z, Az\rangle + n \int_{|x|\leq\varepsilon} \langle z, x\rangle^2 \nu^{(1)}\left(n^{1/\alpha} dx\right) \right| \\
&\qquad \leq \lim_{n\to\infty} n^{1-2/\alpha} \left|\langle z, Az\rangle\right| + \lim_{\varepsilon\downarrow 0} \int_{|x|\leq\varepsilon} \langle z, x\rangle^2 \nu^{(1)}(dx) = 0.
\end{aligned}$$

Then, it follows from Theorem 8.7 of [22] that $\lim_{n\to\infty} \widehat{\mu}^{(1)}(n^{-1/\alpha} z)^n e^{i\langle c_n, z\rangle} = 1$. □

Theorem 4.1 also implies the following.

Proposition 4.4. *Let* $0 < \beta < \alpha$ *and* $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$*. Then,* $\int_{\mathbb{R}^d} |x|^\beta \mu(dx) < \infty$.

Proof. Let $\nu = (\lambda, \nu_\xi)$ be the Lévy measure of $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$. Then, Theorem 4.1 yields that ν_ξ is expressible as (4.1) for some $\ell \in \mathcal{H}$. It follows that

$$\infty > \int_{|x|>1} \nu(dx) = \int_S \lambda(d\xi) \int_1^\infty r^{-\alpha-1}\ell_\xi(r)dr = \int_1^\infty r^{-\alpha-1}dr \int_S \ell_\xi(r)\lambda(d\xi),$$

which entails that $\int_S \ell_\xi(r_0)\lambda(d\xi) < \infty$ for some $r_0 > 1$. If $\beta < \alpha$, then there exists $\varepsilon > 0$ satisfying $\beta + \varepsilon < \alpha$. Then,

$$\begin{aligned}\int_{|x|>r_0} |x|^\beta \nu(dx) &= \int_S \lambda(d\xi) \int_{r_0}^\infty r^{\beta-\alpha-1}\ell_\xi(r)dr \\ &\le \int_S \lambda(d\xi) \int_{r_0}^\infty r^{-\varepsilon-1}\ell_\xi(r_0)dr = \frac{r_0^{-\varepsilon}}{\varepsilon} \int_S \ell_\xi(r_0)\lambda(d\xi) < \infty.\end{aligned}$$

This yields that $\int_{\mathbb{R}^d} |x|^\beta \mu(dx) < \infty$, by Corollary 25.8 of [22]. □

5. Stochastic integral characterizations of α-selfdecomposable distributions

We next consider the characterizations of the classes $L^{\langle\alpha\rangle}(\mathbb{R}^d), \alpha \in (-\infty, 0) \cup (0, 2)$, by stochastic integrals with respect to Lévy processes. Similar results to the following theorem were proved by Jurek [7, 8, 9] and Jurek and Schreiber [11], but the case $1 \le \alpha < 2$ was not completed, and their form of the mappings are slightly different from ours. Thus we show the following theorem. In what follows, we use the mappings $\Phi_\alpha, \alpha \in (-\infty, 2)$, defined by

$$\Phi_\alpha(\mu) = \begin{cases} \mathcal{L}\left(\int_0^{-1/\alpha} (1+\alpha t)^{-1/\alpha} dX_t^{(\mu)}\right), & \text{when } \alpha < 0, \\ \mathcal{L}\left(\int_0^\infty e^{-t} dX_t^{(\mu)}\right), & \text{when } \alpha = 0, \\ \mathcal{L}\left(\int_0^\infty (1+\alpha t)^{-1/\alpha} dX_t^{(\mu)}\right), & \text{when } 0 < \alpha < 2. \end{cases} \tag{5.1}$$

By [26], the domains $\mathfrak{D}(\Phi_\alpha), \alpha \in (-\infty, 2)$, are the following.

$$\mathfrak{D}(\Phi_\alpha) = \begin{cases} I(\mathbb{R}^d), & \text{when } \alpha < 0, \\ I_{\log}(\mathbb{R}^d), & \text{when } \alpha = 0, \\ I_\alpha(\mathbb{R}^d), & \text{when } 0 < \alpha < 1, \\ I_1^*(\mathbb{R}^d), & \text{when } \alpha = 1, \\ I_\alpha^0(\mathbb{R}^d), & \text{when } 1 < \alpha < 2, \end{cases}$$

where

$$I_\alpha(\mathbb{R}^d) = \left\{ \mu \in I(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} |x|^\alpha \mu(dx) < \infty \right\}, \quad \text{for } \alpha > 0,$$

$$I_\alpha^0(\mathbb{R}^d) = \left\{ \mu \in I_\alpha(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} x\mu(dx) = 0 \right\}, \quad \text{for } \alpha \geq 1,$$

$$I_1^*(\mathbb{R}^d) = \left\{ \mu = \mu_{(A,\nu,\gamma)} \in I_1^0(\mathbb{R}^d) \colon \lim_{T\to\infty} \int_1^T t^{-1} dt \int_{|x|>t} x\nu(dx) \text{ exists in } \mathbb{R}^d \right\}.$$

We also denote $\Phi_{f,\mathrm{es}}$ with $f(t) = (1+t)^{-1}$ by $\Phi_{1,\mathrm{es}}$.

Theorem 5.1. *Let $\alpha \in (-\infty, 0) \cup (0, 2)$.*

(i) *When $\alpha < 0$, $\widetilde{\mu} \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ if and only if $\widetilde{\mu} = \Phi_\alpha(\mu)$ for some $\mu \in I(\mathbb{R}^d)$.*

(ii) *When $0 < \alpha < 1$, $\widetilde{\mu} \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ if and only if*

$$\widetilde{\mu} = \sigma_\alpha * \Phi_\alpha(\mu), \tag{5.2}$$

where $\mu \in I_\alpha(\mathbb{R}^d)$ and σ_α is a strictly α-stable distribution.

(iii) *When $\alpha = 1$, $\widetilde{\mu} \in L^{\langle 1\rangle}(\mathbb{R}^d)$ if and only if*

$$\widetilde{\mu} = \sigma_1 * \widetilde{\rho},$$

where $\widetilde{\rho} \in \Phi_{1,\mathrm{es}}(\rho)$ for some $\rho \in I_1(\mathbb{R}^d)$ and σ_1 is a 1-stable distribution.

(iv) *When $1 < \alpha < 2$, $\widetilde{\mu} \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ if and only if (5.2) holds for some $\mu \in I_\alpha^0(\mathbb{R}^d)$ and some α-stable distribution σ_α.*

Proof. (i) See Theorem 4.6 of [13].

(ii) The "if" part is obvious, since α-stable distributions and the images of Φ_α are α-selfdecomposable and $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ is closed under convolution. Let us show the "only if" part. If $\widetilde{\mu} = \widetilde{\mu}_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})} \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ and $\widetilde{\nu} = (\widetilde{\lambda}, \widetilde{\nu}_\xi)$, then

$\widetilde{\nu}_\xi(dr) = r^{-\alpha-1}\widetilde{\ell}_\xi(r)dr$ for some $\widetilde{\ell} \in \mathcal{H}$ by Theorem 4.1. By virtue of the properties of $\widetilde{\ell}_\xi(r)$, $\widetilde{\ell}_\xi(\infty) := \lim_{r\to\infty} \widetilde{\ell}_\xi(r) \geq 0$ exists and is measurable in ξ. Due to Theorem 14.3, Proposition 14.5 and Theorem 14.7 of [22], letting

$$A_{\sigma_\alpha} := 0,\ \nu_{\sigma_\alpha}(B) := \int_S \widetilde{\ell}_\xi(\infty)\widetilde{\lambda}(d\xi) \int_0^\infty 1\!\!1_B(r\xi)r^{-\alpha-1}dr,\ \gamma_{\sigma_\alpha} := \int_{\mathbb{R}^d} \frac{x}{1+|x|^2}\nu_{\sigma_\alpha}(dx),$$

and $\sigma_\alpha = \sigma_{\alpha(A_{\sigma_\alpha},\nu_{\sigma_\alpha},\gamma_{\sigma_\alpha})}$, we have the strict α-stability of σ_α. Furthermore, it follows that $\widetilde{\mu} = \sigma_\alpha * \widetilde{\rho}_{(A_{\widetilde{\rho}},\nu_{\widetilde{\rho}},\gamma_{\widetilde{\rho}})}$, where

$$A_{\widetilde{\rho}} = \widetilde{A}, \quad \nu_{\widetilde{\rho}}(B) = \int_S \widetilde{\lambda}(d\xi) \int_0^\infty 1\!\!1_B(r\xi)r^{-\alpha-1} \left\{\widetilde{\ell}_\xi(r) - \widetilde{\ell}_\xi(\infty)\right\} dr, \quad \gamma_{\widetilde{\rho}} = \widetilde{\gamma} - \gamma_{\sigma_\alpha}.$$

Then $\widetilde{\rho} \in L^{\langle\alpha\rangle}(\mathbb{R}^d) \cap \mathcal{C}_\alpha(\mathbb{R}^d)$, and thus Theorem 4.6 of [13] yields that $\widetilde{\rho} = \Phi_\alpha(\mu)$ for some $\mu \in I_\alpha(\mathbb{R}^d)$. Then (5.2) holds.

(iii) Note that $\mathfrak{D}(\Phi_{1,\mathrm{es}}) = I_1(\mathbb{R}^d)$ due to Theorem 2.8 of [26]. If $\rho = \rho_{(A_\rho,\nu_\rho,\gamma_\rho)} \in I_1(\mathbb{R}^d)$, then $\Phi_{1,\mathrm{es}}(\rho)$ is the set of all $\widetilde{\rho} = \widetilde{\rho}_{(A_{\widetilde{\rho}},\nu_{\widetilde{\rho}},\gamma_{\widetilde{\rho}})} \in I(\mathbb{R}^d)$ such that

$$A_{\widetilde{\rho}} = \int_0^\infty (1+s)^{-2}A_\rho ds = A_\rho, \quad \nu_{\widetilde{\rho}}(B) = \int_0^\infty \nu_\rho\left((1+s)B\right) ds = \int_0^1 \nu_\rho\left(u^{-1}B\right) u^{-2}du,$$

and $\gamma_{\widetilde{\rho}} \in \mathbb{R}^d$ is arbitrary, due to Theorem 3.11 of [27]. We first show the "if" part. $\widetilde{\rho}$ with the form above is α-selfdecomposable, because of Lemma 5.1 of [13] and Theorem 4.1 of this paper. Thus the "if" part is proved. We next show the "only if" part. If $\widetilde{\mu} = \widetilde{\mu}_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})} \in L^{\langle 1\rangle}(\mathbb{R}^d)$ and $\widetilde{\nu} = (\widetilde{\lambda}, \widetilde{\nu}_\xi)$, then $\widetilde{\nu}_\xi(dr) = r^{-2}\widetilde{\ell}_\xi(r)dr$ for some $\ell \in \mathcal{H}$ in view of Theorem 4.1. By virtue of the properties of $\widetilde{\ell}_\xi(r)$, $\widetilde{\ell}_\xi(\infty) := \lim_{r\to\infty} \widetilde{\ell}_\xi(r) \geq 0$ exists and is measurable in ξ. Letting

$$\nu_{\sigma_1}(B) := \int_S \widetilde{\ell}_\xi(\infty)\widetilde{\lambda}(d\xi) \int_0^\infty 1\!\!1_B(r\xi)r^{-2}dr$$

and $\sigma_1 = \sigma_{1(0,\nu_{\sigma_1},0)}$, we have the 1-stability of σ_1 due to Theorem 14.3 of [22]. Furthermore, it follows that $\widetilde{\mu} = \sigma_1 * \widetilde{\rho}_{(A_{\widetilde{\rho}},\nu_{\widetilde{\rho}},\gamma_{\widetilde{\rho}})}$, where

$$A_{\widetilde{\rho}} = \widetilde{A}, \quad \nu_{\widetilde{\rho}}(B) = \int_S \widetilde{\lambda}(d\xi) \int_0^\infty 1\!\!1_B(r\xi)r^{-2} \left\{\widetilde{\ell}_\xi(r) - \widetilde{\ell}_\xi(\infty)\right\} dr, \quad \gamma_{\widetilde{\rho}} = \widetilde{\gamma}.$$

Then Lemma 5.1 of [13] yields that $\widetilde{\rho} \in \Phi_{1,\mathrm{es}}(\rho)$ for some $\rho \in I_1(\mathbb{R}^d)$.

(iv) The proof of the "if" part is the same as that in (ii). Let us show the "only if" part. If $\widetilde{\mu} = \widetilde{\mu}_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})} \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ and $\widetilde{\nu} = (\widetilde{\lambda}, \widetilde{\nu}_\xi)$, then $\widetilde{\nu}_\xi(dr) =$

$r^{-\alpha-1}\widetilde{\ell}_\xi(r)dr$ for some $\ell \in \mathcal{H}$ by Theorem 4.1. By virtue of the properties of $\widetilde{\ell}_\xi(r)$, $\widetilde{\ell}_\xi(\infty) := \lim_{r\to\infty} \widetilde{\ell}_\xi(r) \geq 0$ exists and is measurable in ξ. If we let

$$\nu_{\widetilde{\rho}}(B) := \int_S \widetilde{\lambda}(d\xi) \int_0^\infty 1\!\!1_B(r\xi) r^{-\alpha-1} \left\{ \widetilde{\ell}_\xi(r) - \widetilde{\ell}_\xi(\infty) \right\} dr,$$

then $\int_{\mathbb{R}^d} \frac{|x|^3}{1+|x|^2} \nu_{\widetilde{\rho}}(dx) \leq \int_{|x|\leq 1} |x|^2 \nu_{\widetilde{\rho}}(dx) + \int_{|x|>1} |x| \nu_{\widetilde{\rho}}(dx) < \infty$ by Proposition 4.4. Then letting

$$A_{\widetilde{\rho}} := \widetilde{A}, \quad \gamma_{\widetilde{\rho}} := -\int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2} \nu_{\widetilde{\rho}}(dx),$$

we have that $\widetilde{\rho} = \widetilde{\rho}_{(A_{\widetilde{\rho}}, \nu_{\widetilde{\rho}}, \gamma_{\widetilde{\rho}})} \in L^{\langle\alpha\rangle}(\mathbb{R}^d) \cap \mathcal{C}_\alpha(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} x\widetilde{\rho}(dx) = 0$. Hence Theorem 4.6 of [13] yields that $\widetilde{\rho} = \Phi_\alpha(\mu)$ for some $\mu \in I^0_\alpha(\mathbb{R}^d)$. Furthermore, we have $\widetilde{\mu} = \sigma_{\alpha(A_{\sigma_\alpha}, \nu_{\sigma_\alpha}, \gamma_{\sigma_\alpha})} * \Phi_\alpha(\mu)$, where

$$A_{\sigma_\alpha} = 0, \quad \nu_{\sigma_\alpha}(B) = \int_S \widetilde{\ell}_\xi(\infty)\widetilde{\lambda}(d\xi) \int_0^\infty 1\!\!1_B(r\xi) r^{-\alpha-1} dr, \quad \gamma_{\sigma_\alpha} = \widetilde{\gamma} - \gamma_{\widetilde{\rho}}.$$

This σ_α is α-stable, because of Theorem 14.3 of [22]. □

The stochastic integral characterization of an α-selfdecomposable distribution is unique in the following sense.

Theorem 5.2. (i) *Let $\alpha < 0$. Then, $\mu \in I(\mathbb{R}^d)$ in Theorem 5.1 is uniquely determined by $\widetilde{\mu} \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$.*

(ii) *Let $0 < \alpha < 1$. Then, the strictly α-stable distribution σ_α and $\mu \in I_\alpha(\mathbb{R}^d)$ in Theorem 5.1 are uniquely determined by $\widetilde{\mu} \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$.*

(iii) *Let $\alpha = 1$. If $\rho_j \in I_1(\mathbb{R}^d)$, $\widetilde{\rho}_j \in \Phi_{1,\mathrm{es}}(\rho_j)$, $\sigma_{1,j}$ is a 1-stable distribution for $j = 1, 2$, and $\sigma_{1,1} * \widetilde{\rho}_1 = \sigma_{1,2} * \widetilde{\rho}_2$, then $\sigma_{1,1} = \sigma_{1,2} * \delta_{-\widetilde{c}}$, $\widetilde{\rho}_1 = \widetilde{\rho}_2 * \delta_{\widetilde{c}}$ and $\rho_1 = \rho_2 * \delta_c$ for some $\widetilde{c}, c \in \mathbb{R}^d$.*

(iv) *Let $1 < \alpha < 2$. Then, the α-stable distribution σ_α and $\mu \in I^0_\alpha(\mathbb{R}^d)$ in Theorem 5.1 are uniquely determined by $\widetilde{\mu} \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$.*

To prove this theorem, we need the following lemma.

Lemma 5.3. (i) *For $\alpha \in (-\infty, 0) \cup (0, 2)$, the mapping Φ_α is injective.*

(ii) *Let $\rho_1, \rho_2 \in I_1(\mathbb{R}^d)$, $\widetilde{\rho}_1 \in \Phi_{1,\mathrm{es}}(\rho_1)$, $\widetilde{\rho}_2 \in \Phi_{1,\mathrm{es}}(\rho_2)$, and $\widetilde{\rho}_1 = \widetilde{\rho}_2$. Then, $\rho_1 = \rho_2 * \delta_c$ for some $c \in \mathbb{R}^d$.*

Proof. (i) Let $\mu_1, \mu_2 \in \mathfrak{D}(\Phi_\alpha)$ and $\Phi_\alpha(\mu_1) = \Phi_\alpha(\mu_2)$. Then,

$$C_{\Phi_\alpha(\mu_j)}(z) - b^\alpha C_{\Phi_\alpha(\mu_j)}(b^{-1}z) = \int_0^{(b^\alpha-1)/\alpha} C_{\mu_j}\left((1+\alpha s)^{-1/\alpha} z\right) ds$$

for $j = 1, 2$ and any $b > 1$. Therefore

$$\int_0^t C_{\mu_1}\left((1+\alpha s)^{-1/\alpha} z\right) ds = \int_0^t C_{\mu_2}\left((1+\alpha s)^{-1/\alpha} z\right) ds$$

for any $t > 0$. Differentiating the equation above in t, we have $C_{\mu_1}(z) = C_{\mu_2}(z)$.

(ii) For $j = 1, 2$, let (A_j, ν_j, γ_j) and $(\widetilde{A}_j, \widetilde{\nu}_j, \widetilde{\gamma}_j)$ be the Lévy-Khintchine triplets of ρ_j and $\widetilde{\rho}_j$, respectively. Then,

$$\widetilde{A}_j = \int_0^\infty (1+s)^{-2} A_j ds = A_j, \quad \widetilde{\nu}_j(B) = \int_0^\infty \nu_j\left((1+s)B\right) ds, \quad B \in \mathcal{B}(\mathbb{R}^d),$$

and $\widetilde{\gamma}_j \in \mathbb{R}^d$ is arbitrary, for $j = 1, 2$. We have

$$\widetilde{\nu}_j(B) - b\widetilde{\nu}_j(bB) = \int_0^{b-1} \nu_j\left((1+s)B\right) ds, \quad B \in \mathcal{B}_0(\mathbb{R}^d),$$

for $j = 1, 2$ and any $b > 1$. If $\widetilde{\rho}_1 = \widetilde{\rho}_2$, then $A_1 = A_2$ and

$$\int_0^t \nu_1\left((1+s)B\right) ds = \int_0^t \nu_2\left((1+s)B\right) ds, \quad \text{for any } t > 0 \text{ and any } B \in \mathcal{B}(\mathbb{R}^d).$$

Differentiating the equation above in t, we have $\nu_1 = \nu_2$. □

Proof of Theorem 5.2. (i) See Lemma 5.3 (i).

(ii) Suppose that $\mu_j \in I_\alpha(\mathbb{R}^d)$, $\widetilde{\nu}_j$ is the Lévy measure of $\Phi_\alpha(\mu_j)$, $\sigma_{\alpha,j}$ is a strictly α-stable distribution on $\mathbb{R}^d$ with Lévy-Khintchine triplet $(0, \nu_{\alpha,j}, \gamma_{\alpha,j})$ for $j = 1, 2$, and $\sigma_{\alpha,1} * \Phi_\alpha(\mu_1) = \sigma_{\alpha,2} * \Phi_\alpha(\mu_2)$. Then

$$\nu_{\alpha,1}(B) + \widetilde{\nu}_1(B) = \nu_{\alpha,2}(B) + \widetilde{\nu}_2(B), \quad B \in \mathcal{B}(\mathbb{R}^d),$$

and therefore

$$\nu_{\alpha,1}(B) + r^\alpha \widetilde{\nu}_1(rB) = \nu_{\alpha,2}(B) + r^\alpha \widetilde{\nu}_2(rB), \quad B \in \mathcal{B}(\mathbb{R}^d)$$

for $r > 0$, where we have used the property of the Lévy measures of α-stable distributions that $r^\alpha \nu_{\alpha,j}(rB) = \nu_{\alpha,j}(B)$. Taking into account that $\Phi_\alpha(\mu_j) \in \mathcal{C}_\alpha(\mathbb{R}^d)$ for $j = 1, 2$ and letting $r \to \infty$, we have $\nu_{\alpha,1}(B) = \nu_{\alpha,2}(B)$ for $B \in \mathcal{B}_0(\mathbb{R}^d)$. Thus $\nu_{\alpha,1} = \nu_{\alpha,2}$. For $j = 1, 2$, the strict stability of $\sigma_{\alpha,j}$ implies that $\gamma_{\alpha,j}$ is uniquely determined by $\nu_{\alpha,j}$. Hence $\gamma_1 = \gamma_2$. Therefore $\sigma_{\alpha,1} = \sigma_{\alpha,2}$. Then, $\Phi_\alpha(\mu_1) = \Phi_\alpha(\mu_2)$ and thus $\mu_1 = \mu_2$ due to Lemma 5.3 (i).

(iii) Suppose that $\rho_j \in I_1(\mathbb{R}^d)$, $\widetilde{\rho}_j \in \Phi_{1,\mathrm{es}}(\rho_j)$, $\sigma_{1,j}$ is a 1-stable distribution for $j = 1, 2$, and $\sigma_{1,1} * \widetilde{\rho}_1 = \sigma_{1,2} * \widetilde{\rho}_2$. Due to Lemma 5.1 of [13] and Lemma 4.2 of this paper, $\widetilde{\rho}_j \in \mathcal{C}_1(\mathbb{R}^d)$ for $j = 1, 2$. Then we have that $\sigma_{1,1}$ and $\sigma_{1,2}$ have the same Lévy measure in the same way as (ii). Therefore $\sigma_{1,1} = \sigma_{1,2} * \delta_{-\widetilde{c}}$ for some $\widetilde{c} \in \mathbb{R}^d$. Then, $\widetilde{\rho}_1 = \widetilde{\rho}_2 * \delta_{\widetilde{c}}$. Since $\widetilde{\rho}_1 \in \Phi_{1,\mathrm{es}}(\rho_1)$ and $\widetilde{\rho}_2 * \delta_{\widetilde{c}} \in \Phi_{1,\mathrm{es}}(\rho_2)$, it follows from Lemma 5.3 (ii) that $\rho_1 = \rho_2 * \delta_c$ for some $c \in \mathbb{R}^d$.

(iv) Suppose that $\mu_j \in I^0_\alpha(\mathbb{R}^d)$, $(\widetilde{A}_j, \widetilde{\nu}_j, \widetilde{\gamma}_j)$ is the Lévy-Khintchine triplet of $\Phi_\alpha(\mu_j)$, $\sigma_{\alpha,j}$ is an α-stable distribution on $\mathbb{R}^d$ for $j = 1, 2$, and $\sigma_{\alpha,1} * \Phi_\alpha(\mu_1) = \sigma_{\alpha,2} * \Phi_\alpha(\mu_2)$. Then we have that $\sigma_{\alpha,1}$ and $\sigma_{\alpha,2}$ have the same Lévy measure in the same way as (ii), and so do $\Phi_\alpha(\mu_1)$ and $\Phi_\alpha(\mu_2)$ since $\sigma_{\alpha,1} * \Phi_\alpha(\mu_1) = \sigma_{\alpha,2} * \Phi_\alpha(\mu_2)$. Since $\Phi_\alpha(\mu_j)$ satisfies $\int_{\mathbb{R}^d} x \Phi_\alpha(\mu_j)(dx) = 0$, $\widetilde{\gamma}_j$ is uniquely determined by $\widetilde{\nu}_j$, for $j = 1, 2$. Then $\widetilde{\gamma}_1 = \widetilde{\gamma}_2$. Since $\sigma_{\alpha,1} * \Phi_\alpha(\mu_1) = \sigma_{\alpha,2} * \Phi_\alpha(\mu_2)$ and α-stable distributions do not have Gaussian matrices, we have $\widetilde{A}_1 = \widetilde{A}_2$. Thus $\Phi_\alpha(\mu_1) = \Phi_\alpha(\mu_2)$, which yields that $\mu_1 = \mu_2$ by virtue of Lemma 5.3 (i), and $\sigma_{\alpha,1} = \sigma_{\alpha,2}$. □

As to the continuity of $\Phi_\alpha(\mu)$ in α for a fixed μ, we have the following, which is partially mentioned in [13].

Proposition 5.4. *$\Phi_\alpha(\mu)$ is continuous in $\alpha \in [\alpha_1, \alpha_2]$ with respect to weak convergence for each fixed $\mu \in \mathfrak{D}(\Phi_{\alpha_2})$, where $[\alpha_1, \alpha_2]$ is an interval included in $(-\infty, 1) \cup (1, 2)$.*

Proof. Note that $C_{\Phi_\alpha(\mu)}(z) = \int_0^1 C_\mu(sz) s^{-\alpha-1} ds$ for $\alpha < 2$. Suppose $\alpha \to \alpha_0$ in $[\alpha_1, \alpha_2]$ and $\mu \in \mathfrak{D}(\Phi_{\alpha_2})$. It follows that $|C_\mu(sz)| s^{-\alpha-1} \le |C_\mu(sz)| s^{-\alpha_2-1}$ for all $s \in (0, 1)$ and $\alpha \in [\alpha_1, \alpha_2]$. Since $\mu \in \mathfrak{D}(\Phi_{\alpha_2})$, Propositions 3.4 and 2.17 of [24] and Theorem 2.4 of [26] yield that $\int_0^1 |C_\mu(sz)| s^{-\alpha_2-1} ds < \infty$. Then we can apply the dominated convergence theorem and we have that $\lim_{[\alpha_1,\alpha_2] \ni \alpha \to \alpha_0} \int_0^1 C_\mu(sz) s^{-\alpha-1} ds = \int_0^1 C_\mu(sz) s^{-\alpha_0-1} ds$. Thus $\Phi_\alpha(\mu) \to \Phi_{\alpha_0}(\mu)$. □

6. A Langevin type equation

The contents of this and the following two sections are the main issues of this paper, finding Langevin type equations like (1.3) and (1.5) related to distributions in the classes $L^{\langle\alpha\rangle}(\mathbb{R}^d), \alpha \in (-\infty, 0) \cup (0, 2)$.

For our purpose, we first consider the following *Langevin type equation* for $\alpha \in (-\infty, 0) \cup (0, 2)$:

$$Z_t = M + \int_{t_0}^{t} X(ds) - \int_{t_0}^{t} (1-\alpha s)^{-1} Z_s ds, \quad \begin{cases} t_0 \le t < \infty, & \text{when } \alpha < 0, \\ t_0 \le t < 1/\alpha, & \text{when } 0 < \alpha < 2, \end{cases} \tag{6.1}$$

where

$$t_0 \in \begin{cases} (1/\alpha, \infty), & \text{when } \alpha < 0, \\ (-\infty, 1/\alpha), & \text{when } 0 < \alpha < 2, \end{cases}$$

X is an $\mathbb{R}^d$-valued homogeneous i.s.r.m. over $\mathbb{R}$, and M is an $\mathbb{R}^d$-valued random variable. A stochastic process $\{Z_t\}$ is said to be a *solution* of the Langevin type equation (6.1) or an *Ornstein-Uhlenbeck type process* (OU type process) generated by (α, X) starting from $Z_{t_0} = M$ if $\{Z_t\}$ is a càdlàg process and satisfies (6.1) almost surely. Then, we have the following.

Theorem 6.1. *The stochastic process* $\{Z_t\}$ *defined by*

$$\begin{aligned} Z_t = (1-\alpha t)^{1/\alpha} \Big\{ & (1-\alpha t_0)^{-1/\alpha} M \\ & + \int_{t_0}^{t} (1-\alpha s)^{-1/\alpha} X(ds) \Big\}, \quad \begin{cases} t_0 \le t < \infty, & \textit{when } \alpha < 0, \\ t_0 \le t < 1/\alpha, & \textit{when } 0 < \alpha < 2, \end{cases} \end{aligned} \tag{6.2}$$

is the almost surely unique solution of the equation (6.1).

Proof. A process $\{Z_t\}$ defined by (6.2) is a càdlàg process. It follows that for each fixed t,

$$
\begin{aligned}
&\int_{t_0}^{t} (1-\alpha s)^{-1} Z_s ds \\
&= \int_{t_0}^{t} (1-\alpha s)^{1/\alpha-1} \left\{ (1-\alpha t_0)^{-1/\alpha} M + \int_{t_0}^{s} (1-\alpha u)^{-1/\alpha} X(du) \right\} ds \\
&= (1-\alpha t_0)^{-1/\alpha} M \int_{t_0}^{t} (1-\alpha s)^{1/\alpha-1} ds + \int_{t_0}^{t} (1-\alpha u)^{-1/\alpha} X(du) \int_{u}^{t} (1-\alpha s)^{1/\alpha-1} ds \\
&= M - (1-\alpha t)^{1/\alpha} (1-\alpha t_0)^{-1/\alpha} M + \int_{t_0}^{t} X(du) - (1-\alpha t)^{1/\alpha} \int_{t_0}^{t} (1-\alpha u)^{-1/\alpha} X(du) \\
&= M + \int_{t_0}^{t} X(du) - Z_t \quad \text{a.s.,}
\end{aligned}
$$

where we have used the Fubini type theorem (Theorem 4.7 of [23] for $t_0 \geq 0$ and the natural extension for $t_0 < 0$). Note that $\int_{t_0}^{t} (1-\alpha s)^{-1} Z_s ds$ is continuous in t because of the continuity of the Lebesgue measure ds and the local boundedness of $s \mapsto (1-\alpha s)^{-1} Z_s(\omega)$ for each fixed ω on

$$
\begin{cases} [t_0, \infty), & \text{when } \alpha < 0, \\ [t_0, 1/\alpha), & \text{when } 0 < \alpha < 2. \end{cases}
$$

Also, $M + \int_{t_0}^{t} X(du) - Z_t$ is a càdlàg process. Hence, almost surely,

$$
\int_{t_0}^{t} (1-\alpha s)^{-1} Z_s ds = M + \int_{t_0}^{t} X(du) - Z_t, \quad \text{for all } t,
$$

which yields (6.1) almost surely.

It remains to prove the uniqueness of solutions of (6.1). Suppose that $\{Z_t^{(1)}\}$ and $\{Z_t^{(2)}\}$ are solutions of (6.1). Setting $Z_t := Z_t^{(1)} - Z_t^{(2)}$, we have, almost surely,

$$
Z_t = -\int_{t_0}^{t} (1-\alpha s)^{-1} Z_s ds, \quad \text{for } \begin{cases} t_0 \leq t < \infty, & \text{when } \alpha < 0, \\ t_0 \leq t < 1/\alpha, & \text{when } 0 < \alpha < 2. \end{cases}
$$

Since the right-hand side is continuous in t in the same way as above, so is the left-hand side, which is Z_t. Then, we can apply Gronwall's inequality and have $Z_t = 0$ for all t. Therefore it holds almost surely that for any t, $Z_t^{(1)} = Z_t^{(2)}$. □

The following holds immediately from Theorem 6.1.

Corollary 6.2. *If M is independent of X and $\mathcal{L}(M) \in I(\mathbb{R}^d)$, then the solution $\{Z_t\}$ of* (6.1) *satisfies $\mathcal{L}(Z_t) \in I(\mathbb{R}^d)$ for all t.*

7. Limiting distributions of Ornstein-Uhlenbeck type processes and normal domains of attraction of α-selfdecomposable distributions

7.1. Limiting distributions of Ornstein-Uhlenbeck type processes

We now consider the Langevin type equation (6.1) with $t_0 = 0$ and the limit of its solution, namely,

$$Z_t = M + X_t - \int_0^t (1-\alpha s)^{-1} Z_s ds, \quad \begin{cases} 0 \le t < \infty, & \text{when } \alpha < 0, \\ 0 \le t < 1/\alpha, & \text{when } 0 < \alpha < 2, \end{cases} \tag{7.1}$$

where $\{X_t, t \ge 0\}$ is a Lévy process on $\mathbb{R}^d$ and M is an $\mathbb{R}^d$-valued random variable. This is an extension of (1.3) with $c = 1$. Theorem 6.1 yields that

$$Z_t = (1-\alpha t)^{1/\alpha} \left\{ M + \int_0^t (1-\alpha s)^{-1/\alpha} dX_s \right\}, \quad \begin{cases} 0 \le t < \infty, & \text{when } \alpha < 0, \\ 0 \le t < 1/\alpha, & \text{when } 0 < \alpha < 2, \end{cases} \tag{7.2}$$

is an almost surely unique solution of (7.1). We start with the following.

Theorem 7.1. *Suppose that $\{X_t, t \ge 0\}$ is a Lévy process on $\mathbb{R}^d$ and M is an $\mathbb{R}^d$-valued infinitely divisible random variable independent of $\{X_t\}$. Let $\{Z_t\}$ be the process in* (7.2).

(i) *Let $\alpha < 0$. Then,*

$$\lim_{t \uparrow \infty} \mathcal{L}(Z_t)^{(1-\alpha t)^{-1}} = \Phi_\alpha(\mathcal{L}(X_1)) \in L^{\langle \alpha \rangle}(\mathbb{R}^d),$$

which does not depend on M. Furthermore, if we choose M such that $\mathcal{L}(M) = \Phi_\alpha(\mathcal{L}(X_1))$, then

$$\mathcal{L}(Z_t)^{(1-\alpha t)^{-1}} = \Phi_\alpha(\mathcal{L}(X_1)), \quad \textit{for all } t \in [0, \infty).$$

(ii) *Let* $0 < \alpha < 2$ *and let* $\mathcal{L}(n^{-1/\alpha}M)^n$ *converge to some (automatically strictly* α*-stable) distribution* σ_α *as* $n \to \infty$*. Then,* $\mathcal{L}(Z_t)^{(1-\alpha t)^{-1}}$ *converges as* $t \uparrow 1/\alpha$ *if and only if* $\mathcal{L}(X_1) \in \mathfrak{D}(\Phi_\alpha)$*, in which case, it follows that*

$$\lim_{t\uparrow 1/\alpha} \mathcal{L}(Z_t)^{(1-\alpha t)^{-1}} = \sigma_\alpha * \Phi_\alpha(\mathcal{L}(X_1)) \in L^{\langle\alpha\rangle}(\mathbb{R}^d).$$

Furthermore, if σ_α *is strictly* α*-stable,* $\mathcal{L}(X_1) \in \mathfrak{D}(\Phi_\alpha)$ *and* $\mathcal{L}(M) = \sigma_\alpha * \Phi_\alpha(\mathcal{L}(X_1))$*, then* $\lim_{n\to\infty} \mathcal{L}(n^{-1/\alpha}M)^n = \sigma_\alpha$ *and*

$$\mathcal{L}(Z_t)^{(1-\alpha t)^{-1}} = \sigma_\alpha * \Phi_\alpha(\mathcal{L}(X_1)) \quad \textit{for all } t \in [0, 1/\alpha).$$

Proof. Recall the mapping Φ_α in (5.1). We have

$$\begin{aligned}
&(1-\alpha t)^{-1} C_{Z_t}(z) \\
&\quad = (1-\alpha t)^{-1} \left\{ C_M\left((1-\alpha t)^{1/\alpha} z\right) + \int_0^t C_{X_1}\left((1-\alpha t)^{1/\alpha}(1-\alpha s)^{-1/\alpha} z\right) ds \right\} \\
&\quad = (1-\alpha t)^{-1} C_M\left((1-\alpha t)^{1/\alpha} z\right) + \int_0^{\{(1-\alpha t)^{-1}-1\}/\alpha} C_{X_1}\left((1+\alpha u)^{-1/\alpha} z\right) du.
\end{aligned} \tag{7.3}$$

If $\alpha < 0$, then $\lim_{t\uparrow\infty}(1-\alpha t)^{-1} C_{Z_t}(z) = C_{\Phi_\alpha(\mathcal{L}(X_1))}(z)$. If $0 < \alpha < 2$, then $(1-\alpha t)^{-1} C_M\left((1-\alpha t)^{1/\alpha} z\right) \to C_{\sigma_\alpha}(z)$ and $\{(1-\alpha t)^{-1} - 1\}/\alpha \to \infty$ as $t \uparrow 1/\alpha$, and thus $\mathcal{L}(X_1) \in \mathfrak{D}(\Phi_\alpha)$ if and only if $\mathcal{L}(Z_t)^{(1-\alpha t)^{-1}}$ converges to some distribution as $t \uparrow 1/\alpha$, and this limit $\lim_{t\uparrow 1/\alpha} \mathcal{L}(Z_t)^{(1-\alpha t)^{-1}}$ is equal to $\sigma_\alpha * \Phi_\alpha(\mathcal{L}(X_1))$.

Let $\alpha < 0$. If $\mathcal{L}(M) = \Phi_\alpha(\mathcal{L}(X_1))$, then (7.3) is

$$\begin{aligned}
&(1-\alpha t)^{-1} C_{Z_t}(z) \\
&\quad = (1-\alpha t)^{-1} \int_0^{-1/\alpha} C_{X_1}\left((1+\alpha s)^{-1/\alpha}(1-\alpha t)^{1/\alpha} z\right) ds + \int_0^{\{(1-\alpha t)^{-1}-1\}/\alpha} C_{X_1}\left((1+\alpha u)^{-1/\alpha} z\right) d \\
&\quad = \int_{\{(1-\alpha t)^{-1}-1\}/\alpha}^{-1/\alpha} C_{X_1}\left((1+\alpha u)^{-1/\alpha} z\right) du + \int_0^{\{(1-\alpha t)^{-1}-1\}/\alpha} C_{X_1}\left((1+\alpha u)^{-1/\alpha} z\right) du = C_{\Phi_\alpha(\mathcal{L}(X_1)}
\end{aligned}$$

which yields that $\mathcal{L}(Z_t)^{(1-\alpha t)^{-1}} = \Phi_\alpha(\mathcal{L}(X_1))$ for all $t \in [0, \infty)$.

Let $0 < \alpha < 2$. If σ_α is strictly α-stable, $\mathcal{L}(X_1) \in \mathfrak{D}(\Phi_\alpha)$ and $\mathcal{L}(M) =$

$\sigma_\alpha * \Phi_\alpha(\mathcal{L}(X_1))$, then

$$
\begin{aligned}
&(1-\alpha t)^{-1} C_M\left((1-\alpha t)^{1/\alpha} z\right) \\
&= (1-\alpha t)^{-1} C_{\sigma_\alpha}\left((1-\alpha t)^{1/\alpha} z\right) + (1-\alpha t)^{-1} \int_0^\infty C_{X_1}\left((1+\alpha s)^{-1/\alpha}(1-\alpha t)^{1/\alpha} z\right) ds \\
&= C_{\sigma_\alpha}(z) + \int_{\{(1-\alpha t)^{-1}-1\}/\alpha}^\infty C_{X_1}\left((1+\alpha u)^{-1/\alpha} z\right) du \to C_{\sigma_\alpha}(z) \quad \text{as } t \uparrow 1/\alpha,
\end{aligned}
$$

which is equivalent to that $\mathcal{L}(n^{-1/\alpha}M)^n \to \sigma_\alpha$ as $n \to \infty$, and (7.3) is

$$
\begin{aligned}
(1-\alpha t)^{-1} C_{Z_t}(z) &= C_{\sigma_\alpha}(z) + \int_{\{(1-\alpha t)^{-1}-1\}/\alpha}^\infty C_{X_1}\left((1+\alpha u)^{-1/\alpha} z\right) du \\
&\quad + \int_0^{\{(1-\alpha t)^{-1}-1\}/\alpha} C_{X_1}\left((1+\alpha u)^{-1/\alpha} z\right) du \\
&= C_{\sigma_\alpha}(z) + C_{\Phi_\alpha(\mathcal{L}(X_1))}(z),
\end{aligned}
$$

which yields that $\mathcal{L}(Z_t)^{(1-\alpha t)^{-1}} = \sigma_\alpha * \Phi_\alpha(\mathcal{L}(X_1))$ for all $t \in [0, 1/\alpha)$. □

The following is one of the main theorems of this paper.

Theorem 7.2. (i) *Let $\alpha < 0$. Then, $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ if and only if there exists a Lévy process $\{X_t, t \geq 0\}$ on $\mathbb{R}^d$ such that for one and hence any $\mathbb{R}^d$-valued infinitely divisible random variable M independent of $\{X_t\}$,*

$$\lim_{t\uparrow\infty} \mathcal{L}(Z_t)^{(1-\alpha t)^{-1}} = \mu,$$

where $\{Z_t\}$ is the process in (7.2).

(ii) *Let $0 < \alpha < 1$. Then, $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ if and only if there exist a Lévy process $\{X_t, t \geq 0\}$ on $\mathbb{R}^d$ with $\mathcal{L}(X_1) \in I_\alpha(\mathbb{R}^d)$ and a strictly α-stable distribution σ_α on $\mathbb{R}^d$ such that for one and hence any $\mathbb{R}^d$-valued random variable M independent of $\{X_t\}$ with $\mathcal{L}(M) \in \mathrm{NDA}(0, \sigma_\alpha) \cap I(\mathbb{R}^d)$,*

$$\lim_{t\uparrow 1/\alpha} \mathcal{L}(Z_t)^{(1-\alpha t)^{-1}} = \mu,$$

where $\{Z_t\}$ is the process in (7.2).

(iii) *Let $\alpha = 1$. Then, $\mu \in L^{\langle 1\rangle}(\mathbb{R}^d)$ if and only if there exist a Lévy process $\{X_t, t \geq 0\}$ on $\mathbb{R}^d$ with $\mathcal{L}(X_1) \in I_1(\mathbb{R}^d)$, a 1-stable distribution σ_1 on*

$\mathbb{R}^d$ *such that for one and hence any* $\mathbb{R}^d$*-valued random variable* M *independent of* $\{X_t\}$ *with* $\mathcal{L}(M) \in \mathrm{NDA}(\sigma_1) \cap I(\mathbb{R}^d)$*, there is a nonrandom function* $q\colon [0,1) \to \mathbb{R}^d$ *satisfying*

$$\lim_{t\uparrow 1} \mathcal{L}\left(Z_t - q(t)\right)^{(1-t)^{-1}} = \mu,$$

where $\{Z_t\}$ *is the process in* (7.2).

(iv) *Let* $1 < \alpha < 2$*. Then,* $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ *if and only if there exist a Lévy process* $\{X_t, t \geq 0\}$ *on* $\mathbb{R}^d$ *with* $\mathcal{L}(X_1) \in I^0_\alpha(\mathbb{R}^d)$*, a strictly* α*-stable distribution* σ_α *on* $\mathbb{R}^d$ *and some* $c \in \mathbb{R}^d$ *such that for one and hence any* $\mathbb{R}^d$*-valued random variable* M *independent of* $\{X_t\}$ *with* $\mathcal{L}(M) \in \mathrm{NDA}(0, \sigma_\alpha) \cap I(\mathbb{R}^d)$*,*

$$\lim_{t\uparrow 1/\alpha} \mathcal{L}\left(Z_t + (1-\alpha t)c\right)^{(1-\alpha t)^{-1}} = \mu,$$

where $\{Z_t\}$ *is the process in* (7.2).

Proof. The statements (i), (ii) and (iv) follow from Theorems 5.1 and 7.1.

Let us prove (iii). We first show the "only if" part. If $\mu \in L^{\langle 1\rangle}(\mathbb{R}^d)$, then $\mu = \sigma_1 * \widetilde{\rho}$, where $\widetilde{\rho} \in \Phi_{1,\mathrm{es}}(\rho)$ for some $\rho \in I_1(\mathbb{R}^d)$ and σ_1 is a 1-stable distribution, in view of Theorem 5.1. If we let $\{X_t\}$ be a Lévy process with $\mathcal{L}(X_1) = \rho$, then there exists a nonrandom function $p\colon [0,\infty) \to \mathbb{R}^d$ satisfying $\lim_{T\to\infty} \mathcal{L}\left(\int_0^T (1+s)^{-1} dX_s - p(T)\right) = \widetilde{\rho}$, namely,

$$\int_0^{(1-t)^{-1}-1} C_{X_1}\left((1+u)^{-1}z\right) du - i\left\langle p\left((1-t)^{-1}-1\right), z\right\rangle \to C_{\widetilde{\rho}}(z) \quad \text{as } t \uparrow 1.$$

Let M be an arbitrary $\mathbb{R}^d$-valued random variable independent of $\{X_t\}$ with $\mathcal{L}(M) \in \mathrm{NDA}(\sigma_1) \cap I(\mathbb{R}^d)$. Then there exists a sequence $\{c_n\} \subset \mathbb{R}^d$ satisfying

$$\lim_{n\to\infty} \widehat{\mathcal{L}}(M)\left(n^{-1}z\right)^n e^{i\langle c_n, z\rangle} = \widehat{\sigma}_1(z).$$

If we let $c_s := c_{[s]}$ for $s \in [1,\infty) \setminus \mathbb{N}$, where $[s]$ denotes the largest integer not greater than s, it follows that

$$\lim_{[1,\infty)\ni s\to\infty} \widehat{\mathcal{L}}(M)\left(s^{-1}z\right)^s e^{i\langle c_s, z\rangle} = \widehat{\sigma}_1(z),$$

that is,

$$\lim_{t\uparrow 1} \left\{(1-t)^{-1} C_M\left((1-t)z\right) + i\langle c_{(1-t)^{-1}}, z\rangle\right\} = C_{\sigma_1}(z). \tag{7.4}$$

Letting

$$q(t) := -(1-t)\left\{c_{(1-t)^{-1}} - p\left((1-t)^{-1}-1\right)\right\},$$

we have

$$\begin{aligned}
&C_{\mathcal{L}(Z_t-q(t))^{(1-t)^{-1}}}(z) = (1-t)^{-1}C_{Z_t}(z) - i\left\langle(1-t)^{-1}q(t), z\right\rangle \\
&= (1-t)^{-1}C_M\left((1-t)z\right) + \int_0^{(1-t)^{-1}-1} C_{X_1}\left((1+u)^{-1}z\right)du - i\left\langle(1-t)^{-1}q(t), z\right\rangle \\
&\to C_{\sigma_1}(z) + C_{\widetilde{\rho}}(z) = C_\mu(z) \quad \text{as } t \uparrow 1.
\end{aligned}$$

We next show the "if" part. Assume that there exist a Lévy process $\{X_t, t \geq 0\}$ on $\mathbb{R}^d$ with $\mathcal{L}(X_1) \in I_1(\mathbb{R}^d)$, a 1-stable distribution σ_1 on $\mathbb{R}^d$ such that for some $\mathbb{R}^d$-valued random variable M independent of $\{X_t\}$ with $\mathcal{L}(M) \in \mathrm{NDA}(\sigma_1) \cap I(\mathbb{R}^d)$, there is a nonrandom function $q\colon [0,\infty) \to \mathbb{R}^d$ satisfying $\lim_{t\uparrow 1}\mathcal{L}\left(Z_t - q(t)\right)^{(1-t)^{-1}} = \mu$. Then (7.4) holds for some $c_s, s \in [1,\infty)$. It follows from (7.3) that

$$\begin{aligned}
C_{\mathcal{L}(Z_t-q(t))^{(1-t)^{-1}}}(z) &= (1-t)^{-1}C_M\left((1-t)z\right) + i\langle c_{(1-t)^{-1}}, z\rangle \\
&\quad + \int_0^{(1-t)^{-1}-1} C_{X_1}\left((1+u)^{-1}z\right)du - i\langle(1-t)^{-1}q(t) + c_{(1-t)^{-1}}, z\rangle.
\end{aligned}$$

Since $C_{\mathcal{L}(Z_t-q(t))^{(1-t)^{-1}}}(z)$ and $(1-t)^{-1}C_M\left((1-t)z\right) + i\langle c_{(1-t)^{-1}}, z\rangle$ converge as $t \uparrow 1$, so does $\int_0^{(1-t)^{-1}-1} C_{X_1}\left((1+u)^{-1}z\right)du - i\langle(1-t)^{-1}q(t) + c_{(1-t)^{-1}}, z\rangle$. Therefore, $\int_0^{(1-t)^{-1}-1}(1+u)^{-1}dX_u - (1-t)^{-1}q(t) - c_{(1-t)^{-1}}$ converges in probability as $t \uparrow 1$, since, in general, $\int_0^T f(s)dX_s - g(T)$ converges in probability as $T \to \infty$ if and only if $\int_0^T f(s)dX_s - g(T)$ converges in law as $T \to \infty$. Let $\widetilde{\rho} \in \Phi_{1,\mathrm{es}}(\mathcal{L}(X_1))$ be this limiting distribution. Then, letting $t \uparrow 1$ in the equation above, we have $\mu = \sigma_1 * \widetilde{\rho}$, which belongs to $L^{\langle 1\rangle}(\mathbb{R}^d)$ due to Theorem 5.1. □

7.2. Normal domains of attraction of α-selfdecomposable distributions

Jurek [7] showed that for $\alpha \in \mathbb{R}$, $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ if and only if there exists a sequence $\{\rho_j\} \subset I(\mathbb{R}^d)$ such that

$$\prod_{j=1}^n \widehat{\rho}_j\left(n^{-1}z\right)^{n^\alpha} \to \widehat{\mu}(z) \quad \text{as } n \to \infty, \quad z \in \mathbb{R}^d. \tag{7.5}$$

He constructed $\{\rho_j\}$ by using μ itself and ρ_b in the decomposability (1.7). Our next concern is a normal domain of attraction of $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$, when we regard (7.5) as a limit theorem. In other words, we are interested in finding a concrete nontrivial example of $\{\rho_j\}$ in (7.5), without using μ itself and ρ_b in the decomposability (1.7). In fact, O'Connor [18, 17] did it when he proved a limit theorem like above for $L^{\langle\alpha\rangle}(\mathbb{R}), \alpha \in [-1, 0)$. His method of constructing $\{\rho_j\}$ is to use stochastic integral characterizations. Here, we extend O'Connor's result to the case $\alpha < 2$ and general dimensions $d \in \mathbb{N}$. Furthermore, our method is related to the Langevin type equation (7.1). More precisely, Jurek's result above and Theorem 7.2 entail the following.

Corollary 7.3. (i) *Let $\alpha < 0$. Then, $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ if and only if there exists a Lévy process $\{X_t, t \geq 0\}$ on $\mathbb{R}^d$ such that for some and hence any $\rho_1 \in I(\mathbb{R}^d)$ and for*

$$\rho_j = \mathcal{L}\left(\int_{\{1-(j-1)^{-\alpha}\}/\alpha}^{(1-j^{-\alpha})/\alpha} (1-\alpha s)^{-1/\alpha} dX_s\right), \quad j = 2, 3, \ldots,$$

(7.5) holds.

(ii) *Let $0 < \alpha < 1$. Then, $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ if and only if there exist a Lévy process $\{X_t, t \geq 0\}$ on $\mathbb{R}^d$ with $\mathcal{L}(X_1) \in I_\alpha(\mathbb{R}^d)$ and a strictly α-stable distribution σ_α on $\mathbb{R}^d$ such that for some and hence any $\rho_1 \in$ NDA$(0, \sigma_\alpha) \cap I(\mathbb{R}^d)$ and for*

$$\rho_j = \mathcal{L}\left(\int_{\{1-(j-1)^{-\alpha}\}/\alpha}^{(1-j^{-\alpha})/\alpha} (1-\alpha s)^{-1/\alpha} dX_s\right), \quad j = 2, 3, \ldots,$$

(7.5) holds.

(iii) *Let $\alpha = 1$. Then, $\mu \in L^{\langle 1\rangle}(\mathbb{R}^d)$ if and only if there exist a Lévy process $\{X_t, t \geq 0\}$ on $\mathbb{R}^d$ with $\mathcal{L}(X_1) \in I_1(\mathbb{R}^d)$, a 1-stable distribution σ_1 on $\mathbb{R}^d$ such that for some and hence any $\rho_1 \in$ NDA$(\sigma_1) \cap I(\mathbb{R}^d)$, there is a nonrandom function $q\colon [0,1) \to \mathbb{R}^d$ satisfying that for ρ_1 above and for*

$$\rho_2 = \mathcal{L}\left(\int_0^{1/2} (1-s)^{-1} dX_s - 2\,q\,(1/2)\right),$$

$$\rho_j = \mathcal{L}\left(\int_{1-(j-1)^{-1}}^{1-j^{-1}} (1-s)^{-1} dX_s + (j-1)\,q\left(1-(j-1)^{-1}\right) - j\,q\left(1-j^{-1}\right)\right), \quad j = 3, 4, \ldots$$

(7.5) with $\alpha = 1$ holds.

(iv) *Let* $1 < \alpha < 2$. *Then,* $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ *if and only if there exist a Lévy process* $\{X_t, t \geq 0\}$ *on* $\mathbb{R}^d$ *with* $\mathcal{L}(X_1) \in I^0_\alpha(\mathbb{R}^d)$, *a strictly* α*-stable distribution* σ_α *on* $\mathbb{R}^d$ *and some* $c \in \mathbb{R}^d$ *such that for some and hence any* $\rho_1 \in \mathrm{NDA}(0, \sigma_\alpha) \cap I(\mathbb{R}^d)$ *and for*

$$\rho_2 = \mathcal{L}\left(\int_0^{(1-2^{-\alpha})/\alpha} (1-\alpha s)^{-1/\alpha} dX_s + 2^{1-\alpha} c\right),$$

$$\rho_j = \mathcal{L}\left(\int_{\{1-(j-1)^{-\alpha}\}/\alpha}^{(1-j^{-\alpha})/\alpha} (1-\alpha s)^{-1/\alpha} dX_s - (j-1)^{1-\alpha} c + j^{1-\alpha} c\right), \quad j = 3, 4, \ldots,$$

(7.5) *holds.*

8. Mild Ornstein-Uhlenbeck type processes

This section is concerned with the following *Langevin type equation* which is a version of (1.5) with $c = 1$ to general $\alpha \in (-\infty, 0) \cup (0, 2)$:

$$Z_t - Z_s = \int_s^t X(du) - \int_s^t (1-\alpha u)^{-1} Z_u du, \quad \begin{cases} 1/\alpha < s \leq t < \infty, & \text{when } \alpha < 0, \\ -\infty < s \leq t < 1/\alpha, & \text{when } 0 < \alpha < 2, \end{cases} \tag{8.1}$$

where X is an $\mathbb{R}^d$-valued homogeneous i.s.r.m. over $\mathbb{R}$. A stochastic process $\{Z_t\}$ is said to be a *solution* of the Langevin type equation (8.1) or an *Ornstein-Uhlenbeck type process* (OU type process) generated by (α, X) if $\{Z_t\}$ is a càdlàg process and satisfies (8.1) almost surely.

Maejima and Sato [14] introduced the concept of mild solutions to investigate the semi-version of the Langevin equation (1.5), and proved the equivalence between semi-stationarity and mildness of solutions of the Langevin equation. Now we introduce mildness of solutions of (8.1) in a similar way. For $\mu = \mu_{(A,\nu,\gamma)} \in I(\mathbb{R}^d)$, we define a nonrandom continuous function $q_\mu \colon (-\infty, 1) \to \mathbb{R}^d$ by

$$q_\mu(t) := \begin{cases} \displaystyle\int_t^0 (1-u)^{-1} du \left(\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+(1-u)^{-2}|x|^2} - \frac{1}{1+|x|^2}\right) \nu(dx)\right), & t \leq 0, \\ 0, & 0 < t < 1. \end{cases}$$

For a random variable X with $\mathcal{L}(X) = \mu \in I(\mathbb{R}^d)$, we may also write q_X for q_μ.

Definition 8.1 (Mild OU type processes). Suppose that X is an $\mathbb{R}^d$-valued homogeneous i.s.r.m. over $\mathbb{R}$ with $\mathcal{L}(X((0,1])) = \mu$.

(i) Let $\alpha < 0$. Then, an OU type process $\{Z_t\}$ generated by (α, X) is said to be *mild* if $\text{p-lim}_{t\downarrow 1/\alpha}(1-\alpha t)^{-1/\alpha} Z_t = 0$.

(ii) Let $0 < \alpha < 1$. Suppose that S_α is a strictly α-stable random variable independent of X. Then, an OU type process $\{Z_t\}$ generated by (α, X) is said to be *mild associated with* S_α if $\text{p-lim}_{t\downarrow -\infty}(1-\alpha t)^{-1/\alpha} Z_t = S_\alpha$.

(iii) Let $\alpha = 1$. Suppose that S_1 is a 1-stable random variable independent of X. Then, an OU type process $\{Z_t\}$ generated by $(1, X)$ is said to be *mild associated with* S_1 if $\text{p-lim}_{t\downarrow -\infty}\{(1-t)^{-1} Z_t + q_\mu(t)\} = S_1$.

(iv) Let $1 < \alpha < 2$. Suppose that S_α is an α-stable random variable independent of X. Then, an OU type process $\{Z_t\}$ generated by (α, X) is said to be *mild associated with* S_α if $\text{p-lim}_{t\downarrow -\infty}(1-\alpha t)^{-1/\alpha} Z_t = S_\alpha$.

Before stating the main theorem of this section, we prepare the following lemma, as mentioned in Section 2.

Lemma 8.2. *Let J be an interval in $\mathbb{R}$ which is infinite to the left, X an $\mathbb{R}^d$-valued i.s.r.m. over J, $f\colon J \to \mathbb{R}$ a nonrandom function, and $q\colon J \to \mathbb{R}^d$ a nonrandom continuous function. Assume that* $\text{p-lim}_{s\downarrow -\infty}\left(\int_s^t f(u)X(du) - q(s)\right)$ *exists for each $t \in J$. Then,* $\left\{\text{p-lim}_{s\downarrow -\infty}\left(\int_s^t f(u)X(du) - q(s)\right), t \in J\right\}$ *has a càdlàg modification.*

Proof. Define a process $\{Y_t, t \geq 0\}$ by

$$Y_t = \begin{cases} \text{p-lim}_{s\downarrow -\infty}\left(\int_s^{\tau(t)} f(u)X(du) - q(s)\right) + q(\tau(t)), & \text{for } t > 0, \\ 0, & \text{for } t = 0, \end{cases}$$

where

$$\tau(t) = \begin{cases} \log t, & \text{if } J = \mathbb{R}, \\ a \wedge \log t, & \text{if } J = (-\infty, a] \text{ with } a \in \mathbb{R}, \\ \mathbb{1}_{(0,1]}(t)\log t + \mathbb{1}_{(1,\infty)}(t) 2\pi^{-1} a \arctan(t-1), & \text{if } J = (-\infty, a) \text{ with } a \in \mathbb{R}, \end{cases}$$

which is an nondecreasing continuous function on $(0,\infty)$ onto J. Then $\{Y_t\}$ is an additive process in law and thus it has a càdlàg modification. This implies the existence of càdlàg modification of the process $\left\{\text{p-lim}_{s\downarrow -\infty}\left(\int_s^t f(u)X(du) - q(s)\right), t \in J\right\}$. $\square$

By the lemma above, $\left\{\text{p-lim}_{s\downarrow-\infty}\left(\int_s^t f(u)X(du) - q(s)\right), t\in J\right\}$ is regarded as a càdlàg process, as mentioned in Section 2.

The following is the main theorem of this section.

Theorem 8.3. *Suppose that X is an $\mathbb{R}^d$-valued homogeneous i.s.r.m. over $\mathbb{R}$ with $\mathcal{L}(X((0,1])) = \mu$.*

(i) *Let $\alpha < 0$. Then, $\{Z_t\}$ defined by*

$$Z_t = (1-\alpha t)^{1/\alpha}\int_{1/\alpha}^t (1-\alpha u)^{-1/\alpha} X(du), \quad 1/\alpha < t < \infty, \tag{8.2}$$

is an almost surely unique mild OU type process generated by (α, X) and satisfies that for all $t \in (1/\alpha, \infty)$,

$$\mathcal{L}(Z_t)^{(1-\alpha t)^{-1}} = \Phi_\alpha(\mu) \in L^{\langle\alpha\rangle}(\mathbb{R}^d). \tag{8.3}$$

(ii) *Let $0 < \alpha < 1$ and let S_α be a strictly α-stable random variable independent of X. Then, there exists a mild OU type process generated by (α, X) associated with S_α if and only if $\mu \in I_\alpha(\mathbb{R}^d)$, in which case,*

$$Z_t = (1-\alpha t)^{1/\alpha}\left\{S_\alpha + \int_{-\infty}^t (1-\alpha u)^{-1/\alpha} X(du)\right\}, \quad -\infty < t < 1/\alpha, \tag{8.4}$$

is an almost surely unique mild OU type process generated by (α, X) associated with S_α, and satisfies that for all $t \in (-\infty, 1/\alpha)$,

$$\mathcal{L}(Z_t)^{(1-\alpha t)^{-1}} = \mathcal{L}(S_\alpha) * \Phi_\alpha(\mu) \in L^{\langle\alpha\rangle}(\mathbb{R}^d). \tag{8.5}$$

(iii) *Let $\alpha = 1$ and let S_1 be a 1-stable random variable independent of X. Then, there exists a mild OU type process generated by $(1, X)$ associated with S_1 if and only if $\mu \in I_1(\mathbb{R}^d)$, in which case,*

$$Z_t = (1-t)\left\{S_1 + \underset{s\downarrow-\infty}{\text{p-lim}}\left(\int_s^t (1-u)^{-1} X(du) - q_\mu(s)\right)\right\}, \quad -\infty < t < 1, \tag{8.6}$$

is an almost surely unique mild OU type process generated by $(1, X)$ associated with S_1, and there exist $\widetilde{\rho} \in \Phi_{1,\mathrm{es}}(\mu)$ and a nonrandom function $p\colon (-\infty, 1) \to \mathbb{R}^d$ satisfying $p(0) = 0$ such that for all $t \in (-\infty, 1)$,

$$\mathcal{L}(Z_t - p(t))^{(1-t)^{-1}} = \mathcal{L}(S_1) * \widetilde{\rho} \in L^{\langle 1\rangle}(\mathbb{R}^d). \tag{8.7}$$

(iv) *Let $1 < \alpha < 2$ and let S_α be an α-stable random variable independent of X. Then, there exists a mild OU type process generated by (α, X) associated with S_α if and only if $\mu \in I^0_\alpha(\mathbb{R}^d)$, in which case, $\{Z_t\}$ having the same form of that in (8.4) is an almost surely unique mild OU type process generated by (α, X) associated with S_α, and satisfies that for all $t \in (-\infty, 1/\alpha)$,*

$$\mathcal{L}\left(Z_t - \left\{(1-\alpha t)^{1/\alpha} - (1-\alpha t)\right\} c\right)^{(1-\alpha t)^{-1}} = \mathcal{L}(S_\alpha) * \Phi_\alpha(\mu) \in L^{\langle\alpha\rangle}(\mathbb{R}^d),$$

where $c \in \mathbb{R}^d$ is a constant for which $\mathcal{L}(S_\alpha - c)$ is strictly α-stable.

Proof. (i) $\{Z_t\}$ in (8.2) is a càdlàg process. Note that for every $1/\alpha < s \le t < \infty$, this $\{Z_t\}$ satisfies

$$Z_t = (1-\alpha t)^{1/\alpha} \left\{(1-\alpha s)^{-1/\alpha} Z_s + \int_s^t (1-\alpha u)^{-1/\alpha} X(du)\right\} \quad \text{a.s.}$$

Since the both sides of the equation above have càdlàg sample paths in t, it follows that for each fixed $s \in (1/\alpha, \infty)$, almost surely,

$$Z_t = (1-\alpha t)^{1/\alpha} \left\{(1-\alpha s)^{-1/\alpha} Z_s + \int_s^t (1-\alpha u)^{-1/\alpha} X(du)\right\}, \quad \text{for } t \in [s, \infty). \tag{8.8}$$

This yields that almost surely,

$$Z_t - Z_s = \int_s^t X(du) - \int_s^t (1-\alpha u)^{-1} Z_u du, \quad \text{for } t \in [s, \infty),$$

due to Theorem 6.1 by letting $t_0 = s$ and $M = Z_s$. Since the both sides of the equation above have càdlàg sample paths in s, we have (8.1) almost surely. Looking at the form (8.2), we have the mildness of $\{Z_t\}$. Furthermore, it follows that

$$\begin{aligned}(1-\alpha t)^{-1} C_{Z_t}(z) &= (1-\alpha t)^{-1} \int_{1/\alpha}^t C_\mu\left((1-\alpha t)^{1/\alpha}(1-\alpha u)^{-1/\alpha} z\right) du \\ &= \int_0^{-1/\alpha} C_\mu\left((1+\alpha v)^{-1/\alpha} z\right) dv = C_{\Phi_\alpha(\mu)}(z),\end{aligned}$$

which yields (8.3). We next show the almost sure uniqueness of mild OU type processes generated by (α, X). Let $\{Z_t\}$ be a mild OU type process generated by (α, X). Theorem 6.1 yields (8.8) a.s. for each fixed $s \in (1/\alpha, \infty)$. Then for each (s,t) with $1/\alpha < s \le t < \infty$, we have

$$(1-\alpha t)^{-1/\alpha} Z_t - (1-\alpha s)^{-1/\alpha} Z_s = \int_s^t (1-\alpha u)^{-1/\alpha} X(du) \quad \text{a.s.}$$

Letting $s \downarrow 1/\alpha$, we have that for each $t \in (1/\alpha, \infty)$,

$$(1-\alpha t)^{-1/\alpha} Z_t = \int_{1/\alpha}^t (1-\alpha u)^{-1/\alpha} X(du) \quad \text{a.s.}$$

However, since the both sides of the equation above have càdlàg sample paths, it holds almost surely that

$$(1-\alpha t)^{-1/\alpha} Z_t = \int_{1/\alpha}^t (1-\alpha u)^{-1/\alpha} X(du), \quad \text{for } t \in (1/\alpha, \infty),$$

which yields the uniqueness.

(ii) Note that $\mu \in I_\alpha(\mathbb{R}^d) = \mathfrak{D}(\Phi_\alpha)$ if and only if $\int_{-\infty}^0 (1-\alpha u)^{-1/\alpha} X(du)$ is definable due to Lemma 4.8 of [14]. Let $\mu \in I_\alpha(\mathbb{R}^d)$. Then $\{Z_t\}$ in (8.4) is a càdlàg process, and satisfies (8.1) a.s. in a similar way to (i). Looking at the form (8.4), we have the mildness associated with S_α of $\{Z_t\}$. Furthermore, it follows that

$$\begin{aligned}
&(1-\alpha t)^{-1} C_{Z_t}(z) \\
&\quad = (1-\alpha t)^{-1} \left\{ C_{S_\alpha}\left((1-\alpha t)^{1/\alpha} z\right) + \int_{-\infty}^t C_\mu\left((1-\alpha t)^{1/\alpha}(1-\alpha u)^{-1/\alpha} z\right) du \right\} \\
&\quad = C_{S_\alpha}(z) + \int_0^\infty C_\mu\left((1+\alpha v)^{-1/\alpha} z\right) dv = C_{S_\alpha}(z) + C_{\Phi_\alpha(\mu)}(z),
\end{aligned}$$

which yields (8.5). The almost sure uniqueness of mild OU type processes generated by (α, X) associated with S_α are obtained in a similar way to (i). We next show that the existence of a mild OU type process generated by (α, X) associated with S_α implies that $\mu \in I_\alpha(\mathbb{R}^d)$. If $\{Z_t\}$ is a mild OU type process generated by (α, X) associated with S_α, then by Theorem 6.1, for each (s,t) with $-\infty < s \le t < 1/\alpha$, we have

$$(1-\alpha t)^{-1/\alpha} Z_t - (1-\alpha s)^{-1/\alpha} Z_s = \int_s^t (1-\alpha u)^{-1/\alpha} X(du) \quad \text{a.s.}$$

Letting $t = 0$ and $s \downarrow -\infty$, we have the existence of the limit in probability of $\int_s^0 (1-\alpha u)^{-1/\alpha} X(du)$ as $s \downarrow -\infty$ since $\text{p-lim}_{t\downarrow -\infty}(1-\alpha t)^{-1/\alpha} Z_t = S_\alpha$. This implies that $\mu \in I_\alpha(\mathbb{R}^d)$.

(iii) Due to Lemma 4.8 of [14], $\int_s^0 (1-u)^{-1} X(du) - q_\mu(s)$ converges in probability as $s \downarrow -\infty$ if and only if $\int_0^t (1+u)^{-1} X(-du) - q_\mu(-t)$ converges in probability as $t \uparrow \infty$. If (A_t, ν_t, γ_t) denotes the Lévy-Khintchine triplet of $\mathcal{L}\left(\int_0^t (1+u)^{-1} X(-du)\right)$, then

$$A_t = \int_0^t (1+u)^{-2} A du, \quad \nu_t(B) = \int_0^t \nu\left((1+u)B\right) du, \quad \gamma_t = q_\mu(-t),$$

where (A, ν, γ) is the Lévy-Khintchine triplet of $\mu = \mathcal{L}(X((0,1]))$. Then, it follows from Lemma 5.4 and Proposition 5.6 of [24] that $\int_s^0 (1-u)^{-1} X(du) - q_\mu(s)$ converges in probability as $s \downarrow -\infty$ if and only if $\mu \in \mathfrak{D}(\Phi_{1,\mathrm{es}}) = I_1(\mathbb{R}^d)$. Let $\mu \in I_1(\mathbb{R}^d)$. Then $\{Z_t\}$ in (8.6) is a càdlàg process, and satisfies (8.1) a.s. in a similar way to (i). Since

$$\begin{aligned}
&(1-t)^{-1} Z_t + q_\mu(t) \\
&\quad = S_1 + \operatorname*{p-lim}_{s\downarrow -\infty}\left(\int_s^0 (1-u)^{-1} X(du) - q_\mu(s)\right) - \left(\int_t^0 (1-u)^{-1} X(du) - q_\mu(t)\right) \\
&\quad \to S_1 \quad \text{in probability as } t \downarrow -\infty,
\end{aligned}$$

we have the mildness associated with S_1 of $\{Z_t\}$. The 1-stability of $\mathcal{L}(S_1)$ yields that $(1-t)^{-1} C_{S_1}\left((1-t)z\right) = C_{S_1}(z) + i\langle c(t), z\rangle$ for some function $c(t)$ satisfying $c(0) = 0$. Then, it follows that

$$\begin{aligned}
&(1-t)^{-1} C_{Z_t}(z) \\
&\quad = (1-t)^{-1}\left\{ C_{S_1}\left((1-t)z\right) + \lim_{s\downarrow -\infty}\left(\int_s^t C_\mu\left((1-t)(1-u)^{-1} z\right) du - i\langle (1-t) q_\mu(s), z\rangle\right)\right\} \\
&\quad = C_{S_1}(z) + i\langle c(t), z\rangle + \lim_{s\downarrow -\infty}\left(\int_0^{(1-s)(1-t)^{-1}-1} C_\mu\left((1+v)^{-1} z\right) dv - i\langle q_\mu(s), z\rangle\right) \\
&\quad = C_{S_1}(z) + \lim_{s\downarrow -\infty}\left\{\int_0^{(1-s)(1-t)^{-1}-1} C_\mu\left((1+v)^{-1} z\right) dv - i\langle q_\mu\left(-(1-s)(1-t)^{-1}+1\right), z\rangle\right. \\
&\qquad\qquad \left. + i\left\langle c(t) - q_\mu(s) + q_\mu\left(-(1-s)(1-t)^{-1}+1\right), z\right\rangle\right\}.
\end{aligned}$$

Note that the law of $\int_0^{(1-s)(1-t)^{-1}-1}(1+v)^{-1}X(dv)-q_\mu\left(-(1-s)(1-t)^{-1}+1\right)$ tends to some $\widetilde{\rho}\in\Phi_{1,\mathrm{es}}(\mu)$ as $s\downarrow-\infty$ and this limit does not depend on t. Then $p(t):=(1-t)\lim_{s\downarrow-\infty}\{c(t)-q_\mu(s)+q_\mu\left(-(1-s)(1-t)^{-1}+1\right)\}$ exists in $\mathbb{R}^d$. This function satisfies that $p(0)=0$, since $c(0)=0$. Then we have (8.7). To prove the almost sure uniqueness of mild OU type processes generated by $(1,X)$ associated with S_1, let $\{Z_t\}$ be such a process. Then, Theorem 6.1 yields that, for each fixed $s\in(-\infty,1)$, almost surely,

$$Z_t=(1-t)\left\{(1-s)^{-1}Z_s+\int_s^t(1-u)^{-1}X(du)\right\},\quad\text{for } t\in[s,1).$$

Then for each (s,t) with $-\infty<s\le t<1$, we have

$$(1-t)^{-1}Z_t-(1-s)^{-1}Z_s-q_\mu(s)=\int_s^t(1-u)^{-1}X(du)-q_\mu(s)\quad\text{a.s.}\tag{8.9}$$

Letting $s\downarrow-\infty$, we have that for each $t\in(-\infty,1)$,

$$(1-t)^{-1}Z_t-S_1=\operatorname*{p-lim}_{s\downarrow-\infty}\left(\int_s^t(1-u)^{-1}X(du)-q_\mu(s)\right)\quad\text{a.s.}$$

However, since the both sides of the equation above have càdlàg sample paths, it holds almost surely that

$$(1-t)^{-1}Z_t-S_1=\operatorname*{p-lim}_{s\downarrow-\infty}\left(\int_s^t(1-u)^{-1}X(du)-q_\mu(s)\right),\quad\text{for } t\in(-\infty,1),$$

which yields the uniqueness. We next show that the existence of a mild OU type process generated by $(1,X)$ associated with S_1 implies that $\mu\in I_1(\mathbb{R}^d)$. If $\{Z_t\}$ is a mild OU type process generated by $(1,X)$ associated with S_1, then (8.9) holds in the same way as above. Letting $t=0$ and $s\downarrow-\infty$, we have the existence of the limit in probability of $\int_s^0(1-u)^{-1}X(du)-q_\mu(s)$ as $s\downarrow-\infty$, which implies that $\mu\in I_1(\mathbb{R}^d)$.

(iv) It is proved in a similar way to (ii). □

We conclude this paper with the continuity in $\alpha\in(-\infty,1)\cup(1,2)$ of mild OU type processes. Let

$$\mathbb{T}_\alpha=\begin{cases}(1/\alpha,\infty), & \text{when } \alpha\in(-\infty,0),\\ \mathbb{R}, & \text{when } \alpha=0,\\ (-\infty,1/\alpha), & \text{when } \alpha\in(0,1)\cup(1,2),\end{cases}$$

and let X be an $\mathbb{R}^d$-valued homogeneous i.s.r.m. over $\mathbb{R}$. Define $\{Z_t^{(\alpha,X)}, t \in \mathbb{T}_\alpha\}$ by

$$Z_t^{(\alpha,X)} = \begin{cases} (1-\alpha t)^{1/\alpha} \displaystyle\int_{1/\alpha}^{t} (1-\alpha u)^{-1/\alpha} X(du), & \text{for } \begin{cases} \alpha < 0, \\ \mathcal{L}(X((0,1])) \in I(\mathbb{R}^d), \end{cases} \\ e^{-t} \displaystyle\int_{-\infty}^{t} e^{u} X(du), & \text{for } \begin{cases} \alpha = 0, \\ \mathcal{L}(X((0,1])) \in I_{\log}(\mathbb{R}^d), \end{cases} \\ (1-\alpha t)^{1/\alpha} \displaystyle\int_{-\infty}^{t} (1-\alpha u)^{-1/\alpha} X(du), & \text{for } \begin{cases} 0 < \alpha < 1, \\ \mathcal{L}(X((0,1])) \in I_\alpha(\mathbb{R}^d), \end{cases} \\ (1-\alpha t)^{1/\alpha} \displaystyle\int_{-\infty}^{t} (1-\alpha u)^{-1/\alpha} X(du), & \text{for } \begin{cases} 1 < \alpha < 2, \\ \mathcal{L}(X((0,1])) \in I_\alpha^0(\mathbb{R}^d). \end{cases} \end{cases}$$

Then, $\{Z_t^{(\alpha,X)}, t \in \mathbb{T}_\alpha\}$ is the unique mild OU type process generated by (α, X) when $\alpha \in (-\infty, 0)$, the unique stationary OU type process (1.6) with $c = 1$ when $\alpha = 0$, and the unique mild OU type process generated by (α, X) associated with 0 when $\alpha \in (0,1) \cup (1,2)$. Proposition 5.4 and Theorem 8.3 implies the continuity of $\mathcal{L}(Z_t^{(\alpha,X)})$ in $\alpha \in [\alpha_1, \alpha_2]$ with respect to weak convergence for each fixed X with

$$\mathcal{L}(X((0,1])) \in \begin{cases} I(\mathbb{R}^d), & \text{when } \alpha_2 < 0, \\ I_{\log}(\mathbb{R}^d), & \text{when } \alpha_2 = 0, \\ I_{\alpha_2}(\mathbb{R}^d), & \text{when } 0 < \alpha_2 < 1, \\ I_{\alpha_2}^0(\mathbb{R}^d), & \text{when } 1 < \alpha_2 < 2, \end{cases} \tag{8.10}$$

and any fixed $t \in \mathbb{T}_{\alpha_1} \cap \mathbb{T}_{\alpha_2}$, where $[\alpha_1, \alpha_2]$ is an interval included in $(-\infty, 1) \cup (1, 2)$. However, we can get a stronger result as follows.

Theorem 8.4. *Let $\alpha, \alpha_2 \in (-\infty, 1) \cup (1, 2)$, $\alpha \le \alpha_2$, and let X satisfy* (8.10). *Let $n \in \mathbb{N}$ and fix $t_1, t_2, \ldots, t_n \in \mathbb{T}_\alpha$ satisfying $t_1 < t_2 < \cdots < t_n$. Then*

$$\mathcal{L}\left(\left(Z_{t_1}^{(\beta,X)}, Z_{t_2}^{(\beta,X)}, \ldots, Z_{t_n}^{(\beta,X)}\right)\right) \to \mathcal{L}\left(\left((Z_{t_1}^{(\alpha,X)}, Z_{t_2}^{(\alpha,X)}, \ldots, Z_{t_n}^{(\alpha,X)}\right)\right), \tag{8.11}$$

as $(-\infty, \alpha_2) \cap \{\beta' \in (-\infty, 1) \cup (1, 2) \colon t_1, t_2, \ldots, t_n \in \mathbb{T}_{\beta'}\} \ni \beta \to \alpha$.

Proof. Define a function φ_α on $\mathbb{T}_\alpha$ by

$$\varphi_\alpha(u) = \begin{cases} (1-\alpha u)^{-1/\alpha}, & \text{when } \alpha \neq 0, \\ e^u, & \text{when } \alpha = 0. \end{cases}$$

Then,

$$\begin{pmatrix} Z_{t_1}^{(\beta,X)} \\ Z_{t_2}^{(\beta,X)} \\ \vdots \\ Z_{t_n}^{(\beta,X)} \end{pmatrix} = \begin{pmatrix} \varphi_\beta(t_1)^{-1} & & & 0 \\ \varphi_\beta(t_2)^{-1} & \varphi_\beta(t_2)^{-1} & & \\ \vdots & & \ddots & \\ \varphi_\beta(t_n)^{-1} & \varphi_\beta(t_n)^{-1} & \cdots & \varphi_\beta(t_n)^{-1} \end{pmatrix} \begin{pmatrix} \int_{\inf \mathbb{T}_\beta}^{t_1} \varphi_\beta(u) X(du) \\ \int_{t_1}^{t_2} \varphi_\beta(u) X(du) \\ \vdots \\ \int_{t_{n-1}}^{t_n} \varphi_\beta(u) X(du) \end{pmatrix}.$$

Note that $\varphi_\beta(t_k)^{-1} \to \varphi_\alpha(t_k)^{-1}$ as $\beta \to \alpha$ for all $k = 1, 2, \dots, n$ and that

$$\int_{\inf \mathbb{T}_\beta}^{t_1} \varphi_\beta(u) X(du),\ \int_{t_1}^{t_2} \varphi_\beta(u) X(du), \dots,\ \int_{t_{n-1}}^{t_n} \varphi_\beta(u) X(du),$$

are independent. Hence it suffices to prove that $\int_{\inf \mathbb{T}_\beta}^{t_1} \varphi_\beta(u) X(du) \to \int_{\inf \mathbb{T}_\alpha}^{t_1} \varphi_\alpha(u) X(du)$ and $\int_{t_k}^{t_{k+1}} \varphi_\beta(u) X(du) \to \int_{t_k}^{t_{k+1}} \varphi_\alpha(u) X(du)$ in law as $\beta \to \alpha$ for each $k = 1, 2, \dots, n-1$. We have

$$\begin{aligned} C_{\int_{\inf \mathbb{T}_\beta}^{t_1} \varphi_\beta(u) X(du)}(z) &= \int_{\inf \mathbb{T}_\beta}^{t_1} C_{X((0,1])}(\varphi_\beta(u) z) du = \int_0^{\varphi_\beta(t_1)} C_{X((0,1])}(sz) s^{-\beta-1} ds \\ &= \int_0^{\varphi_\beta(t_1)\wedge 1} C_{X((0,1])}(sz) s^{-\beta-1} ds + \int_{\varphi_\beta(t_1)\wedge 1}^{\varphi_\beta(t_1)} C_{X((0,1])}(sz) s^{-\beta-1} ds \\ &=: I_1(\beta) + I_2(\beta) \quad \text{say.} \end{aligned}$$

Then $I_1(\beta) \to I_1(\alpha)$ as $\beta \to \alpha$ by a similar argument as that in the proof of Proposition 5.4. If $\varphi_\beta(t_1) > 1$, then $I_2(\beta) = \int_1^{\varphi_\beta(t_1)} C_{X((0,1])}(sz) s^{-\beta-1} ds$. We may assume that $t_1 \in \mathbb{T}_{\alpha_2}$ and $(-\infty, 1) \cup (1, 2) \supset [\alpha_1, \alpha_2] \ni \beta \to \alpha$ with α_1 satisfying $t_1 \in \mathbb{T}_{\alpha_1}$. Note that $\varphi_\beta(t_1)$ is continuous in β on $[\alpha_1, \alpha_2]$. Then $\mathbb{1}_{(1, \varphi_\beta(t_1))}(s) |C_{X((0,1])}(sz)| s^{-\beta-1} \le |C_{X((0,1])}(sz)| s^{-\alpha_1-1}$ for all $s \in \big(1, \max_{\beta' \in [\alpha_1, \alpha_2]} \varphi_{\beta'}(t_1)\big)$ and all $\beta \in [\alpha_1, \alpha_2]$, and $\int_1^{\max_{\beta' \in [\alpha_1, \alpha_2]} \varphi_{\beta'}(t_1)} |C_{X((0,1])}(sz)| s^{-\alpha_1-1} ds < \infty$. Therefore we can apply the dominated convergence theorem and we have $I_2(\beta) \to I_2(\alpha)$ as $\beta \to \alpha$. Thus

$$C_{\int_{\inf \mathbb{T}_\beta}^{t_1} \varphi_\beta(u) X(du)}(z) \to I_1(\alpha) + I_2(\alpha) = C_{\int_{\inf \mathbb{T}_\alpha}^{t_1} \varphi_\alpha(u) X(du)}(z),$$

as $\beta \to \alpha$. In a similar way, we also have that $C_{\int_{t_k}^{t_{k+1}} \varphi_\beta(u) X(du)}(z) \to C_{\int_{t_k}^{t_{k+1}} \varphi_\alpha(u) X(du)}(z)$ as $\beta \to \alpha$ for each $k = 1, 2, \dots, n-1$. □

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